UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

SYLLABUS DEL CORSO

Metodi Matematici per L'analisi Economica – Controllo Ottimo

2223-1-F4001Q094

Aims

In line with the educational objectives of the Master Degree in Mathematics, the course aims at providing the *knowledge* about the fundamental concepts and statements of the theory of optimal control using the varionational approach and using the dynamic programming. Moreover, we will introduce the students to the theory of the differential games. It will also build the *skills* needed to understand and use the most important proving arguments and techniques in the theory and the *ability* to solve exercises and deal with several models and applications.

Contents

Optimal control problems with the variational approach: theory and economic models. Optimal control problems with dynamic programming: theory and economic models. Introduction to differential games.

Detailed program

1. INTRODUCTION TO THE OPTIMAL CONTROL
a. Some problems. The moon landing problem, in boat with Pontryagin, a model of optimal consumption, the "lady in the lake".
b. Statement of an optimal control problem Definition of control, dynamics, trajectory, control set. Admissible control. Importance of the case of linear dynamics.

2. THE OPTIMAL CONTROL WITH THE VARIATIONAL APPROACH
a. The simplest problem of optimal control The theorem of Pontryagin (PROOF in the case of control set U = R): comments and consequences of the Maximum principle. Extremal control, associated multiplier. Normal and abnormal controls. Sufficient conditions of optimality: the condition of Mangasarian (PROOF). Concave functions,

upgradient, upgradiente for differentiable function, theorem of Rockafellar. The sufficient condition of Arrow (PROOF). Transversality conditions for problems with fixed initial/final points. On the minimum problems. An example of abnormal control. *A two sector model with investment and consumption goods. A model of inventory and production I.*

b. The simplest problem of the calculus of variations Euler's theorem (PROOF as a particular case of the theorem of Pontryagin). Transversality conditions for problems with fixed initial/final points. Sufficient conditions for the simplest problem using concavity/convexity. *Curve of minimal length*.

c. Singular and bang-bang controls. Definition of bang-bang control, switching time and singular controls. *The construction of a mountain road with minimal cost.*

d. More general problem of optimal control Problems of Mayer, Bolza and Lagrange: their equivalence (PROOF). A necessary condition for the problem of Bolza with final time fixed/free. *The adjustment model of labor demand (Hamermesh).* Time optimal problems: *In boat with Pontryagin.* Singular time optimal problems: *The Dubin car.* Problems with infinite horizon: counterexample of Halkin; sufficient condition (PROOF). Current Hamiltonian and current multiplier. Models of economic growth: the utility functions. *A model of optimal consumption with log-utility.*

e. Problems of existence and controllability Examples. Gronwall inequality. Theorem of existence of optimal control for the problems of Bolza: the case of closed control set and the case of compact control set.

3. OPTIMAL CONTROL WITH THE METHOD OF DYNAMIC PROGRAMMING

a. The value function of its properties in the simplest problem of optimal control. Definition of the function value. The first necessary condition for the value function (PROOF). Bellman’s principle of optimality (PROOF). The properties of the value function: the equation of Hamlton-Jacobi-Bellmann (PROOF). The Hamiltonian of Dynamic Programming. Sufficient conditions of optimality (PROOF). On the problems of minimum.

The value function for problem with fixed final time and free final value, is Lipschitz (PROOF). Definition of viscosity solution; the vaue function is the unique viscosity solution of BHJ equation. *Problem of business strategy of production / sale II.*

b. More general problem of optimal control Necessary and sufficient conditions for more general optimal control problems. *A model of inventory and production II.* Infinite horizon problems: the current value function and its BHJ equation. *A problem of optimal consumption with HARA utility.* An idea of the stochastic situation: *the model of Merton.*

c. Relations between the variational approach and Dynamic Programming Interpretation of the multiplier as a shadow price (PROOF).

4. DIFFERENTIAL GAMES

a. Introduction Statement of a differential game with two players. Symmetric games, fully cooperative games, zero-sum games. Concepts of solutions: Nash equilibrium, Stackelberg equilibrium. Types of strategies: open-loop and Markov.

b. Nash equilibrium * Open loop strategies. Definition, use of the variational approach and sufficient condition for an open-loop strategy. *The model “workers versus capitalists” of Lancaster. Two fishermen at the lake I. A model on international pollution.* ** Markovian strategies. Because the variation technique is not particularly useful. Definition of the value function on a Nash feedback equilibrium. Necessary and sufficient conditions with dynamic programming. Value Functions for Affine-Quadratic Two-Player Differential Game Problems. The current value functions for infinite and discounted horizon games. *A problem of production for two competing companie*s. *Two fishermen at the lake II.*

c. Stackelberg equilibrium. Leader and follower players, the set of best replies. Open-loop solution with the variational approach. *A model on International pollution with hierarchical relations.*

d. Zero sum games Nash equilibrium optimal control as a saddle point. Non anticipative strategies. Upper value function V?, lower value V- and their relation.

Upper Hamiltonian of Dynamic Programming *H*?PD and Lower *H*?PD*:* their relation (PROOF). Properties of V? and V?: Lipschitz properties ans viscosity solution for the Isaacs’s equations.

Isaacs condition (minimax condition) and the Hamiltonian of Dynamic Programming *H*PD. Definition of value function V. A geometric proof that V satisfies the Isaacs equation (PROOF). *War of attrition and attack.*

e. Pursuit and evasion games. Statement, target set, exit time. The value function and the Isaacs’ equation for the autonomous case (PROOF). *Lady in the lake.*

Prerequisites

The tools and the knowledge of the courses of the first degree are a sufficient basis.

Teaching form

Lectures with exercises.

Textbook and teaching resource

[C1] A. Calogero *“Notes on optimal control theory”,* avaible in web site.
[C2] A. Calogero *“A very short tour on differential games*”, avaible in web site.
[C3] A. Calogero “*Exercises of dynamic optimization*”, avaible in web site.

Other references: [BO] T. Ba?ar, G.O. Olsder “*Dynamic noncooperative game theory*”, SIAM Classic in Applied Mathematics, 1998
[B] A. Bressan “*Noncooperative differential games. A Tutorial*”, Milan Journal of Nathematics, vol 79, pag 357-427, 2011.
[E] L.C. Evans “*An introduction to mathematical optimal control theory”*, disponibile gratuitamente in rete.
[FR] W.H. Fleming, R.W. Rishel *"Deterministic and stochastic optimal control*", Springer-Verlag, 1975
[KS] M.I. Kamien, N.L. Schwartz “*Dynamic optimization*” Elsevier, second edition, 2006
[SS] A. Seierstad, K Sydsæter “*Optimal control theory with economics applications*” Elsevier Science, 1987

Semester

First semester.

Assessment method

The examination starts with a written test. The evaluation will be in thirty: it is possible to take the oral examination only with evalutation major then 26 in the written part.

WRITTEN TEST: (3 hours) contains the following arguments:

- definitions, theorems and proofs (the proofs required are denoted with PROOF in the program of the course);
- economic and more general models, as in the program of the course;
- exercises of optimal control, using the variational approach and the dynamic programming.
  The exercises will be chosen in the list of exercise [C3] located in the page of the course.

ORAL TEST: (the date will be fixed together with the student). It consists in a discussion starting to the written part.

The student can reject a sufficient evaluation at most 2 times.

Office hours

On appointment.

Sustainable Development Goals

QUALITY EDUCATION