SOLUTION HOMEWORK 1
GAME THEORY Ph.D. 2023

December 5, 2022

1 EXERCISE 1

Consider the extensive form games pictured in figure 1 and in figure 2.

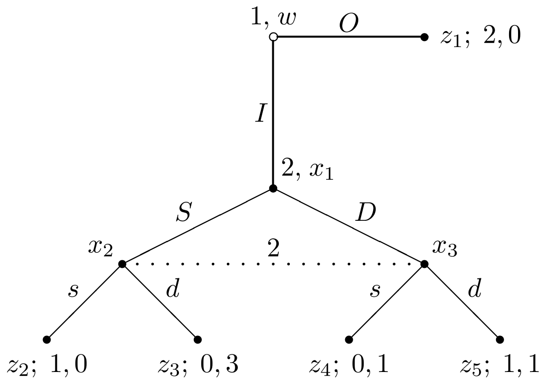

Figure 1

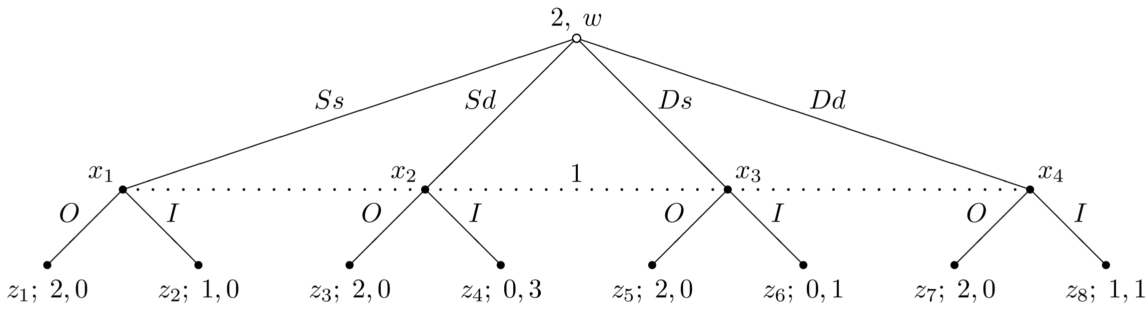

Figure 2

1. Describe formally all the elements of the extensive form game represented in the tree of figure 1;
2. Describe formally all the elements of the extensive form game represented in the tree of figure 2;
3. Find the strategic and the reduced strategic forms associated to the games of figure 1 and of figure 2;
4. Describe the sets of pure, mixed and behavioral strategies for the extensive form games of figure 1 and of figure 2;
5. If possible, for both games find a mixed strategy profile that has no outcome equivalent behavioral strategy profile. Explain the reasons of your result;
6. For the game of figure 1 find the set of rationalizable strategies;
7. For the game of figure 2 find the set of iteratively undominated strategy.

2 SOLUTION TO EXERCISE 1

1. Formally a finite extensive form game is so defined:

$$E := (N; T, \prec; A, \alpha; \iota; H_i; \rho, v_i),$$

where:

(a) N is the finite **set of players** (n is the number of players);

(b) T is a finite **set of nodes**, that together with the binary relation on T represents $\prec$ and form an arborescence, i.e. it totally orders the predecessors of each member of T;

(c) A is the set of **actions** and $\alpha : T \setminus W \to A$ is a function that labels each non-initial node with the **last action taken** to reach it;

(d) $\iota : X \to N$ represents the rules for determining **whose move it is** at a decision nodes x;

(e) information is represented by a partition H of X that divides the decision nodes into **information sets**;

(f) $\rho \in \Delta(W)$ is a **probability distribution on initial nodes**;

(g) $v_i : Z \to \mathbf{R}$ is the **utility** function of player i.

