LECTURE 3

Nash and Bayes-Nash Equilibria

MAIN POINTS OF PREVIOUS LECTURE

CRUCIAL PROBLEM

Intelligent players anticipate opponents' rational behavior implying **iterative solutions**

HOWEVER TO MAKE OPERATIVE THIS ANTICIPATION OF OPPONENTS' RATIONAL BEHAVIOR, PLAYERS NEED TO KNOW

1. **OPPONENTS' STRATEGY SETS**
2. **OPPONENTS' PAYOFF FUNCTIONS**

i.e.

THE GAME

However standard models do not specify players' information on the game itself: *information sets regard actions only*

IMPERFECT INFORMATION

VS

INCOMPLETE INFORMATION

Imperfect Information vs. Incomplete Information

- Standard models do not specify players' information on the game itself***: information sets regard actions only***
- ***In standard game theory there is no formal tools to model information about the game***
- ***Standard informal assumption***:

The **game** is **common knowledge**, i.e.

1. all the players know the **game**
2. All the players know that all the players know the **game**
3. Etc. ad infinitum

- If a game satisfies this assumption is called **complete information game**

Imperfect Information vs. Incomplete Information

Definitions

- Game of ***imperfect information***: one or more players do not know the full history of the game, i.e. previous moves.
- Game of ***incomplete information***: the players have private information about the **game**, which we will call the **state of nature.**
- We need new formal tools to deal with incomplete information: **information sets are not enough since they regard players' actions**

Example 1: the problem when the true game being played is unknown - 1

State of nature 1

Player 1 \ Player 2	L	R
T	0, 1	1, 0
B	1, 0	0, 1

State of nature 2

Player 1 \ Player 2	L	R
T	1, 0	0, 1
B	0, 1	1, 0

Example 1: players' best response as function of: Prior belief Opponent's strategy

Player 1 rational behavior

Player 2

	L	R
p<0.5	T	B
p>0.5	B	T

p=Pr¹{s=State of nature 1} by player 1

Player 2 rational behavior

Player 1	q<0.5	q>0.5
T	R	L
B	L	R

q=Pr²{s=State of nature 1} by player 2

Example 1: the problem when the true game being played is unknown - 3

- As the previous slide shows
  - 1's optimal strategy depends on
    1. Prior belief p **and**
    2. The strategy of 2, which in turn depend on
       1. Prior belief q **and**
       2. The strategy of 1, which in turn depend on
          1. Prior belief p **and**
          2. The strategy of 2, which in turn depend on **…**
- Therefore when we don't know the s.o.n., it is not enough to have beliefs on it (first order beliefs), but we need beliefs on beliefs (second order beliefs), etc. i.e. we need
  - **Infinite hierarchy of beliefs**

Example 1: the problem when the true game being played is unknown - 4

- According to the **Bayesian approach**, each player has a belief on the unknown s.o.n.
- But unlike to decision making problem, in an interactive situation we are naturally lead, as previously shown, to
  - **Infinite hierarchies of beliefs**
- But this object is cumbersome and hardly manageable
- This is the **explicit approach** and its complexity was the main obstacle to the development of the theory of games of incomplete information
- Till a breakthrough by Harsanyi

Bayesian games
and
the Harsanyi approach

Imperfect Information vs. Incomplete Information: Harsanyi idea

- The key to analyze games of incomplete information is to transform them into games of imperfect information by letting nature move first, randomly selecting each possible "state of nature" and "players' information on it", i.e.
- Nature selects each player's possible **type** **(Harsanyi transformation).**

