UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

COURSE SYLLABUS

Geometry II

2122-2-E3501Q011

Aims

The aim of the course is to introduce the foundation of the theory and of the use of differential forms on open sets of Euclidean spaces, as a basis for the general treatment in the context of differentiable manifolds.

Differential forms are a tool of pervasive and fundamental importance in Geometry, Differential Topology, and Analysis; they are furthermore unavoidable in the modern formulation of physical laws.

The theory will be developed from its algebraic first principles, that is, from the basic notion of a tensor in linear algebra.

The expected learning outcomes include the following:

- the knowledge and understanding of the basic definitions and statements, as well as of the basic strategies of proof in the theory of differential forms; the knowledge and understanding of some of the most relevant basic applications, notably to the study of smooth proper maps between opens sets of Euclidean spaces; the knowledge and understanding of some of the key foundational examples of the theory;
- the ability to apply the acquired abstract knowledge to the solution of simple computational exercises and theoretical problems, referring in a precise and well-organized manner to the pertinent results being used; the ability to master the algebraic, differential and integral calculus of differential forms, and to use it in the some simple practical situations, such as the study of proper maps between open sets of Euclidean spaces; the ability to apply the theoretical background to the construction and discussion of simple examples and solution of exercises; the ability to expose and communicate effectively and clearly the theoretical content of the course.

Contents

Alternating multilinear algebra; differential forms on Euclidean space and their operations; Poincaré Lemma; applications to physics; integration; change of variables; degree of a proper differentiable map between open sets in Euclidean spaces and applications; Theorems of Gauss-Green and Stokes; De Rham Theory (brief outline).

Detailed program

Exterior algebra of a vector space and its operations: exterior product, contractions; oriented Euclidean vector spaces and their volume elements; vector fields and differential forms; exterior differential; closed and exact forms; winding number and applications; gradient, rotor and divergence; differential forms under smooth maps: pull-back; integration; integration and homotopy; change of variable formula; Poincaré Lemma; Poincaré Lemma with compact support; integration on oriented parametrized varieties; Theorems of Gauss-Green and Stokes; degree of a proper smooth map between open sets in an Euclidean space and its computation; invariance under smooth proper homotopy; applications: the Fundamental Theorem of Algebra and the Brower Fixed Point Theorem.

Prerequisites

The content of the courses of Analysis I and (in part) II, Linear Algebra and Geometry, Geometry I.

Teaching form

Live lessons at the blackboard (6 CFU) and live exercise sessions at the blackboard (2 CFU).

Textbook and teaching resource

Reference text: teacher's notes on e-learning

Recommended reading:

the following book is especially pertinent to the content of this course:

- V. Guillemin and P. Haine, Differential forms, World Scientific 2019

Further recommended textbooks are:

- M. Do Carmo, Differential forms and applications, Springer Verlag 1996;

- V. Guillemin, A. Pollack, Differential Topology 1974;
- W. Fulton, Differential Topology, a first course, Springer Verlag 1995.

Semester

2nd semester

Assessment method

The exam may be passed either by taking two written partial tests during the course, or in the regular exam sessions following the course.

The partial tests consist in a flexible combination of exercises and theoretical questions, and each only covers a part of the program; the exact subdivsion will be comminicated well in advance during the course. They will be followed by a conclusive oral discussion (see below). To pass the exam, a minimum passing grade of 18 is required in both parts.

The regular exam sessions, on the other hand, comprise two written tests, a practical and a theoretical one, each referred to the whole course, followed by an oral discussion. In the practical test, the student will be asked to solve various computational exercises, while in the theoretical test there will be questions involving definitions, statement's of theorems, proofs, construction of examples and counterexamples, and simple theoretical problems.

The aim of the final discussion is typically to expose the evaluation of the student's script; only in special cases, where the student's competence can't be clearly assessed by the scripts, will the discussion contribute to the final evaluation.

The practical tests will measure the student's ability to master the acquired formalism and apply it to some simple computations, to build on the acquired theoretical knowledge, and to invoke it in a pertinent and precise manner.

The theoretical tests will evaluate the knowledge and understanding of the conceptual framework of the course, as well as the ability to expose it in a well-organized, consistent and effective manner.

In order to successfully complete the exam, the student needs to first pass the practical test, thus obtaining a grade of at least 18/30, and then to also obtain the passing grade in the theoretical test of the same session or, upon his/her choice, of the session immediately following.

To each exercise/theoretical question (or problem) a maximum partial grade will be assigned by the commission, depending on its difficulty and length; in the evaluation, every student will be given a grade in correspondence to each exercise/theoretical question (or problem) up to the maximum one, measuring the exactness, the completeness, the rigour, the clarity and the overall coherence of the development.

Office hours

Upon appointment.