Therefore the game of figure 1 is described by the following elements:

$$N = \{1, 2\};$$

$$T = \{w, x_1, x_2, x_3, z_1, z_2, z_3, z_4, z_5\},$$

$$w \prec x_1 \prec x_2 \prec z_2, w \prec x_1 \prec x_2 \prec z_3, w \prec x_1 \prec x_3 \prec z_4, w \prec x_1 \prec x_3 \prec z_5, w \prec z_1;$$

$$A := \{O, I, S, D, s, d\},$$

$$\alpha(t) = \begin{cases} I & \text{if } t = x_1 \\ O & \text{if } t = z_1 \\ S & \text{if } t = x_2 \\ D & \text{if } t = x_3 \\ s & \text{if } t = z_2 \\ d & \text{if } t = z_3 \\ s & \text{if } t = z_4 \\ d & \text{if } t = z_5. \end{cases}$$

$$H_1 = \{\{w\}\} \quad and \quad H_2 = \{\{x_1\}, \{x_2, x_3\}\}$$

$$\iota(t) = \begin{cases} 1 & \text{if } t = w \\ 2 & \text{if } t = x_1 \\ 2 & \text{if } t = x_2 \\ 2 & \text{if } t = x_3. \end{cases}$$

$$\rho(w) = 1$$

$$v_1(t) = \begin{cases} 2 & \text{if } t = z_1 \\ 1 & \text{if } t = z_2 \\ 0 & \text{if } t = z_3 \\ 0 & \text{if } t = z_4 \\ 1 & \text{if } t = z_5. \end{cases}$$

$$v_2(t) = \begin{cases} 0 & \text{if } t = z_1 \\ 0 & \text{if } t = z_2 \\ 3 & \text{if } t = z_3 \\ 1 & \text{if } t = z_4 \\ 1 & \text{if } t = z_5. \end{cases}$$

2. Similarly, the game of figure 2 is described by the following elements:

$$N = \{1, 2\};$$

$$T = \{w, x_1, x_2, x_3, x_4, z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\}$$

$$w \prec x_1 \prec z_1, w \prec x_1 \prec z_2, w \prec x_2 \prec z_3, w \prec x_2 \prec z_4,$$

$$w \prec x_3 \prec z_5, w \prec x_3 \prec z_6, w \prec x_4 \prec z_7, w \prec x_4 \prec z_8;$$

$$A := \{O, I, Ss, Dd, Ds, Sd\},$$

$$\alpha(t) = \begin{cases} Ss & \text{if } t = x_1 \\ O & \text{if } t = z_1 \\ I & \text{if } t = z_2 \\ Sd & \text{if } t = x_2 \\ O & \text{if } t = z_3 \\ I & \text{if } t = z_4 \\ Ds & \text{if } t = x_3 \\ O & \text{if } t = z_5 \\ I & \text{if } t = z_6 \\ Dd & \text{if } t = x_4 \\ O & \text{if } t = z_7 \\ I & \text{if } t = z_8. \end{cases}$$

$$H_2 = \{\{w\}\} \quad and \;\; H_1 = \{\{x_1, x_2, x_3, x_4\}\};$$

$$\iota(t) = \begin{cases} 2 & \text{if } t = w \\ 1 & \text{if } t = x_1 \\ 1 & \text{if } t = x_2 \\ 1 & \text{if } t = x_3 \\ 1 & \text{if } t = x_4. \end{cases}$$

$$\rho(w) = 1$$

$$v_1(t) = \begin{cases} 2 & \text{if } t = z_1 \\ 1 & \text{if } t = z_2 \\ 2 & \text{if } t = z_3 \\ 0 & \text{if } t = z_4 \\ 2 & \text{if } t = z_5 \\ 0 & \text{if } t = z_6 \\ 2 & \text{if } t = z_7 \\ 1 & \text{if } t = z_8. \end{cases}$$

$$v_2(t) = \begin{cases} 0 & \text{if } t = z_1 \\ 0 & \text{if } t = z_2 \\ 0 & \text{if } t = z_3 \\ 3 & \text{if } t = z_4 \\ 0 & \text{if } t = z_5 \\ 1 & \text{if } t = z_6 \\ 0 & \text{if } t = z_7 \\ 1 & \text{if } t = z_8. \end{cases}$$