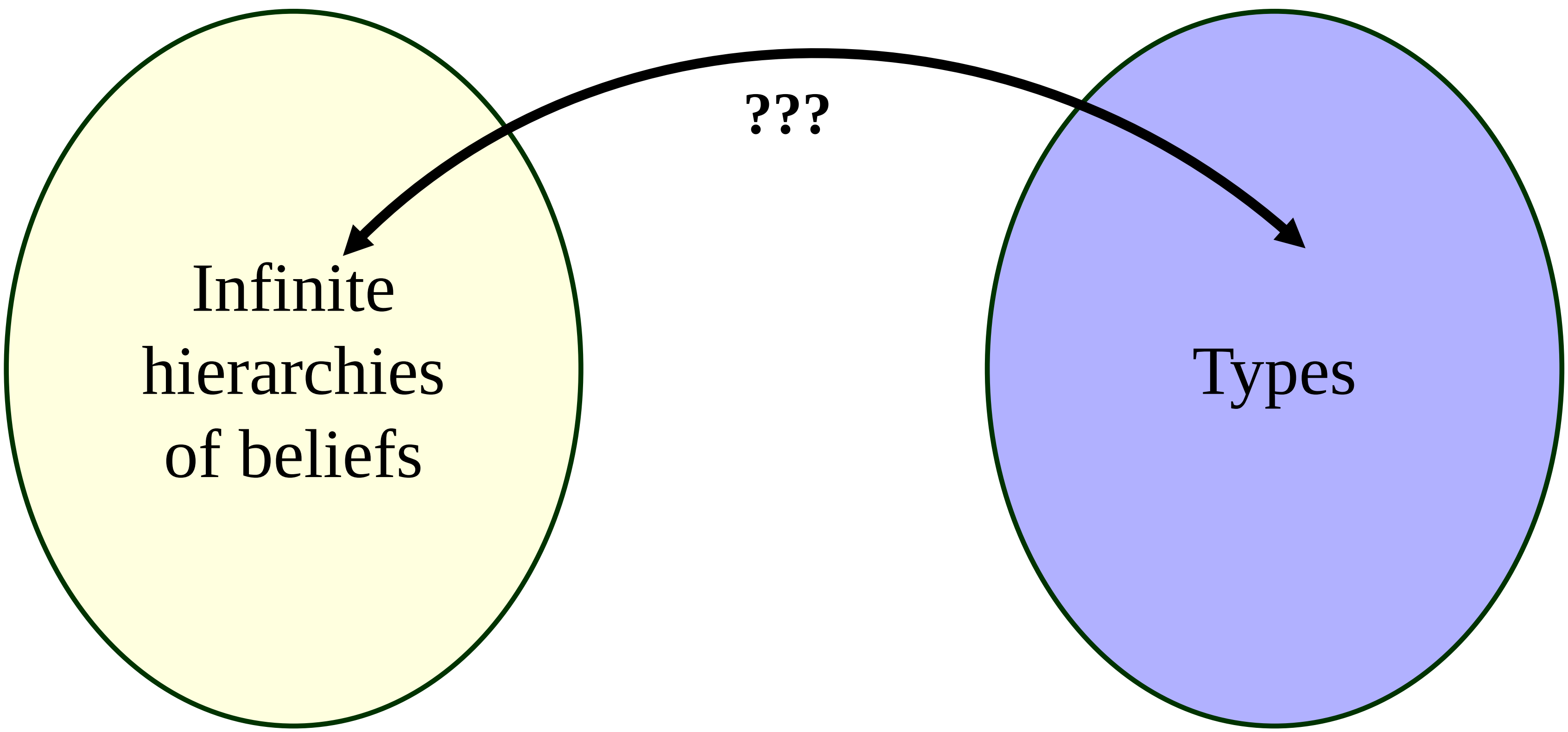
The notion of Bayesian game

- Using the Harsanyi approach, the situation of incomplete information is reinterpreted as a game of imperfect information
- Nature makes the first move, choosing realizations of the random variables that determine
  - each player's **TYPE,**
  - **i.e. each player's PRIVATE INFORMATION ON THE RULE OF THE GAME, INCLUDING OTHER PLAYERS' POSSIBLE PRIVATE INFORMATION**
- Each players observes the realization of only his type
- This sort of game is called **BAYESIAN GAME.**

The notion of TYPE

- A **PLAYER'S SET OF TYPES** is a random variable,
- its realization is a **PLAYER'S TYPE** representing the **player's private information.**
- **A type is a full description of**
  - **Player's beliefs on the rule of the game i.e. on state of nature**
  - **Beliefs on other players' beliefs on s.o.n. and its own beliefs**
  - **Etc.**
- **NB: there is a circular element in the definition of type, which is unavoidable in interactive situations**
- **i.e. the Harsanyi approach solves the problem of modelling incomplete information in a simple ingenious way at the cost of making the set of possible types potentially extremely complex**

Types and infinite hierarchies of beliefs



Bayesian Games

(Harsanyi, *Management Science* 1967-8)

- u_i = utility function for i, $u_i(a,t)$ depends on both actions a and types t.
- normal form game $G = \{N; A_1,\ldots,A_n; u_1,\ldots,u_n\}$
- Bayesian game $\Gamma = \{N; A_1,\ldots,A_n; T_1,\ldots,T_n; p_1,\ldots,p_n; u_1,\ldots,u_n\}$
- A_i = strategy set for i, actions in the Bayesian Game:

  $a = (a_1,\ldots,a_n) \in A = A_1 \times \ldots \times A_n$.
- T_i = type set for i, types: $t = (t_1,\ldots,t_n) \in T = T_1 \times \ldots \times T_n$
- p_i = beliefs for i, $p_i(t_{-i} \mid t_i)$ = i's belief about types t_{-i} given type t_i.

Bayesian Games
(Harsanyi, *Management Science* 1967-8)

- Beliefs $\{p_1,\ldots,p_n\}$ are ***consistent*** if they can be derived using Bayes' rule from a **common joint distribution p(t) on T**; i.e., there exists p(t) such that

$$p_i(t_{-i}|t_i) = \frac{p(t)}{p(t_i)} \quad \text{where} \qquad p(t_i) = \sum_{t_{-i} \in T_{-i}} p(t_{-i}, t_i)$$

  for all i and t_i.

