UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

# SYLLABUS DEL CORSO

## Geometria Simplettica

**2627-1-F4002Q036**

### Aims

A symplectic manifold is a differentiable manifold endowed with a nondegenerate closed 2-form (and has therefore even dimension). For instance, the 2-sphere in 3-dimensional euclidean space has a standard symplectic structure, as do all cotangent bundles and complex projective space. Symplectic Geometry studies the properties of sympectic manifolds.

Interest for Symplectic Geometry originates in Hamiltonian dynamics, which makes this field tightly related to Mathematical Physics and Classical Physics: the symplectic structure naturally associated a vector field to any smooth function (the 'Hamiltonian'), and Hamilton's equations describe the corresponding flow.

Besides being intensely studied for its intrinsic interest, Symplectic Geometry is intimately related to other branches of mathematics. For example, the symplectic structure of cotangent bundles is crucial in the study of pseudodifferential and Fourier integral operators, and is the unavoidable geometric basis of Semiclassical Analysis, while the symplectic properties of coadjoint orbits of a Lie group offer a geometric interpretation of its irreducible representations.

Furthermore, Symplectic Geometry is an indispensable tool and vantange point in both real and complex Differential Geometry. For instance, the geodesic flow of a Riemannian manifold is a special type of Hamiltonian system, and can most naturally be approached from the symplectic viewpoint. In addition, an extremely important class of complex manifolds, encompassing the complex projective category, is given by the so-called Kaehler manifolds, that are characterized by the existence of a symplectic structure suitably compatible with the complex one.

The symplectic approach has played a crucial role in many of the most innovative recent developments in complex Differential Geometry.

We aim to discuss and elaborate on the basic concepts of Symplectic Geometry, starting with the local aspects and then moving on to the more global properties.

Special emphasis will be given to the discussion of Lagrangian submanifolds and their functorial properties, which are essential in the discussion of Fourier integral operators.
Time permitting, we shall approach the theme of Hamiltonian action, moment map and symplectic reduction. The latter contruction leads to a 'quotient' symplectic structure starting from a symplectic manifold endowed with a group of symmetries.

The expected learning outcomes include:

- the knowledge and understanding of the fundamental definitions and statements, as well as of the basic strategies of proof in symplectic geometry; the knowledge and understanding of some crucial examples in which the theory manifests itself;
- the ability to recognize the role that concepts and techniques from symplectic geometry play in various areas of mathematics (such as differential equations, Riemannian geometry, complex geometry, representation theory), and in the mathematical modelling of physical situations (mathematical physics); the skill to apply such conceptual background to the construction of concrete examples and to the solution of exercises; the ability to communicate and explain in a clear and precise manner both the theoretical aspects of the course and their applications to specific situations, possibly to different contexts.

## Contents

Symplectic vector spaces, symplectic manifolds, Hamiltonian flows and symplectomorphisms, canonical forms of symplectic structures, moment maps and symplectic reductions. Basic examples of symplectic reductions. Lagrangian submanifolds and their functorial properties; phase functions.

## Detailed program

· Symplectic linear algebra.

· Cotangent bundles, Hamiltonian equations, Poisson brackets.

· Symplectic manifolds and special submanifolds, neighborhood theorems.

· Isotopies and theorems of Darboux and Moser.

· Lagrangian submanifolds and their functorial properties; Weinstein's tubular enighbourhood Theorem;

· Moment maps and their properties, symplectic reduction;

· Compatible complex and almost complex structures; Kähler and quasi-Kähler manifolds.

· Coadjoint orbits and their natural symplectic structure.

## Prerequisites

Prerequisites are: a good familiarity with the concepts of linear algebra offered during the Laurea Triennale in Mathematics, since the study of symplectic linear algebra will play a foundational role in the development of the course; the most basic notions on differentiable manifolds and differential forms, as they are commonly treated say in the courses of Geometry II and III. Brief recalls will be given as needed.

## Teaching form

This course will be normally be taught entirely by live lectures at the blackboard, which will also video-recorded and made available to the students through the elearning platform.

## Textbook and teaching resource

The hand-written notes by the teacher and the notes in latex by Dr. Massimo Frigerio, who took the course in 2018/19.

**Further bibliographic references:**

V. Guillemin, S. Sternberg, Geometric Asymptotics, AMS 1990

V. Guillemin, S. Sternberg, Symplectic Techniques in Physics, Cambridge University Press

D. McDuff, D. Salamon, Introduction to Symplectic Topology, Clarendon Press, Oxford

**Recommended readings:**

V. Guillemin, S. Sternberg, Semiclassical Analysis, International Press

J. J. Duistermaat, Fourier Integral Operators, Birkhäuser

## Semester

1st semester

## Assessment method

The exam will comprise two written tests, followed by an oral discussion of them. Each test will deal with a part of the course (I and II), and will be aimed at the evaluation of the knowledge, understanding and skills forming the expected learning outcomes of the course. The precise subdivision of the topics between the two tests will be communicated during the course well in advance with respect to the tests themselves. Each test consists of a flexible combination of theoretical questions (including definitions, statements and proofs) and more practical ones (including exercises, examples and counterexamples). Each test will be evalueated independently, and will concur in the same amount to the determination of the final grade; in order for the student to pass the exam, both tests will need to get a pass grade.

The two written tests may be taken in different exam sessions. In each session, it will be possible to sign up for either test, but it is only the registration to the second one that makes it possible to record the vote. The date of the oral discussion of the tests will be announced following their correction.

For all the duration of the current health crisis, the final oral discussion will take place remotely on the WeBex platform, and the link will be made available through the e-learning page of the course. The modality of the written tests will be detailed later.

## Office hours

Upon appointment.

## Sustainable Development Goals

QUALITY EDUCATION