3. From both extensive form games we obtain the same strategic form game that can not be reduced.

	Ss	Sd	Ds	Dd
O	$2, 0$	$2, 0$	$2, 0$	$2, 0$
I	$1, 0$	$0, 3$	$0, 1$	$1, 1$

Figure 5

4. (a) For the game of figure 1, we get

$$S_1 = \{O, I\}$$

$$S_2 = \{Ss, Sd, Ds, Dd\};$$

$$\Sigma_1 = \Delta(S_1) = \{\sigma_1(O), \sigma_1(I) | \sigma_1(\cdot) \geq 0 \ \ \& \sigma_1(O) + \sigma_1(I) = 1\}$$

$$\Sigma_2 = \Delta(S_2) = \{\sigma_2(Ss), \sigma_2(Sd), \sigma_2(Ds), \sigma_2(Dd) \ \ |$$

$$\sigma_2(\cdot) \geq 0 \ \ \& \ \ \sigma_2(Ss) + \sigma_2(Sd) + \sigma_2(Ds) + \sigma_2(Dd) = 1\},$$

$$\Pi_1 = \{(\pi_1(O), \pi_1(I)) \mid \pi_1(\cdot) \geq 0, \ \ \& \ \ \pi_1(O) + \pi_1(I) = 1$$

$$\Pi_2 = \{(\pi_2(S), \pi_2(D)), \ (\pi_2(s), \pi_2(d)) \ \mid \ \pi_2(\cdot) \geq 0, \ \ \pi_2(S) + \pi_2(D) = 1, \ \ \pi_2(s) + \pi_2(d) = 1\}.$$

(b) For the game of figure 2, we get

$$S_1 = \{O, I\}$$

$$S_2 = \{Ss, Sd, Ds, Dd\};$$

$$\Sigma_1 = \Pi_1 = \Delta(S_1) = \{\sigma_1(O), \sigma_1(I) \mid \sigma_1(\cdot) \geq 0 \ \& \sigma_1(O) + \sigma_1(I) = 1\}$$

$$\Sigma_2 = \Pi_2 = \Delta(S_2) = \{\sigma_2(Ss), \sigma_2(Sd), \sigma_2(Ds), \sigma_2(Dd) \mid$$

$$\sigma_2(\cdot) \geq 0 \ \& \ \sigma_2(Ss) + \sigma_2(Sd) + \sigma_2(Ds) + \sigma_2(Dd) = 1\}.$$

Behavioral and mixed strategy coincide since both players have only one information set.

5. (a) Consider the following mixed strategy profile: $\hat{\sigma}_2(Ss) = \hat{\sigma}_2(Dd) = 1/2$ e $\hat{\sigma}_1(I) = 1$. To find the probability distribution on outcomes generated by $\hat{\sigma}$, we first need to construct the outcome function:

$$\xi(s) = \begin{cases} z_1 & \text{if } s = (O, Ss), \text{ either } s = (O, Sd) \text{ or } s = (O, Ds) \text{ or } s = (O, Dd) \\ z_2 & \text{if } s = (I, Ss) \\ z_3 & \text{if } s = (I, Sd) \\ z_4 & \text{if } s = (I, Ds) \\ z_5 & \text{if } s = (I, Dd). \end{cases}$$

Consequently we get the following probability distribution on outcomes generated by $\hat{\sigma}$:

$$\mathbf{P}(z|\hat{\sigma}) = \begin{cases} 0 \times 1/2 + 0 \times 0 + 0 \times 0 + 0 \times 1/2 = 0 & \text{if } z = z_1 \\ 1 \times 1/2 = 1/2 & \text{if } z = z_2 \\ 0 \times 0 = 0 & \text{if } z = z_3 \\ 0 \times 0 = 0 & \text{if } z = z_4 \\ 1 \times 1/2 = 1/2 & \text{if } z = z_5. \end{cases}$$