- Beliefs are **consistent** if nature moves first and types are determined according to the **common prior** p(t) and each i is informed only of t_i.
- Plausible (?) if types are interpreted as full description of a player's private information

Beliefs derived from common prior - 1

- **EXAMPLE: joint & marginal probability**

	A low costs	A high costs	Marginal Pr of B costs
B low costs	0.45	0.05	0.5
B high costs	0.15	0.35	0.5
Marginal Pr of A costs	0.6	0.4	1

Beliefs derived from common prior - 2

- **EXAMPLE: conditional probability**

Pr{B cost | A cost}

		A INFORMATION	
		A low costs	A high costs
UNCERETAIN EVENT	B low costs	0.45/0.6= =0.75	0.05/0.4= =0.125
	B high costs	0.15/0.6= =0.25	0.35/0.4= =0.875

Beliefs derived from common prior - 3

- **EXAMPLE: conditional probability**

Pr{A cost | B cost}

		B INFORMATION	
		B low costs	B high costs
UNCERETAIN EVENT	A low costs	0.45/0.5= =0.9	0.15/0.5= =0.3
	A high costs	0.05/0.5= =0.1	0.35/0.5= =0.7

Definition

- A ***strategy in a Bayesian game*** for i is a plan of action for each of i's possible types

$$d_i: T_i \rightarrow A_i$$

- As usual it says what to do in every possible contingency (each of the possible types).

Example 1: a modified prisoner's dilemma with different possible payoffs

- Prisoner 2 has two possible different payoffs:
  - With probability m the players' payoffs are that of figure 1
  - With probability 1-m the players' payoffs are that of figure 2
  - Player 2 knows his own payoffs
- Thus the players are possibly playing two different games, with player 2 informed of the true game (asymmetric information).

The possible payoffs of player 2

Figure 1

Player 1 \ **Player 2**	DC	C
DC	0, -2	-10, -1
C	-1, -10	-5, -5

Figure 2

Player 1 \ **Player 2**	DC	C
DC	0, -2	-10, **-7**
C	-1, -10	-5, **-11**

The Harsanyi approach applied to example 1

- According to this approach each player's preferences are determined by the realization of a random variable;
- The random variable's actual realization is observed only by the player
- **Its ex ante probability distribution is assumed to be common knowledge among all the players**
- **Players' types:**
  - player 1 set of types is the null set since player 1 has no private information: $T_1 = \{\varnothing\}$
  - player 2 set of types has two element, the payoffs of figure 1 and figure 2: $T_2 = \{t', t''\}$
- **Players' Beliefs:**
  - $p_1\{t' \mid \varnothing\} = m$
  - $p_2\{\varnothing \mid t'\} = p_2\{\varnothing \mid t''\} = 1.$

Beliefs derived from common prior - 1

- **EXAMPLE 1: joint & marginal probability**

1 type / 2 type	∅	Marginal Pr of 2 type
t'	m	m
t"	1-m	1-m
Marginal Pr of 1 type	1	1

