UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

COURSE SYLLABUS

Stochastic Calculus and Finance

2526-1-F4002Q027

Aims

Learning Objectives (Dublin Descriptors):

At the end of the course, students will have achieved the following learning outcomes:

1. Knowledge and Understanding. Students will acquire advanced knowledge of stochastic calculus, with particular reference to Brownian motion, stochastic differential equations (SDEs), and their connections with partial differential equations (PDEs), as well as applications to the mathematical modeling of financial markets.

2. Applying Knowledge and Understanding. Students will be able to apply the theoretical framework of stochastic calculus to the formulation and analysis of mathematical models for stochastic phenomena, including applications to PDEs and problems arising in mathematical finance.

3. Making Judgements. Students will develop the ability to critically assess the applicability of stochastic models and techniques in various contexts.

4. Communication Skills. Students will acquire the ability to communicate, in a clear and rigorous manner, mathematical arguments, proofs, and results related to stochastic calculus, using appropriate terminology and formal mathematical language.

5. Learning Skills. Students will develop the methodological tools and autonomy necessary to undertake further study and research in stochastic analysis, (stochastic) PDEs, mathematical finance, or related areas.

Contents

- Introduction to stochastic processes in continuous time

- Brownian motion
- Basics on Levy processes
- Ito stochastic integral
- Ito's formula
- Stochastic differential equations (SDEs)
- The Kolmogorov differential operator and the Feynman-Kac formula
- Introduction to continuous time financial markets
- The Black and Scholes formula and the pricing of European options

Detailed program

Brownian motion. Stochastic processes, path space, cylinder sets, product sigma-algebra. Law of a process and finite-dimensional laws. Normal random vectors. Gaussian processes. Definition of Brownian motion (BM). Construction of BM from the Daniell-Kolmogorov existence theorem and the Kolmogorov continuity theorem. Characterization of BM as a Gaussian process. Invariance properties of BM (space reflection, time translation and reflection, diffusive rescaling, time inversion). BM with respect to a filtration. Path properties of BM: non differentiability. Quadratic variation of BM. Law of the iterated logarigthm. BM in higher dimension.

Lévy processes. Generalities on filtrations (F_t) in continuous time. Natural filtration of a stochastic process, adapted processes. Right continuity and completeness of a filtration (definition of F_t+), standard extension. Lévy processes with respect to a filtration. Examples: Poisson process, compounded Poisson process. A Lévy process with respect to a filtration (F_t) is independent of F_0. Blumenthal 0-1 law. Stopping times and strong Markov property.

The Ito integral. Modification and indistinguishability for stochastic processes. Continuity and measurability for stochastic processes. Sigma-algebra of events before a stopping time. Continuous-time martingales, examples, right-continuous modifications, stopping times and maximal inequality. Progressively measurable processes. Ito integral for simple processes. Extension to M^2 and M^2_loc. Properties: locality, existence of a version with continuous paths, martingale property. Quadratic variation. Riemann sums for the Ito integral of processes with continuous paths. Local martingales.

The Ito formula. Ito formula for BM. Ito processes. Ito formula for general Ito processes. Applications of Ito formula, geometric BM, exponential super-martingale. Multi-dimensional Ito formula. Harmonic functions and the Dirichlet problem. Giranov's theorem. Representation theorem for Brownian martingales.

Stochastic differential equations. Strong and weak existence, pathwise uniqueness and uniqneness in law. Strong existence and pathwise uniqueness under Lipschitz assumptions. Example: the Ornstein-ühlenbeck process. The Kolmogorov semigroup. The Kolmogorov partial differential equation. Feynman-Kac formula.

Application to financial markets. Financial assets, call and put options, payoff. Hedging of options. Continuous-time market model based on a non-risky asset (bond) and d risky assets (stocks) driven by d independent BMs. Equivalent local martingale measure. Self-financing investing strategies and admissible strategies. The theorem of absence of arbitrage. Pricing and hedging of European options. The Bkach&Scholes model (one dimensional with constant drift, volatility, interest rate). Explicit formula for the price of a European call option. A Markovian model of financial market with drift and volatility depending on an underling asset and on time (local volatility). Representation formula for the price of European options and for hedging strategies.

Prerequisites

Measure-theoretic probability theory. Stochastic processes in discrete time. Basic properties of Hilbert spaces and L^p spaces.

Teaching form

In-person, lecture-based teaching in slots of 2 hours

Textbook and teaching resource

Lecture notes by the teacher

Book **Brownian Motion, Martingales, and Stochastic Calculus** by Jean-François Le Gall, Springer series Graduate Texts in Mathematics (Volume 274, 2016)

Semester

First (Fall) semester

Assessment method

Oral exam, with a possible preliminary written part, to assess the student knowledge and ability to critically discuss definitions, statemets, examples and proofs presented in the course.

Office hours

By appointment

Sustainable Development Goals

QUALITY EDUCATION