Denote by $\hat{\pi}$ a behavioral strategy profile outcome equivalent to $\hat{\sigma}$. From the just calculated probability distribution we know that $\mathbf{P}(z_2|\hat{\sigma}) > 0$ and $\mathbf{P}(z_3|\hat{\sigma}) = 0$, implying $\hat{\pi}_2(d) = 0$. Moreover $\mathbf{P}(z_5|\hat{\sigma}) > 0$ implying $\hat{\pi}(d) > 0$, a contradiction. Therefore there exists no behavioral strategy profile out come equivalent to $\hat{\sigma}$. This result is possible since the game of figure 1 has imperfect recall: actually when player 2 has to take its move in $\{x_2, x_3\}$, it has forgotten whether it has previously played S or D.

(b) The game of figure 2 has perfect memory and consequently Kuhn's theorem holds and therefore for all possible mixed strategy profiles there is an outcome equivalent behavioural strategy profile. Therefore for the game of figure 2 it is not possible to find a mixed strategy that has no outcome equivalent behavioural strategy.

Note that even if the games of figure 1 and of figure 2 have the same reduced strategic form, they have different characteristics of the players memory capabilities and thus different relationships between mixed and behavioral strategies.

6. From the strategic form game of the extensive form game 1

	Ss	Sd	Ds	Dd
O	$2,0$	$2,0$	$2,0$	$2,0$
I	$1,0$	$0,3$	$0,1$	$1,1$

Figure 5

it is immediate that I is never a best reply, thus that it is not rationalizable, while all strategies of player 2 are possible best responses. Thus, the set of rationalizable strategies is

$$R = \{0\} \times \{Ss, Sd, Ds, Dd\};$$

7. Since both the extensive form games of figure 1 and 2 have the same strategic form and because rationalizable and iterated strictly undominate strategies coincide, the solution is the same of point 6.

3 EXERCISE 2

Consider the extensive form games in figure 2 and in figure 3

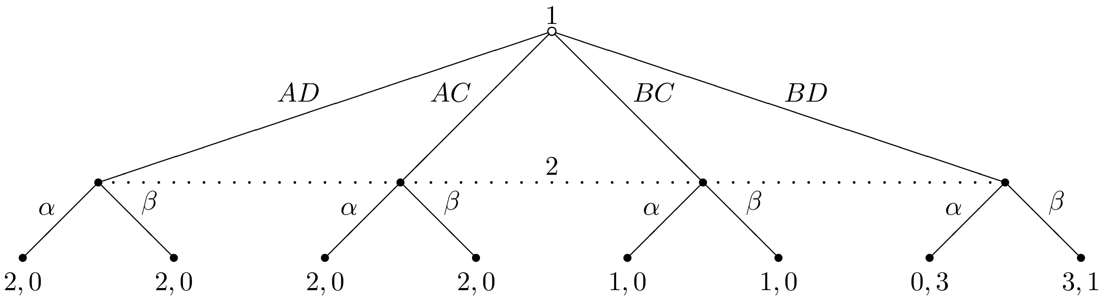

Figure 3

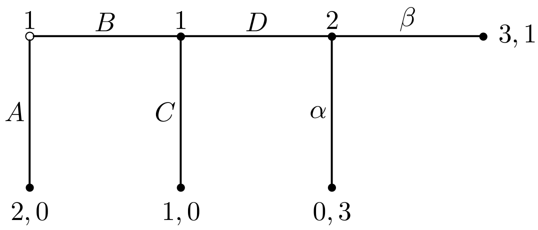

Figure 4

1. Find the strategic and reduced strategic forms associated to the games of figure 3 and of figure 4 and discuss the result;
2. For the game of figure 3 find the set of rationalizable strategies;
3. For the game of figure 4 find the set of iteratively undominated strategy.