Beliefs derived from common prior - 2

- **EXAMPLE 1: conditional probability**

Pr{2 type | 1 type}

		1 INFORMATION
UNCERETAIN EVENT	2 type	∅
	t'	m/1 = m
	t''	(1-m)/1 = 1-m

Beliefs derived from common prior - 3

- **EXAMPLE: conditional probability**

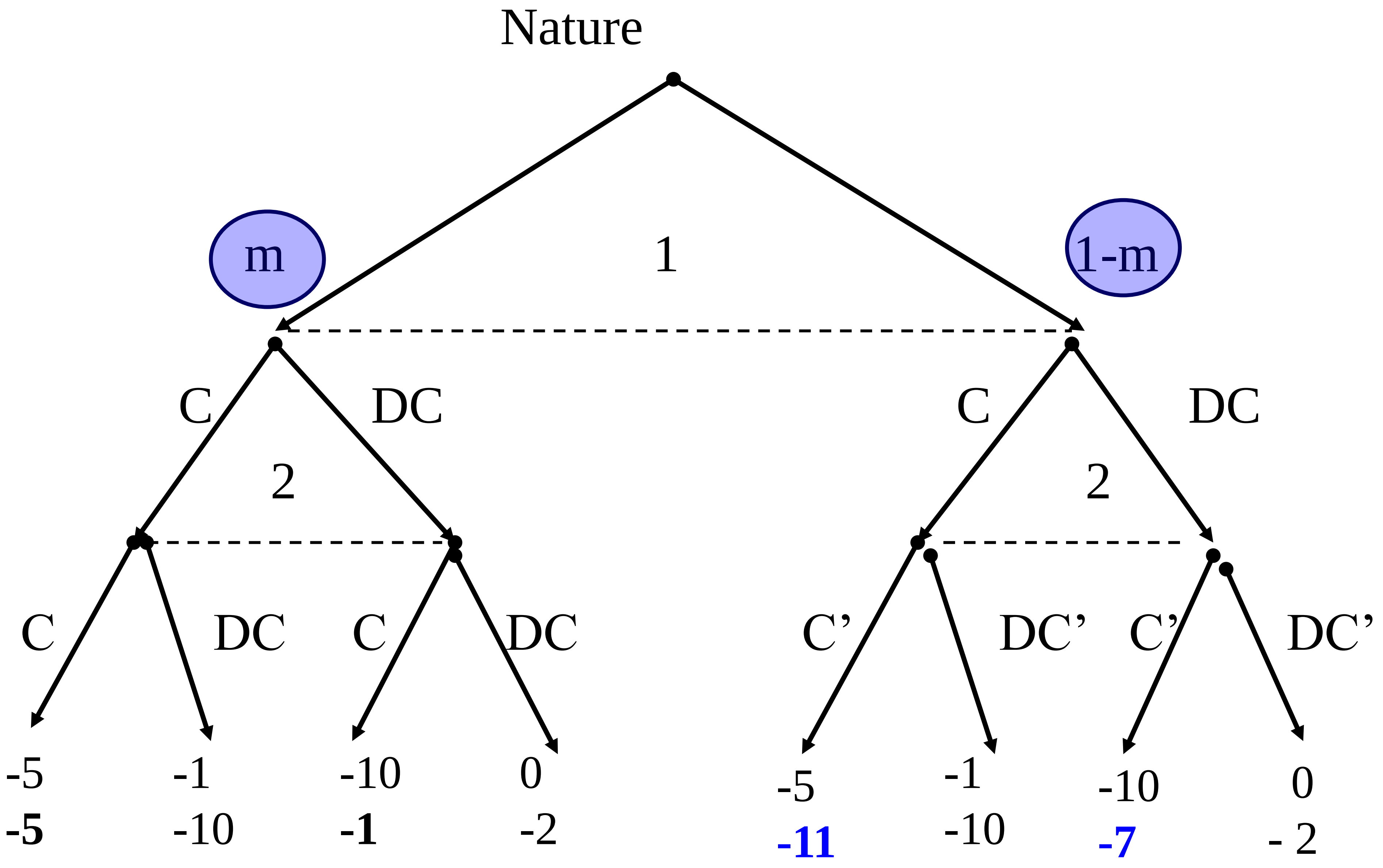
Pr{1 type | 2 type}

		2 INFORMATION	
UNCERETAI N EVENT	**1 type**	**t’**	**”**
	∅	**m/m = 1**	**(1-m)/(1-m)=1**

The Extensive Form of example 1

$$T = \{(Fig\ 1, \emptyset), (Fig\ 2, \emptyset)\} \quad p(t') = Pr\{(Fig\ 1, \emptyset)\} = m \Rightarrow$$

$$p_1(t'|t_1) = Pr\{Fig\ 1|\emptyset\} = m \ \& \ p_2(t'|t_2) = Pr\{\emptyset|Fig\ 2\} = Pr\{\emptyset|Fig\ 1\} = 1$$



The Bayesian strategic form of example 1

1 \ 2	C-C’	C-DC’	DC-C’	DC-DC’
C	-5, -5m-11(1-m)	-5m-1(1-m), -5m-10(1-m)	-5, -5m-11(1-m)	-1, -10
DC	-10, -1m-7(1-m)	-10m+0(1-m), -1m-2(1-m)	0m-10(1-m), -2m-7(1-m)	0, -2

SUMMING UP - 1

- **BAYESIAN GAME:** a game in which players are uncertain on payoff relevant parameters
- **STATE OF NATURE:** payoff relevant data. It is convenient to think of a s.o.n. as a full description of a game form
- **TYPE**: full description of player's relevant characteristics, therefore it fully describes
  1. **Player's beliefs (i.e. information) on s.o.n.**
  2. **Player's beliefs on others' beliefs**
  3. **Player's beliefs on others' beliefs on its beliefs**
  4. **Etc. ad infinitum**

SUMMING UP - 2

- **STATE OF THE WORLD:** a specification of s.o.n. ***and*** players' types. i.e. of
  1. Payoff relevant parameters
  2. Beliefs of all levels
- **COMMON PRIOR AND CONSISTENT BELIEFS:** players' beliefs are said to be **consistent** if they are derived from the same probability distribution (the **common prior**) by conditioning on each player's private information. Therefore if beliefs are consistent, the only source of differences in beliefs is difference in information

Nash equilibria

Example 1

	A	B	C	D
A	**0, 7**	**2, 5**	**7, 0**	0, 1
B	**5, 2**	**3, 3**	**5, 2**	0, 1
C	**7, 0**	**2, 5**	**0, 7**	0, 1
D	0, 0	0, -2	0, 0	10, -1

Solution by ISUS and by rationalizability

- With both concepts, the solution is

$$\{A, B, C\} \times \{A, B, C\}$$

i.e. there are 9 likely outcomes.

- PROBLEM: the forecast is too vague, we would like a more sharp prediction

Example 1 again

	A	B	C	D
A	**0, 7**	**2, 5**	**7, 0**	0, 1
B	**5, 2**	**3, 3**	**5, 2**	0, 1
C	**7, 0**	**2, 5**	**0, 7**	0, 1
D	0, 0	0, -2	0, 0	10, -1

Remark

Note that (B, B) has a peculiar characteristic:

1. it is a profile such that each strategy is a best reply to the other strategy
2. Therefore if, for some unspecified reason, this profile is played, then nobody has an incentive to deviate.

This is not true for the other rationalizable outcomes.

Example 1 another time

	A	B	C	D
A	0, 7	2, 5	7, 0	0, 1
B	5, 2	3, 3	5, 2	0, 1
C	7, 0	2, 5	0, 7	0, 1
D	0, 0	0, -2	0, 0	10, -1

An equilibrium concept as solution: Nash Equilibrium

For an n-person game in Normal form, **a strategy profile** $s^* \in S$ is a *Nash equilibrium* in pure strategies if **for all i**

$$u_i(s^*) \geq u_i(s^*_{-i}, s_i) \quad \text{for all } s_i \in S_i$$

where $s^*_{-i} = (s^*_1, \ldots, s^*_{i-1}, s^*_{i+1}, \ldots, s^*_n)$.

