UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

COURSE SYLLABUS

Riemannian Geometry

2627-1-F4002Q012

Aims

1. Knowledge and understanding
By the end of the course, students will have acquired a solid understanding of the foundations of classical Riemannian geometry, including the concepts of Riemannian metric, Levi-Civita connection, geodesics, and curvature. They will be able to grasp the relationship between the local (differential) structure and the global (topological) shape of Riemannian manifolds.

2. Applying knowledge and understanding
Students will be able to apply the acquired concepts to concrete examples, verifying the geometric properties of Riemannian manifolds. They will be capable of computing geodesics and curvature, and of analyzing significant examples such as spheres, hyperbolic spaces, (warped) products and and model manifolds.

3. Making judgements
The course aims to develop the ability to critically analyze the studied geometric structures and to make independent judgements about the validity of the properties and results learned. Students will be encouraged to reflect on the geometric meaning of theoretical concepts and their interaction with the topology of the manifold.

4. Communication skills
Through theoretical discussions and take-home assignments, students will be encouraged to express the concepts of Riemannian geometry clearly and rigorously, both in written and oral form. They will be able to effectively present proofs, examples, and logical-mathematical arguments.

5. Learning skills
The course will provide the theoretical and methodological tools needed to independently pursue the study of advanced developments in Riemannian geometry. Students will be able to delve into topics not explicitly covered in class, complete proofs, consult specialized texts, and integrate new knowledge within a process of continuous learning.

Contents

In order to provide the intuitive support necessary to approach the study of Riemannian manifolds, the course will begin with some introductory aspects of the classical local and global theory of regular surfaces in three-dimensional Euclidean space. Then, starting from the problem of the existence of a Riemannian metric on a generic differentiable manifold, the course will move on to the notion of the Levi-Civita connection and the corresponding parallel transport, which will allow for the definition of a geodesic as a curve with zero acceleration. Through the concept of curve length, an intrinsic distance will be introduced, which will induce the topology of the underlying manifold. From there, the course will delve into the global aspects of geodesics, culminating in the notion of completeness and its metric characterizations (Hopf–Rinow Theorem). The study of the Riemann curvature tensor and its traces will precede the culminating part of the course, which, time permitting, will be dedicated to the relationship between the sign of curvature and the topology of a complete Riemannian manifold.

Detailed program

1. Outline of regular surfaces in Euclidean space and their curvatures
2. Definition and existence of Riemannian metrics
3. Levi-Civita connection and parallel transport
4. Geodesics and expontential map
5. The intrinsic metric structure of a Riemannian manifold
6. Curvatures of a Riemannian manifold
7. Jacobi fields and conjugate points
8. Global results
   8.1) Global theory of geodesics and completeness
   8.2) The Bonnet-Myers Theorem
   8.3) (time permitting) The Cartan-Hadamard Theorem

Prerequisites

Differential calculus in several variables, basic notions of differentiable manifolds, linear and multilinear algebra.

Teaching form

56 hours of of in-person, lecture-based teaching (8 ects)

Lectures are primarily in Italian, and when necessary, in English.

Textbook and teaching resource

Texbooks for the introductory part on the theory of surfaces

M. P. do Carmo, *Differential geometry of curves & surfaces*. Dover Publications, Inc., Mineola, NY, 2016.

M. Abate; F. Tovena, *Curves and surfaces*. Unitext, 55 Springer, Milan, 2012.

Basic textbooks on Riemannian Geometry

M. P. do Carmo *Riemannian geometry*. Birkhäuser Boston, Inc., Boston, MA, 1992.

Lee, John M. *Introduction to Riemannian manifolds*. Second edition. Graduate Texts in Mathematics, 176. Springer, Cham, 2018.

Additional teaching material (such as lecture notes) will be provided during the course

Texbooks for further studies

S. Gallot, D. Hulin, J. Lafontaine *Riemannian geometry*. Third edition. Universitext. Springer-Verlag, Berlin, 2004.

P. Petersen *Riemannian Geometry*. Graduate Texts in Mathematics, 171. Springer, 2006.

T. Sakai, *Riemannian geometry*. Transl. Math. Monogr., 149 American Mathematical Society, Providence, RI, 1996.

Semester

Second semester

Assessment method

The assessment of learning is conducted through a traditional oral exam, during which the student must demonstrate their acquisition of basic concepts, the proofs of the main theorems, and the ability to analyze and perform calculations on some concrete examples. The introductory aspects of the geometry of Euclidean surfaces will not be subject to assessment. At their discretion, students may begin the exam with a brief seminar focusing on an in-depth topic not covered during the course. The list of possible seminars will be made available well in advance on the course's E-Learning page.

Office hours

By appointment

Sustainable Development Goals

QUALITY EDUCATION