4 SOLUTION TO EXERCISE 2

1. The strategic and the reduced strategic form of both games are in figure 6 and 7, respectively.

	α	β
AD	2, 0	2, 0
AC	2, 0	2, 0
BC	1, 0	1, 0
BD	0, 3	3, 1

Figure 6

	α	β
$A\cdot$	2, 0	2, 0
BC	1, 0	1, 0
BD	0, 3	3, 1

Figure 7

This exercise and the previous one show that extremely different extensive games may have the same reduced strategic form, in particular properties such as perfect memory and perfect information are not immediately reflected in the structure of reduced strategic form.

2. From the strategic form game, it is immediate that BC is never a best reply, thus that it is not rationalizable, while all strategies of player 2 are possible best responses. Thus, the set of rationalizable strategies is

$$R = \{AD, AC, BD\} \times \{\alpha, \beta\};$$

3. Since both the extensive form games of figure 3 and 4 have the same strategic form and because rationalizable and iterated strictly undominate strategies coincide, the solution is the same of point 6.

5 Exercise 3

Consider the following perfectly competitive market:

- there is a continuum of producers indexed by the parameter i distributed with a density $f(i)$on the interval $[i_0, i_1] \subseteq \mathbf{R}^{++}$such that $\int_{i_0}^{i_1} i\, f(i)di = K$;
- the producers choose the quantity $q(i)$before the market opens;
- the cost function of each firm $i$is $C(q, i) = \frac{q(i)^2}{2i}$;
- the demand curve for this market is $D(p) = max\{0, a - bp\}$;
- the timing is such that first the firms simultaneously choose the production quantities, then the price clears the market.

1. Find the Walrasian equilibrium of this economy;
2. Describe formally the strategic form game associated to this perfectly competitive economy;
3. Find the set of rationalizable strategies of the game described at point 2;
4. Find the set of iterated strictly undominated strategies of the game described at point 2.

6 Solution to exercise 3

1. To find the Walrasian equilibrium, we should first determine the aggregate supply in a perfectly competitive setting. Firm i in this setting will maximise its profit taking the price as given:

$$max_{q(i)}[pq(i) - \frac{(q(i))^2}{2i}].$$

The FOC is $p - \frac{q(i)}{i} = 0$, which implies the firm's supply

$$q(i)^s = ip.$$

Consequently the aggregate supply is

$$S(p) = \int_{i_0}^{i_1} q(i)^s f(i)di = \int_{i_0}^{i_1} ip\, f(i)di = p\int_{i_0}^{i_1} i\, f(i)di = Kp.$$

Then the Walrasian equilibrium of this market is the couple (Q^W, p^W) such that aggregate supply is equal to the aggregate demand, i.e.

$$Kp = a - bp$$

Which implies

$$p^W = \frac{a}{K+b} \;\;\&\;\; Q^W = \frac{Ka}{K+b}.$$

Clearly any single firm i will supply

$$q(i)^W = i\frac{a}{K+b}.$$

2. A strategic form game is defined by $G = (N, S_i, u_i)$ where

   (a) N is the set of players

   (b) S_i is the set of pure strategies of player $i \in N$

   (c) $u_i : S \to \mathbf{R}$ is the payoff function of player i.

   In this example:

   (a) $N = [i_0, i_1] \subset \mathbf{R}^+$ where the players are distributed according to the density $f(i)$;

   (b) $S_i \subseteq \mathbf{R}^+$

   (c) $u(q(i), Q) = q(i)\frac{a-Q}{b} - \frac{q(i)^2}{2i}$ where $Q = \int_{i_0}^{i_1} q(i)\, f(i)\, di$ is the aggregate production.