NASH EQUILIBRIUM

- NB: it is defined as **A STRATEGY PROFILE**, not as a Cartesian product, like Rationalizability.
- This depends on the fact that we are dealing with an **equilibrium concept.**

Non existence of pure strategy NE Example 2: Matching Pennies

- Each of two players simultaneously show a penny.
- If the pennies match (both heads or both tails), player 1 gets 2's penny. Otherwise, player 2 gets 1's penny.

Example 2 in Normal Form

		2	
		H	T
1	H	1, -1	-1, 1
	T	-1, 1	1,-1

No equilibrium in pure strategies

Definition

An n-tuple of mixed strategies $\hat{\sigma} = \left(\hat{\sigma}_1, ..., \hat{\sigma}_n\right)$ is a Nash Equilibrium if for every i,

$$u_i(\hat{\sigma}) \geq u_i(\hat{\sigma}_{-i}, \sigma_i) \qquad \forall \sigma_i \in \Sigma_i$$

Definitions - 1

- The best reply correspondence to σ_{-i} for a strategic form game G=(S,u) for each player *i* is

$$BR_i(\sigma_{-i}) = \arg\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}) =$$

$$= \{\sigma_i \in \Sigma_i \mid u_i(\sigma) \geq u_i(\sigma_{-i}, \hat{\sigma}_i) \quad \forall \hat{\sigma}_i \in \Sigma_i\}$$

$$BR_i(\Sigma_{-i}) = \bigcup_{\sigma_{-i} \in \Sigma_{-i}} BR_i(\sigma_{-i}) =$$

$$= \{\sigma_i \in \Sigma_i \mid \exists \sigma_{-i} \in \Sigma_{-i} : \; u_i(\sigma) \geq u_i(\sigma_{-i}, \hat{\sigma}_i) \quad \forall \hat{\sigma}_i \in \Sigma_i \}$$

- The best reply correspondence of the strategic form game G=(S,u) is

$$BR = \prod_{i \in N} BR_i(\Sigma_{-i}) \quad \text{therefore} \quad BR : \Sigma \Rightarrow \Sigma$$

Definitions

- σ^* is a Nash equilibrium iff $\sigma^* \in BR(\sigma^*)$

- In words:
  - σ^* is a Nash equilibrium

    iff

  - σ^* is a fixed point of the best reply correspondence

Three different topics

1. Existence of Nash equilibria ⇔ existence of a fixed point

2. Why Nash equilibria?

3. Calculations of Nash equilibria

CONSIDERATIONS ON NASH EQUILIBRIA

INTERPRETATION OF NASH EQUILIBRIUM

- FOUR POSSIBLE **INTERPRETATIONS:**

1. Nash equilibrium as eductive (introspective) solution
2. Nash equilibrium as rest point of some dynamic process
3. Nash equilibrium as Rational Expectations Equilibrium
4. **Strategy configuration such that no players has an incentive to deviate**

Why Nash Equilibria?

- Once one has selected the appropriate game, attention typically turns to equilibrium behavior.
- Under the classical view of game theory, one should be able to deduce the equilibrium play from the specification of the game and the hypothesis that it is commonly known that the players are rational.
- An analyst observing the game should be able to make such a deduction, as should the players in the game.
- This immediately answers an obvious question:
- Why are we interested in the equilibrium of a game?
- **In the classical view, the equilibrium implication of a game will be obvious to rational players, and will just as obviously be reflected in their behavior**.

Problems with Nash Equilibrium

Deduction from complete information

- Difficulties appeared in the attempt to show that Nash equilibrium could be deduced from the specification of the game and the hypotheses that the players are commonly known to be rational:
- common knowledge of rationality allowed one to infer only that players will restrict attention to rationalizable strategies.

Problems with Nash Equilibrium

Multiplicity

- Multiple equilibria arise in many settings for many reasons.
- How are we to identify the equilibrium implication of the game in the presence of multiple equilibria?
- **A response** to this question was the **equilibrium refinements** literature, which sought "refinement" criteria for limiting attention to a subset of the set of Nash equilibria.
- For example, one might restrict attention to Nash equilibria that do not play weakly dominated strategies
- A different response is to accept multiplicity
  1. focusing on results that depend only on the presumption that some equilibrium is chosen, without being specific as to which equilibrium
  2. using empirical methods to point the way to an equilibrium
  3. noting that in some cases, models with multiple equilibria may provide the best match for the interaction being studied

Answer to problems with Nash Equilibrium

- In response, the classical view of game theory gave way to an ***instrumental view***.
- In this view, the game is not a literal description of an interaction, but it is a model that one hopes is useful in studying that interaction.
- Game-theoretic solution concepts should be understood in terms of their applications, and should be judged by the quantity and quality of their applications.
- The game is thus a deliberate approximation, designed to include important aspects of the interaction and exclude unimportant ones

Example of the instrumental view

- The choice between the Cournot and Bertrand models hinges not on what one thinks firms actually do, but on which model gives the most useful insights.
- Are we working in a setting in which competition between even two firms is enough to drive prices to marginal cost? If so, the Bertrand model may be appropriate.
- Do we think that the entry of a new firm into the market is likely to decrease the profits of existing firms? If so, the Cournot model is likely to be appropriate

