UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

SYLLABUS DEL CORSO

Analisi Superiore

2122-1-F4001Q055

Aims

The aim of the course is to introduce some basic material of wide use in analysis. This is done by illustrating how the theories explored interact with the Dirichlet problem for the Laplacian.

The expected learning outcomes include:

- the knowledge and understanding of the fundamental definitions and statements, as well as of the basic strategies of proof of modern analysis; the knowledge and understanding of some crucial examples in which the theory manifests itself;
- the ability to recognize the role that concepts and techniques from modern analysis introduced in the lectures (such as convolution, distributions, Sobolev spaces) play in various areas of pure and applied mathematics (numerical analysis, mathematical physics, probability); the skill to apply such conceptual background to the construction of concrete examples and to the solution of exercises; the ability to communicate and explain in a clear and precise manner both the theoretical aspects of the course and their applications to specific situations, possibly to different contexts.

Contents

Basics on convolution, the Dirichlet problem in the ball and in the half space, distributions, regularity of distributions, Sobolev spaces, second order elliptic problems.

Detailed program

Chapter 0. **Preliminaries**
Convolution. Hypersurfaces of class C^k in R^n. The divergence theorem and Green formulas. Complex measures.

Chapter 1. **The classical Dirichlet problem**
Harmonic functions. Mean value theorems for harmonic functions. Characterization of harmonic functions via mean value theorems.The maximum principle for harmonic functions and the uniqueness of the Dirichlet problem.The Newtonian potential. Green's representation formula. The Green's function. The Poisson kernel. Green's function and the Poisson kernel for the half-space. Further properties of harmonic funcitons: estimates for derivative, Schwarz's reflection principle and Liouville's theorem. Classical solution for the Dirichlet problem on the sphere and on the half-spce.

Chapter 2. **L^p data and convergence to boundary**
Poisson integral of measures and L^p functions. Weak ? convergence. Solution of the Dirichlet problem with L^p boundary data. Operators of weak type (1, 1). Marcinkiewicz interpolation theorem. The Hardy–Littlewood maximal operator. A covering lemma. Boundedness properties of the Hardy–Littlewood maximal function. Lebesgue's differentiation theorem. Nontangential convergence of Poisson integrals.

Chapter 3. **Distributions and their derivatives**
Distributions. Examples. Derivatives of distributions. Examples.

Chapter 4. **Sobolev spaces**
Motivations, definitions and properties. Properties of Sobolev spaces: $W^{k,p}$(?) is a Banach space, approximation by smooth functions, product and composition of Sobolev spaces. Sobolev spaces in dimension 1: existence of a continuous representative and fundamental theorem of calculus for $W^{1,p}$(a, b) functions. Morrey Theorem. Sobolev inequality (Sobolev-Gagliardo-Nirenberg Theorem). Sobolev embeddings. Extension operator and Extension Theorem for half-spaces and bounded regular domains. Global approximation by smooth functions. Sobolev embeddings for extension domains. Embeddings for higher order Sobolev spaces. Rellich-Kondrachov Theorem. Existence of the Trace Operator ?$_0^p$: $W^{1,p}$(?) ? L^p(??) with 1? p < +? and ? being a half-space or bounded regular domain. Fractional Sobolev spaces and Gagliardo Theorem (hints). Characterization of $W^{1,p}_0$(?) by traces.

Chapter 5. **Second order elliptic problems**
Lax-Milgram Lemma. Second order elliptic problems: variational formulation, existence of solutions. Poincaré inequality. Dirichlet Principle. Elliptic problems with Neumann boundary conditions: variational formulation and some hints on H(div,?) spaces. Poincaré?-Wirtinger inequality. Existence of solutions for the Neumann problem under compatibility conditions

Prerequisites

Calculus in several variables, linear algebra, basics of Hilbert and L^p spaces.

Teaching form

Lectures with blackboard. The teaching hours will be dedicated either to the illustration of main results in the theory, or to the solution of exercises (previously assigned) containing (possibly fine) applications of the theory.

Textbook and teaching resource

- Notes available on the e-learning page of the course.
- A. Bressan. *Lecture Notes on Functional Analysis*. American Mathematical Society, 2013.
- H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.
- L.C. Evans. *Partial differential equations*, American Mathematical Society.

Semester

I semester.

Assessment method

The exam consists of a written test, aimed at verifying the level of knowledge, the ability to apply it to the resolution of exercises, the student's independence in making judgements, as well as his/her communication skills. The test is divided into two parts: the first part contains theoretical questions (proofs of part of the results illustrated during the course), while the second part contains exercises, often similar to those solved during the class hours. The two parts will contribute equally to the determination of the final grade.

Office hours

Upon appointment.