UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

COURSE SYLLABUS

Stochastic Calculus and Finance

2425-1-F4001Q107

Aims

The course aims at providing the student with the definitions and basic properties of the Brownian motion and the fundamental results of the theory of stochastic differential equations. Particular emphasis will be given on the interactions between stochastic differential equations and partial differential equations, and on the applications to the modeling of financial derivatives.

At the end of the course students will have acquired the following:

- knowledge: language, definitions and statements of the fundamental results about Brownian motion and stochastic differential equations;
- competence: operational understanding of the main proof techniques and of the main financial models to which the theory can be applied;
- skills: ability to apply theoretical notions to the analysis of problems and models.

Contents

- Introduction to stochastic processes in continuous time
- Levy processes and Brownian motion
- Ito stochastic integral
- Ito's formula
- Stochastic differential equations (SDEs)
- The associated Kolmogorov differential operator
- The Kolmogorov PDE and the Feynman-Kac formula
- Introduction to continuous time financial markets
- The Black and Scholes formula and the pricing of European options

Detailed program

Brownian motion. Stochastic processes, path space, cylinder sets, product sigma-algebra. Law of a process and finite-dimensional laws. Normal random vectors. Gaussian processes. Definition of Brownian motion (BM). Construction of BM from the Daniell-Kolmogorov existence theorem and the Kolmogorov continuity theorem. Characterization of BM as a Gaussian process. Invariance properties of BM (space reflection, time translation and reflection, diffusive rescaling, time inversion). BM with respect to a filtration. Path properties of BM: non differentiability. Quadratic variation of BM. Law of the iterated logarigthm. BM in higher dimension.

Lévy processes. Generalities on filtrations (F_t) in continuous time. Natural filtration of a stochastic process, adapted processes. Right continuity and completeness of a filtration (definition of F_t+), standard extension. Lévy processes with respect to a filtration. Examples: Poisson process, compounded Poisson process. A Lévy process with respect to a filtration (F_t) is independent of F_0. Blumenthal 0-1 law. Stopping times and strong Markov property.

The Ito integral. Modification and indistinguishability for stochastic processes. Continuity and measurability for stochastic processes. Sigma-algebra of events before a stopping time. Continuous-time martingales, examples, right-continuous modifications, stopping times and maximal inequality. Progressively measurable processes. Ito integral for simple processes. Extension to M^2 and M^2_loc. Properties: locality, existence of a version with continuous paths, martingale property. Quadratic variation. Riemann sums for the Ito integral of processes with continuous paths. Wiener integral. Local martingales.

The Ito formula. Ito formula for BM. Ito processes. Ito formula for general Ito processes. Applications of Ito formula, geometric BM, exponential super-martingale. Multi-dimensional Ito formula. Representation theorem for Brownian martingales.

Stochastic differential equations. Strong and weak existence, pathwise uniqueness and uniqneness in law. The Yamada-Watanaabe theorem. Strong existence and pathwise uniqueness under Lipschitz assumptions. Flow property. The Kolmogorov semigroup. The Kolmogorov partial differential equation. Feynman-Kac formula.

Application to financial markets. Financial assets, call and put options, payoff. Hedging of options. Continuous-time market model based on a non-risky asset (bond) and d risky assets (stocks).driven by d independent BMs. Equivalent local martingale measure. Self-financing investing strategies and admissible strategies. The theorem of absence of arbitrage. Pricing and hedging of European options. The Bkach&Scholes model (one dimensional with constant drift, volatility, interest rate). Explicit formula for the price of a European call option. A Markovian model of financial market with drift and volatility depending on an underling asset and on time. Representation formula for the price of European options and for hedging strategies.

Prerequisites

Measure-theoretic probability theory, stochastic processes in discrete time, basic properties of Hilbert spaces and L^p spaces.

Teaching form

28 x 2 hours of in-person, lecture-based teaching

Textbook and teaching resource

Lecture notes by the teacher

Book **Brownian Motion, Martingales, and Stochastic Calculus** by Jean-François Le Gall, Springer series Graduate Texts in Mathematics (Volume 274, 2016)

Semester

First (Fall) semester

Assessment method

Oral exam to assess the student knowledge and ability to critically discuss definitions, statemets, examples and proofs presented in the course

Office hours

By appointment

Sustainable Development Goals

QUALITY EDUCATION