Game theoretic models

- An implication of the instrumental view is that making a model more realistic does not necessarily make it a better model.
- It is obvious that making a model more *complicated* does not necessarily make it a better model
- a model as complicated as its intended application is also typically useless.
- Even without extra complication, more realism need not be a step forward for a model

More on the instrumental view

- ***The instrumental view complicates game theory***.
- A world of literal descriptions and perfectly rational players is typically more orderly than are approximations of a complicated world filled with people.
- Consider **the prisoners' dilemma**:
- Will people defect in the prisoners' dilemma?
- In the **classical view** this is obvious since under this interpretation, the numbers in the payoff matrix are utilities indicating that the agents derives higher utility from defecting.
- Asking whether the agent might cooperate is equivalent to asking whether we have gotten the game wrong. If the game is correct, there can be no outcome other than defection.
- Things are more complicated under **an instrumental view**.
- First, the actions cooperate and defect are approximations of alternatives that may be much more complex. Cooperation may involve colluding in an oligopoly market or signing a nuclear arms agreement, while defection may involve flooding the market with increased output or installing an antiballistic missile shield.
- In addition, we typically cannot hope to measure utilities, and the numbers in the cells are instead measures of profits or some other more-readily-measured quantity.
- Will the players defect?
- Equivalently, **have we chosen well in approximating the interaction as a prisoners' dilemma?** This can be a difficult question.

Applications and solution concepts - 1

- In many applications, results hinge on selecting a particular equilibrium for study
- **the choice of equilibrium concept, and the choice between multiple equilibria satisfying that concept, is part of the construction of the model, and should be informed by the details of the application one has in mind**
- Examples:
  - modeling an encounter between two agents that have limited experience and knowledge of one another, such as the US president and a dictator suspected of harboring weapons of mass destruction, a restriction to rationalizable strategies may be too demanding, since one might reasonably question whether there is common knowledge of rationality
  - applying Nash equilibrium, and even applying a particular Nash equilibrium, in settings where the participants have enough historical or cultural experience with the game. We take it for granted that people will drive on the left in the United Kingdom and on the right in the United States

Applications and solution concepts - 2

- Considerable work remains to be done on identifying both
  – which equilibrium concept we should be using and
  – which of the potentially many equilibria consistent with that concept should command our attention.
- Keynes: “**Economics is the science of thinking in terms of models joined to the art of choosing models which are relevant to the contemporary world**”
- Graduate courses in economics tend to focus on the science of working with models.
- Progress on equilibrium selection will come from careful work on the art of choosing models.
- This is **a joint choice involving both the game and the relevant equilibrium**, and will typically depend on the setting to which the analysis is to be applied.

CALCULATIONS OF EQUILIBRIA

- Simple examples of calculations of Nash equilibria

Two-person, zero-sum game

Player 1 \ Player 2	Left	Center	Right
Left	0, 0	1, -1	0, 0
Center	-1, 1	0, 0	-1, 1
Right	0, 0	1, -1	0, 0

Two-person, zero-sum game: best reply strategies for Player 1

Player 1 \ Player 2	Left	Center	Right
Left	**0**, 0	1, -1	**0**, 0
Center	-1, 1	0, 0	-1, 1
Right	**0**, 0	1, -1	**0**, 0

Two-person, zero-sum game: best reply strategies for Player 2

Player 1 \ Player 2	Left	Center	Right
Left	0, **0**	1, -1	0, **0**
Center	-1, **1**	0, 0	-1, **1**
Right	0, **0**	1, -1	0, **0**

Two-person, zero-sum game: Four equilibria

Player 1 \ Player 2	Left	Center	Right
Left	**0**, **0** ★	1, -1	**0**, **0** ★
Center	-1, **1**	0, 0	-1, **1**
Right	**0**, **0** ★	1, -1	**0**, **0** ★

Three players game

Player 3 - Yes

Player 1 \ Player 2	Yes	No
Yes	5, 5, 5	0, 0, 0
No	0, 0, 0	0, 0, 0

Player 3 - No

Player 1 \ Player 2	Yes	No
Yes	0, 0, 0	0, 0, 0
No	0, 0, 0	0, 0, 0

Three players game: best reply strategies for player 1

Player 3 - Yes

Player 1 \ Player 2	Yes	No
Yes	**5**, 5, 5	**0**, 0, 0
No	0, 0, 0	**0**, 0, 0

Player 3 - No

Player 1 \ Player 2	Yes	No
Yes	**0**, 0, 0	**0**, 0, 0
No	**0**, 0, 0	**0**, 0, 0

Three players game: best reply strategies for player 2

Player 3 - Yes

Player 1 \ Player 2	Yes	No
Yes	5, **5**, 5	0, 0, 0
No	0, **0**, 0	0, **0**, 0

Player 3 - No

Player 1 \ Player 2	Yes	No
Yes	0, **0**, 0	0, **0**, 0
No	0, **0**, 0	0, **0**, 0

