UNIVERSITÀ
DEGLI STUDI DI MILANO-BICOCCA

COURSE SYLLABUS

Stochastic Processes

2627-1-F4002Q029

Aims

1. Knowledge and understanding
   The course provides students with an advanced understanding of the fundamental concepts, definitions, and main results in the theory of stochastic processes in discrete time. Particular emphasis will be placed on the theory of martingales, highlighting its central role in modern probability and its applications. Students will deepen their comprehension of key probabilistic structures and will acquire familiarity with advanced mathematical formalism used in the field.

2. Applying knowledge and understanding
   Students will develop the ability to apply the theoretical tools and probabilistic techniques to solve exercises, analyze models, and tackle problems involving discrete-time stochastic processes. They will be able to rigorously apply fundamental results, such as optional stopping theorems, martingale convergence theorems, and other core principles, to concrete problems in probability theory and related areas.

3. Making judgements
   The course will foster the students' capacity for autonomous critical thinking in mathematical reasoning. They will learn to evaluate the validity of arguments, to choose appropriate mathematical tools in solving problems, and to critically assess the hypotheses and limitations of the results within the theory of stochastic processes.

4. Communication skills
   Students will improve their ability to clearly and rigorously communicate advanced mathematical concepts, both orally and in writing, using the specific language and formalism of probability theory. They will be encouraged to present proofs, discuss theoretical results, and explain the application of key concepts to diverse contexts within stochastic processes.

5. Learning skills
   Upon completion of the course, students will have strengthened their ability to engage in autonomous study and to acquire new knowledge in advanced probability and stochastic processes. They will be equipped to

deepen their understanding through further specialized courses, research activities, or applications of stochastic models in mathematics, finance, or other quantitative disciplines.

Contents

Complements of probability, Conditional law and conditional expectation. Martingales in discrete time. Financial markets and Martingales. Examples and applications.

Detailed program

- Advanced probability: Characteristic function, uniqueness and relations with convergence in law. Gaussian vectors. Compactness creteria for the convergence in law.
- Conditional law and expectation. Definitions and properties. Existence of conditional expectation of a random variable with respect to a sigma algebra. Fundamental properties: tower property, Jensen inequality, freezing. Limit theorems.
- Discrete-time Martingales. Definition and examples (sums of independent centered r.v.s, products of independent r.v.s with expectation 1, closed martingales). Integral of a predictable process. Stopped Martingales. Optional stopping theorem. Applications: first hitting time of a random walk on Z; the gambler's ruin problem. Upcrossing Lemma. Almost sure convergence of martingales bounded in L^1 norm. Martingales bounded in L^2 norm. Uniform integrability and convergence in L^1. Proof of the strong law of large numbers. Maximal inequality. Doob's inequality, convergence in L^p. Examples: Galton-Watson branching processes. Applicastion to the convergence of sums of random variables
- Financial markets with discrete time. Arbitrage and equivalent martingale measure.

Prerequisites

Knowledge of differential and integral calculus for functions of one and more real variables, as well as measure-theoretical probability theory is needed. It is also useful to know definitions and basic properties of L^p spaces and Hilbert spaces.

Teaching form

A hybrid teaching approach is used, combining lecture-based (DE) and interactive teaching (DI) methods. The DE includes theoretical lessons in which the knowledge about definitions, results and relevant examples is given, in order to give the skills and abilities needed to use the previous notions to solve exercises and to deal with problems (also related to extra-mathematical applications). The DI involves active student participation through answering questions and problems posed by the instructor, short presentations, and group discussions, usually conducted in the second part of the lesson. The exact number of hours dedicated to DE and DI cannot be predetermined, as the methods intertwine dynamically to adapt to the course needs, fostering participatory and integrated learning by combining theory and practice

Textbook and teaching resource

- Jean Jacod & Philip Protter: Probability essentials
- D. Williams, Probability with Martingales, Cambridge University Press (1991).
- Lecture notes (available on the e-learning platform)
- Written tests from previous years, with detailed solutions (available on the e-learning platform).
- List of proofs that may be requested during the oral examination (available on the e-learning platform).

Semester

First (fall) semester.

Assessment method

Written and oral exam. Mark out of thirty. There are no ongoing tests.

The written test, containing PROBLEMS, EXERCISES, and THEORETICAL QUESTIONS with open answers evaluates the operational *ability* to solve exercises, it receives a mark out of thirty. It is necessary to obtain an evaluation of at least 16/30 in the written test to access the oral exam, that consists in a DISCUSSION OF THE WRITTEN TEST AND OF THE TOPICS TREATED DURING LECTURES. It evaluates the *capacity* to present a selection of proofs and, above all, the critical and operational *knowledge* of the definitions and results presented during the course, also by means of examples and counterexamples. The oral exam will therefore deal with the topics treated during lectures. The final evaluation will result from the combination between the evaluation of the written test and that of the oral examination. The exam is passed if the evaluation is at least 18/30.

There will be 6 exam sessions.

Office hours

By appointment.

Sustainable Development Goals

QUALITY EDUCATION