UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

SYLLABUS DEL CORSO

Geometria II

2627-2-E3502Q009

Aims

The aim of the course is to introduce the theoretical foundations and the use of differential forms on open sets of Euclidean spaces, as a basis for the more general treatment in the context of differentiable manifolds.

Differential forms play a pervasive and foudational role in Geometry, Differential Topology, and Analysis; they are furthermore unavoidable in the coordinate free formulation of physical laws. In particuar, they lead to the introduction of de Rahm cohomology groups, which relate the topological and differentiable properties of a space.

The theory will be developed from its algebraic first principles, that is, from the basic notion of a tensor in linear algebra.

To this end, the first part of the course will dwell on some foundational contents from linear and tensor algebra, such as duality, the determinant and its role in the theory of alternating tensors, orientations of a real vector space, and the volume element of a Euclidean oriented vector space. These concepts,
besides being pervasive in Geometry, illustrate, enrich and complete the previous background from linear algebra.

Differential forms on open sets in Euclidean spaces will be introduced in the second part, as the assignment of an alternating tensor varying smoothly with the base point. The algebraic concepts from the former part will be brought into contact with the tools of Analysis, leading to the discussion of the exterior differential and its functorial properties, to the integration of a differential k-form on certain parametrized subsets, and the introduction of de Rahm cohomology groups.

Special emphasis will be given to the concept of area form and the computation of areas of geometrical loci, and to the concept of tangent space and its determination.

The climax of the course is given by Stokes'Theorem, also known as the divergence Theorem, which plays a crucial conceptual and technical role in several mathematical (and physical) contexts.

The expected learning outcomes include the following:

- The knowledge and understanding of the basic definitions and statements, as well as of the basic strategies of proof in the theory of differential forms; the knowledge and understanding of some of the most relevant basic applications, notably to the study of smooth proper maps between opens sets of Euclidean spaces; the knowledge and understanding of some of the key foundational examples of the theory. This knowledge and understanding will be developed through an in-depth discussion of the key concepts and statements of the theory, and a critical analysis of the proofs and their techniques.
- The ability to apply the acquired abstract knowledge to the solution of simple computational exercises and theoretical problems, referring in a precise and well-organized manner to the pertinent results being used; the ability to master the algebraic, differential and integral calculus of differential forms, and to use it in the some simple practical situations, such as the study of proper maps between open sets of Euclidean spaces; the ability to apply the theoretical background to the construction and discussion of simple examples and solution of exercises; the ability to expose and communicate effectively and clearly the theoretical content of the course. This ability will be acquired through the discussion of salient examples shedding light on the theory, as well as by the solution of exercises suggested to the students, both of theoretical and computational type.
- Soft skills: autonomy of judgment, communication skills and ability to learn. These skills will be stimulated and developed by emphasizing the technical and conceptual sinergy with the content of other courses (of algebra, linear algebra and mathematical analysis), and placing great value on expository clarity of both theoretical results and computational applications.

Contents

Alternating multilinear algebra; differential forms on Euclidean space and their operations; Gauss-Green Theorem; de Rahm cohomology; area and integration of differential forms; orientation; Stokes' theorem.

Detailed program

Exterior algebra of a vector space and its operations: exterior product, contractions; oriented Euclidean vector spaces and their volume elements; vector fields and differential forms; exterior differential; closed and exact forms; gradient, rotor and divergence; pull-back of differential forms under smooth maps: integration; integration and homotopy; change of variable formula; Poincaré Lemma; Poincaré Lemma with compact support; integration on oriented parametrized varieties; Theorems of Gauss-Green and Stokes.

Prerequisites

The content of the courses of Analysis I and (in part) II, Linear Algebra and Geometry, Geometry I.

Teaching form

This course will be normally be taught entirely by live lectures at the blackboard, which will also video-recorded and made available to the students through the elearning platform.
The precise subdivision is as follows:
live lessons at the blackboard (6 CFU)

live exercise sessions at the blackboard (2 CFU).
The course is taught in Italian.

Textbook and teaching resource

Reference text: teacher's notes on e-learning

Recommended reading:

the following book is especially pertinent to the content of this course:

- V. Guillemin and P. Haine, Differential forms, World Scientific 2019

Further recommended textbooks are:

- M. Do Carmo, Differential forms and applications, Springer Verlag 1996;
- V. Guillemin, A. Pollack, Differential Topology 1974;
- W. Fulton, Differential Topology, a first course, Springer Verlag 1995.

Semester

2nd semester

Assessment method

The exam may be passed either by taking two written partial tests during the course, or in the regular exam sessions following the course.

The partial tests consist in a flexible combination of exercises and theoretical questions, and each only covers a part of the program; the exact subdivsion will be comminicated well in advance during the course. To pass the exam, a minimum passing grade of 18 is required in both parts. In specific cases, however, students may be required to integrate the written tests with an oral examination (see below).

The regular exam sessions, on the other hand, comprise two written tests, a practical and a theoretical one, each referred to the whole course; again, in specific cases students may be required to integrate the written tests with an oral examination. In the practical test, the student will be asked to solve various computational exercises, while in the theoretical test there will be questions involving definitions, statement's of theorems, proofs, construction of examples and counterexamples, and simple theoretical problems.

The aim of the final discussion is typically to expose the evaluation of the student's script; only in special cases, where the student's competence can't be clearly assessed by the scripts, will the discussion contribute to the final evaluation.

The practical tests will measure the student's ability to master the acquired formalism and apply it to some simple computations, to build on the acquired theoretical knowledge, and to invoke it in a pertinent and precise manner.

The theoretical tests will evaluate the knowledge and understanding of the conceptual framework of the course, as well as the ability to expose it in a well-organized, consistent and effective manner.

In order to successfully complete the exam, the student needs to first pass the practical test, thus obtaining a grade of at least 18/30, and then to also obtain the passing grade in the theoretical test of the same session or, upon his/her choice, of the session immediately following.

To each exercise/theoretical question (or problem) a maximum partial grade will be assigned by the commission, depending on its difficulty and length; in the evaluation, every student will be given a grade in correspondence to each exercise/theoretical question (or problem) up to the maximum one, measuring the exactness, the completeness, the rigour, the clarity and the overall coherence of the development.

Office hours

Upon appointment.

Sustainable Development Goals

QUALITY EDUCATION