Three players game: best reply strategies for player 3

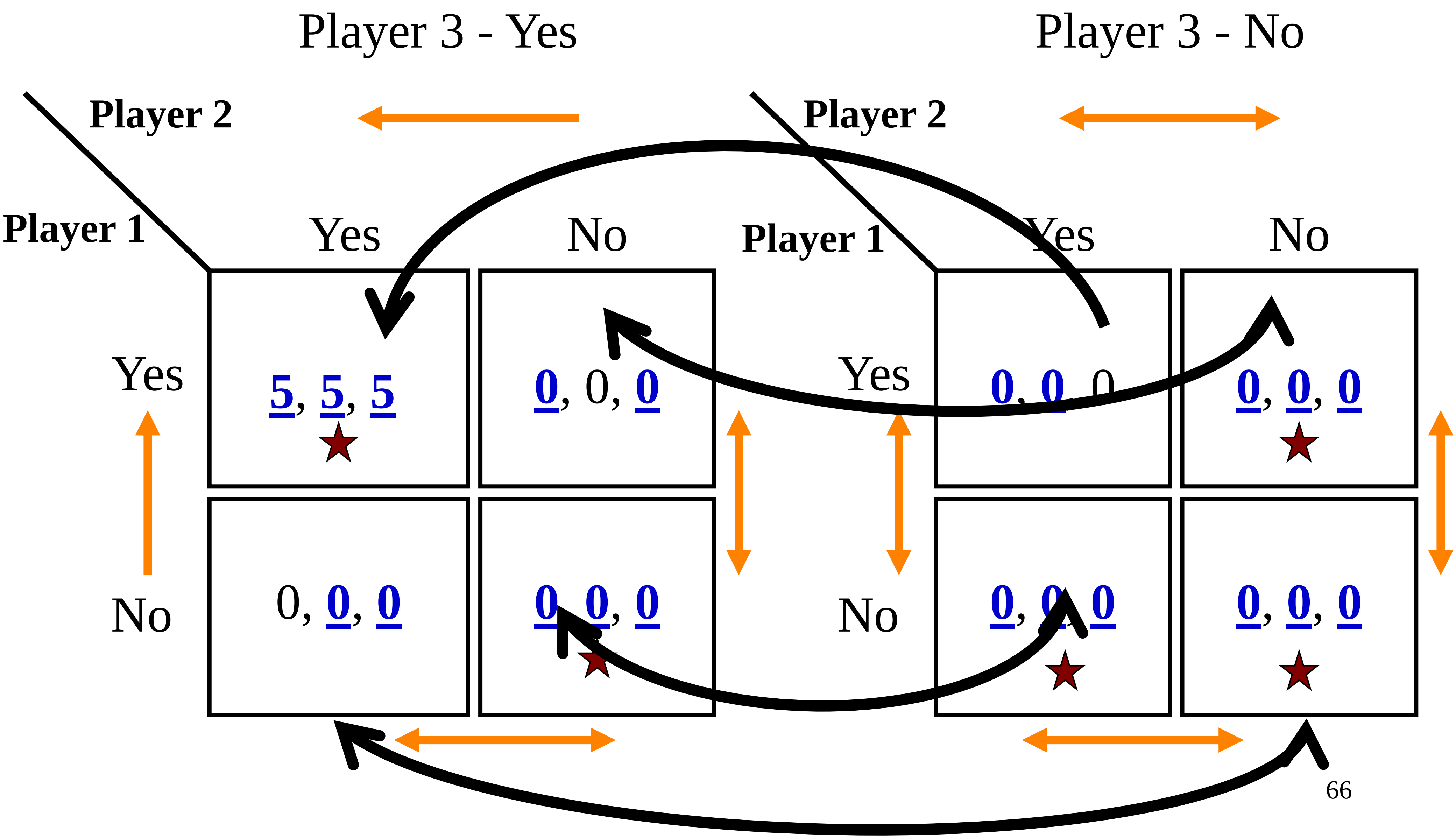
Player 3 - Yes

Player 1 \ Player 2	Yes	No
Yes	5, 5, **5**	0, 0, **0**
No	0, 0, **0**	0, 0, **0**

Player 3 - No

Player 1 \ Player 2	Yes	No
Yes	0, 0, 0	0, 0, **0**
No	0, 0, **0**	0, 0, **0**

Three players game: Five pure strategy equilibria



Matching Pennies: The payoff matrix

Player 1 \ Player 2	Heads	Tails
Heads	+1, -1	-1, +1
Tails	-1, +1	+1, -1

Matching Pennies: No equilibrium in pure strategies

All Best Responses are underlined.

Player 1 \ Player 2	Heads	Tails
Heads	**<u>+1</u>**, -1	-1, **<u>+1</u>**
Tails	-1, **<u>+1</u>**	**<u>+1</u>**, -1

Computing Mixed Strategy Equilibria in 2×2 Games

- Solution criterion: each pure strategy in a mixed strategy equilibrium pays the same at equilibrium
- Each pure strategy not in a mixed strategy equilibrium pays less
- Detailed calculations for Matching Pennies

Matching Pennies: What about mixed strategies?