3. To calculate the set of rationalizable strategies we work iteratively on the players' best reply functions $BR_i(p) = ip$:

(a) i. by definition $R_i(0) = S_i = \mathbf{R}^+$

(b) note that $q(i) \geq 0$ implies $Q = \int_{i_0}^{i_1} q(i)\, f(i)\, di \geq 0$. Therefore $p \leq \frac{a}{b}$ since $p = \frac{a-Q}{b}$. Thus

$$R_i(1) = \left\{ q(i) \in \mathbf{R}^+ | q(i) \in BR_i(p) \text{ for some } p \leq \frac{a}{b} \right\} = \left\{ q(i) \in \mathbf{R}^+ | q(i) \leq i\frac{a}{b} \right\};$$

(c) Since $\forall q(i) \in R_i(1)\ q(i) \leq i\frac{a}{b}$, then $Q \in \int_{i_0}^{i_1} R_i(1)\, f(i)\, di$ implies $Q \leq \frac{aK}{b}$. Therefore $p \geq \frac{a}{b} - \frac{aK}{b^2}$ since $p = \frac{a-Q}{b}$. Thus

$$R_i(2) = \left\{ q(i) \in R_i(1) | q(i) \in BR_i(p) \text{ for some } p \in \left[\frac{a}{b} - \frac{aK}{b^2}, \frac{a}{b} \right] \right\} =$$

$$= \left\{ q(i) \in R_i(1) | q(i) \in \left[i\frac{a}{b} - i\frac{aK}{b^2}, i\frac{a}{b} \right] \right\};$$

(d) Since $\forall q(i) \in R_i(2)\ q(i) \geq i\frac{a}{b}\left(1 - \frac{K}{b}\right)$, then $Q \in \int_{i_0}^{i_1} R_i(2)\, f(i)\, di$ implies $Q \leq \frac{a}{b}\left(1 - \frac{K}{b}\right) K$. Therefore $p \leq \frac{a}{b} - \frac{a}{b}\left(1 - \frac{K}{b}\right)\frac{K}{b} = \frac{a}{b}\left(1 - \frac{K}{b} + \frac{K^2}{b^2}\right)$ since $p = \frac{a-Q}{b}$. Thus

$$R_i(3) = \left\{ q(i) \in R_i(1) | q(i) \in BR_i(p) \text{ for some } p \in \left[\frac{a}{b}\left(1 - \frac{K}{b}\right), \frac{a}{b}\left(1 - \frac{K}{b} + \frac{K^2}{b^2}\right) \right] \right\} =$$

$$= \left\{ q(i) \in R_i(1) | q(i) \in \left[i\frac{a}{b}\left(1 - \frac{K}{b}\right), i\frac{a}{b}\left(1 - \frac{K}{b} + \frac{K^2}{b^2}\right) \right] \right\};$$

(e) n general at stage n even,

$$R_i(n) = \left\{ q(i) \in R_i(n+1) | q(i) \in \left[i\frac{a}{b}\sum_{t=0}^{n-1}\left(-\frac{K}{t}\right)^t, i\frac{a}{b}\sum_{t=0}^{n-2}\left(-\frac{K}{t}\right)^t \right] \right\};$$

(f) in general at stage $n+1$ odd,

$$R_i(n+1) = \left\{ q(i) \in R_i(n+1) | q(i) \in \left[i\frac{a}{b}\sum_{t=0}^{n-1}\left(-\frac{K}{t}\right)^t, i\frac{a}{b}\sum_{t=0}^{n}\left(-\frac{K}{t}\right)^t \right] \right\};$$

(g) Therefore the set of rationalizable strategies shrinks if and only if the sum $\sum_{t=0}^{\infty}(\frac{a}{b})(-\frac{K}{b})^t$ is convergent. Therefore

i. A. if $K < b$, then

$$R_i = \left\{ i\frac{a}{b+K} \right\}$$

if $K \geq b$, then

$$R_i = \left[0, i\frac{a}{b}\right].$$

4. To calculate the set of iterated strictly undominated strategies is equivalent to calculate the set of rationalizable strategies, since we know that they coincide.