UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

COURSE SYLLABUS

Geometric Analysis

2627-1-F4002Q002

Aims

The course aims to provide an introduction to analysis on metric spaces, with particular emphasis on geometric aspects and the development of first-order differential calculus in general metric settings.

The intended learning outcomes (according to the Dublin Descriptors) are

1. **Knowledge and understanding**
   At the end of the course, the student will have acquired a solid understanding of fundamental definitions, main theoretical results, and selected proof strategies in the context of analysis on metric spaces. The student will be familiar with significant classes of spaces (length spaces, intrinsic spaces, metric measure spaces) and the mathematical frameworks in which they arise.

2. **Applying knowledge and understanding**
   The student will be able to apply analytical and geometrical methods to non-standard metric spaces, identifying key geometric and analytic properties. They will also be able to introduce and use tools of first-order differential calculus in these settings.

3. **Making judgements**
   The student will develop the ability to critically assess the hypotheses of key results, evaluate the applicability of theoretical tools in different contexts, and recognize conceptual connections among the various topics covered in the course.

4. Communication skills
   The student will be able to clearly and rigorously present theoretical content, examples, and personal insights, using appropriate mathematical and argumentative language in both written and oral form.

5. **Learning skills**
   The student will acquire the ability to independently study advanced texts, and the conceptual tools required to pursue further studies in geometric analysis and related fields, also in preparation for research activities.

Contents

Basic notions and geometric aspects (including curvature) of the length metric spaces.

Elements of analysis and first order differential calculus on measure metric spaces.

Detailed program

Part I. Metric spaces and curvature.

- Metric spaces: definition, examples, topology; Hausdorff measure and dimension.
- Length spaces, intrinsic metrics, geodesics, length and velocity; constructions and examples.
- Spaces of bounded curvature: some equivalent definitions of curvature bounds (from above or from below); angles; local and global curvature bounds.
- Convergence of metric spaces: uniform convergence; definitions and properties of the Gromov-Hausdorff distance.
- An overview of some properties of metric spaces with positive curvature: volume growth, Hausdorff dimension; globalization; examples (cones, convex surfaces, ...); some compactness results.

Part II. Analysis and differential calculus on metric spaces of measure

- Measure metric spaces; doubling measures; covering lemmas: Vitali's and Lebesgue's theorem.
- Hardy-Littlewood maximal function: boundedness results.
- A review of Sobolev spaces in R^n; Sobolev embeddings; Poincaré inequalities.
- Sobolev spaces on metric spaces: definition based on the maximal function; Lipschitz functions: extension and density theorems.
- Upper gradient, modulus of a curve family, capacity; Newtonian Sobolev spaces on metric spaces: definitions and properties.
- An introduction to differential equations on metric spaces: Poincaré inequalities on metric spaces; harmonic functions and the Dirichlet problem.

Prerequisites

Calculus in several variables; elements of measure theory and of the theory of Lp spaces.

A basic knowledge of the Sobolev spaces in R^n is not required, however can help to familiarize with some of the topics in the second part of the course.

Teaching form

A hybrid teaching approach is used, that combines lecture-based teaching (DE) and interactive teaching (DI). DE constitues the larger part of the cours and involves detailed presentation and explanation of theoretical content and

more significant examples. DI includes active student participation through exercises and problems, short presentations and group discussions. It is not possible to precisely determine in advance the number of hours dedicated to DE and DI, as these methods are dynamically intertwined to adapt to the course's needs and promote a participatory and integrated learning environment, combining theory and practice.

Lessons (56 hours, 8 CFU) are conducted in person and are delivered in Italian or, when necessary, in English.

Some exercise will be assigned during the lectures. Their solutions can be discussed at the request of the students, either in class or during office hours.

Textbook and teaching resource

The main textbook are the following.

For the 1st part of the course:

- D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001 (for the 1st part of the course)

For the 2nd part of the course:

- J. Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001 (for the 2nd part of the course) Supplementary textbooks and resources may be suggested during the course.
- A. Björn, J. Björn, *Nonlinear potential theory on metric spaces*. EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011. xii+403 pp.

These textbooks are complemented by some teacher's lecture notes which present the concepts, results and proofs, as well as most examples, treated during the lessons.

Semester

II semester.

Assessment method

The assessment method consists in an oral exam. Mark out of thirty. There are no ongoing partial test.

The exam aims at verifying the level of knowledge, the student's independence in making judgements, as well as his/her communication skills. During the oral exam, the topics covered by the lectures will be discussed. Some additional easy exercises or examples, not covered in class, can be also discussed.

The course is divided into two main parts (see the "detailed program" for more details).
For the exam it is possible to focus more on one of the two parts, which should be known in detail. Concerning the other part, the student should know the main concepts, objects and results, but not necessarily the rigorous statements and proofs.

It is also possible to choose a short additional topic to start the oral exam with. The topic will be previously agreed with, and possibly (but not necessarily) proposed by, the teacher.

Office hours

By appointment.

Sustainable Development Goals

QUALITY EDUCATION