UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

COURSE SYLLABUS

Probability Theory

2526-3-E3501Q014

Aims

1. Knowledge and understanding (DdD 1)
   At the end of the course, the student will have acquired the formal language of probability theory, its main concepts (sample spaces, random variables, distributions, independence, expectation, variance), and fundamental results (law of large numbers, central limit theorem, rules of probability calculus).

2. Applying knowledge and understanding (DdD 2)
   The student will be able to apply the learned concepts and theorems to solve elementary probability problems, analyze simple stochastic models, and carry out standard exercises in both theoretical and applied contexts.

3. Making judgements (DdD 3)
   The student will develop critical thinking skills to identify the most suitable probabilistic approach to model simple phenomena, assessing assumptions and limitations of the adopted models. This skill is fostered through classroom discussion and guided problem-solving.

4. Communication skills (DdD 4)
   The student will acquire the ability to clearly and rigorously communicate probabilistic arguments, both orally and in writing, using appropriate mathematical language, also within group work or short presentations.

5. Learning skills (DdD 5)
   The course provides the methodological foundations necessary for autonomous learning of probability and its applications, enabling the student to deepen related topics (such as statistics, stochastic processes, or machine learning) in further studies.

Contents

The first part of the course presents the mathematical theory of probability, in order to describe random phenomena by means of **probability spaces**, following N. Kolmogorov's axioms based on measure theory. A great deal of attention is given to **random variabels**, which form the "operational language" of probability theory.

The second part of the course starts discussing the various notions of **convergence of random variables**. The fundamental **limit theorems** in the theory of probability are prsented, namely the *law of large numbers* and the *central limit theorem*. The course is concluded with an introduction to **Markov chains**, one of the simplest yet most important classes of stochastic processes.

Along the whole course, the presentation of the theory is enriched by the discussion of several **models and applications**.

Detailed program

1. *Probability spaces*

- Introduction to probability
- Axioms of probability
- Basic properties of probability
- Combinatorics and uniform spaces
- Conditional probability
- Independence of events

2. *Random variables*

- Reminders of measure theory
- Important distributions, discrete and absolutely continuous
- Random variables
- Marginal laws and joint law
- Independence of random variables
- Transformations of random variables
- Expected value, moments, variance and covariance
- L? spaces and inequalities
- Correlation and linear regression (hints)

3. *Convergence and limit theorems*

- Reminder on convergence theorems
- Borel-Cantelli lemma
- Weak and strong law of large numbers
- Notions of convergence for random variables
- Weak convergence of probabilities
- Law of small numers
- Central limit theorem and normal approximation
- Kolmogorov's 0-1 law

4. *Markov chains*

- Introduction to stochastic processes
- Markov chains and basic properties

- Recurrent and transient states
- Invariant and reversible measures
- Convergence theorem (hints)
- Absorption probabilities (hints)
- Random walks on graphs (hints)

5. *Models and Applications* (presented alongside the theory, if time permits)

- Classical paradoxes (birthdays, Monty-Hall, Borel, Bertrand)
- Random permutation and fixed points
- Concentration of volume in high dimensions
- Weierstrass' approximation theorem
- Simulation of random variables
- Simple random walk
- Gambler's ruin
- The PageRank algorithm

Prerequisites

The knowledge, competences and skills taught in the courses of the first two years, in particular *Linear Algebra, Analysis 1 and 2 (= calculus in one and more variables), Measure Theory*.

Teaching form

The course is composed by lectures and recitations in the classroom:

- theoretical lectures (10 ects) -in presence, delivery-based mode- are focused on the knowledge of definitions, results and relevant examples, as well as the competences linked to their comprehension;
- recitations (2 ects) - in presence, interactive mode- are focused on the skills necessary to apply the theoretical knoledge and competencies to the solution of exercises.

The course is given in Italian.

Textbook and teaching resource

Reference textbooks

- F. Caravenna, P. Dai Pra. *Probabilità. Un'introduzione attraverso modelli e applicazioni.* Seconda Edizione (2021), Springer-Verlag Italia.
- D. F. Anderson, T. Seppäläinen, B. Valkó. *Introduction to Probability*. Cambridge University Press (2018).
- J. Jacod, P. Protter. *Probability Essentials*. 2nd Edition, Springer (2003).
- P. Baldi. *Calcolo delle Probabilità*. McGraw-Hill (2007, 2011).

Other dydactical material (available on the e-learning page of the course)

- Notes by the teacher on specific arguments

- Weekly exercise sheets (with detailed solutions)
- Written exams from previous years (with detailed solutions)
- List of proofs for the oral examination

Semester

Third year, First (Fall) Semester.

Assessment method

Written examination (or midterms) and oral examination, with the rules described in the sequel. The aspects that will be evaluated are the correctness of the answers, the creativity, the precision, the clarity of exposition.

There will be 6 exam sessions.

- The *written examination* lasts 3 hours and gets a mark out of 30. This examination tests both theotetical knoledge and competencies (definitions, examples and counter-examples) and practical skills (solving exercises). The written examination is passed with a minimal mark of 15/30 and allows to be admitted to the oral examination.
- In the middle and at the end of the course there will be two *midterm exams,* structured in analogy with the written examination, which last 1.5 hours each and get a mark out of 30. Passing both midterms with a minimal mark of 15/30 is equivalent to passing the written examination (with the average of the marks) and allows one to be admitted to the oral examination.
- The *oral examination* lasts 30-45 minutes and gets a mark out of 30. It can be given in any exam session of the academic year (after passing the written examination). The oral examination tests the knowledge of a selection of proofs as well as a working knowledge of the notions of the course. The oral examinations is passed with a minimal mark of 15/30.
- The final mark results from the average between the marks of the written and oral examinations. The exam is passed with a minimal mark of 18/30.

Exemption from the oral examination. Students who pass the written examination with a mark in the range 20-27/30 are allowed to be exempted from the oral examination, the final mark being equal to the mark obtained in the written examination; with a mark greater than 27/30, it is still possible to be exempted from the oral examination, however the final mark in this case will be 27/30; finally, with a mark smaller than 20/30 it is compulsory to take the oral examination.

Office hours

Upon appointment.

Sustainable Development Goals

QUALITY EDUCATION