1 \ 2	probability	y	1- y
probability		h	t
x	H	+1, -1	-1, +1
1- x	T	-1, +1	+1, -1

x, y between 0 and 1

That is, $0 \leq x \leq 1$ and $0 \leq y \leq 1$

Need to calculate player 1's expected utility from player 2's mixed strategy

1 \ 2	h (probability y)	t (probability $1-y$)	EU_1:
H	+1, -1	-1, +1	$2y - 1$
T	-1, +1	+1, -1	$1 - 2y$

$$EU_1(H) = y \times 1 + (1-y) \times -1 = 2y - 1$$

$$EU_1(T) = y \times -1 + (1-y) \times 1 = 1 - 2y$$

Need to calculate player 2's expected utility from player 1's mixed strategy

probability	1 \ 2	h	t
x	H	+1, -1	-1, +1
1- x	T	-1, +1	+1, -1
	EU_2:	1 - 2x	2x - 1

$EU_2(h) = x \times -1 + (1- x) \times 1 = 1 - 2x$

$EU_2(t) = x \times 1 + (1- x) \times -1 = 2x - 1$

In equilibrium, Player 1 is willing to randomize only when he is indifferent between H and T

$EU_1(H) = y \times 1 + (1 - y) \times -1 = 2y - 1$
$EU_1(T) = y \times -1 + (1 - y) \times 1 = 1 - 2y$

In equilibrium: $EU_1(H) = EU_1(T)$

$\therefore \quad 2y - 1 = 1 - 2y$

$\Rightarrow \quad 4y = 2$

$\Rightarrow \quad y = ½$

$\Rightarrow \quad 1 - y = 1 - ½ = ½$

$\therefore \quad y = 1 - y = ½$

Similarly, Player 2 is willing to randomize only when she is indifferent between H and T

Player 2's Conditions:

$EU_2(h) = x \times -1 + (1- x) \times 1 = 1 - 2x$

$EU_2(t) = x \times 1 + (1- x) \times -1 = 2x - 1$

In equilibrium: $EU_2(h) = EU_2(t)$

$\therefore \quad 1 - 2x = 2x - 1$

$\Rightarrow \quad x = ½$ and $1 - x = 1 - ½ = ½$

$\therefore \quad x = 1 - x = ½$

Matching Pennies: Equilibrium in mixed strategies

probability	1 \ 2	h	t	EU_1:
	probability	½	½	
½	H	+1, -1	-1, +1	0
				$\parallel$
½	T	-1, +1	+1, -1	0
EU_2:		0	= 0	

Each player is playing a best response to the other!

Mixed strategies are not intuitive: **You randomize to make me indifferent**

Row randomizes to make Column indifferent.

Column randomizes to make Row indifferent.

Then each player is playing a best response to the other.

- **GENERAL WAY OF CALCULATING THE SET OF NASH EQUILIBRIA**
- **THE USE OF BEST REPLY CORRESPONDENCES**

Battle of sexes: the set of Nash equilibria in pure and mixed strategies

probability	1 \ 2	y	1- y
		h	t
x	H	3, 1	0, 0
1- x	T	0, 0	1, 3

x, y between 0 and 1

That is, $0 \leq x \leq 1$ and $0 \leq y \leq 1$

Need to calculate player 1's expected utility from player 2's mixed strategy

1 \ 2 (probability)	h (y)	t ($1-y$)	EU_1:
H	3, 1	0, 0	$3y$
T	0, 0	1, 3	$1 - y$

$EU_1(H) = y \times 3 + (1-y) \times 0 = 3y$

$EU_1(T) = y \times 0 + (1-y) \times 1 = 1 - y$

Player 1 best reply depends on player 2 mixed strategy

$EU_1(H) = y \times 3 + (1- y) \times 0 = 3y$
$EU_1(T) = y \times 0 + (1- y) \times 1 = 1 - y$

1 best reply: H iff $EU_1(H) \geq EU_1(T)$

$\therefore \quad 3y \geq 1 - y$

$\Rightarrow \quad 4y \geq 1$

$\Rightarrow \quad y \geq 1/4$

The best reply correspondence of player 1

$$x = \sigma_1(H) = \begin{cases} 1 & if \quad y = \sigma_2(h) \geq 1/4 \\ \in [0,1] & if \quad y = \sigma_2(h) = 1/4 \\ 0 & if \quad y = \sigma_2(h) \leq 1/4. \end{cases}$$

Need to calculate player 2's expected utility from player 1's mixed strategy

probability	1 \ 2	h	t
x	H	3, 1	0, 0
1- x	T	0, 0	1, 3
	EU_2:	x	3 - 3x

$EU_2(h) = x \times 1 + (1-x) \times 0 = x$

$EU_2(t) = x \times 0 + (1-x) \times 3 = 3 - 3x$

Player 2 best reply depends on player 1 mixed strategy

$EU_2(h) = x \times 1 + (1 - x) \times 0 = x$
$EU_2(t) = x \times 0 + (1 - x) \times 3 = 3 - 3x$

2 best reply: h iff $EU_2(h) \geq EU_2(t)$

$\therefore \quad x \geq 3 - 3x$

$\Rightarrow \quad 4x \geq 3$

$\Rightarrow \quad x \geq 3/4$

The best reply correspondence of player 2

$$y = \sigma_2(h) = \begin{cases} 1 & if \quad x = \sigma_1(H) \geq 3/4 \\ \in [0,1] & if \quad x = \sigma_1(H) = 3/4 \\ 0 & if \quad x = \sigma_1(H) \leq 3/4. \end{cases}$$

The set of Nash equilibria using best reply correspondences

Thus

$$x = \begin{cases} 1 & if \quad y \geq 1/4 \\ \in [0,1] & if \quad y = 1/4 \\ 0 & if \quad y \leq 1/4. \end{cases}$$

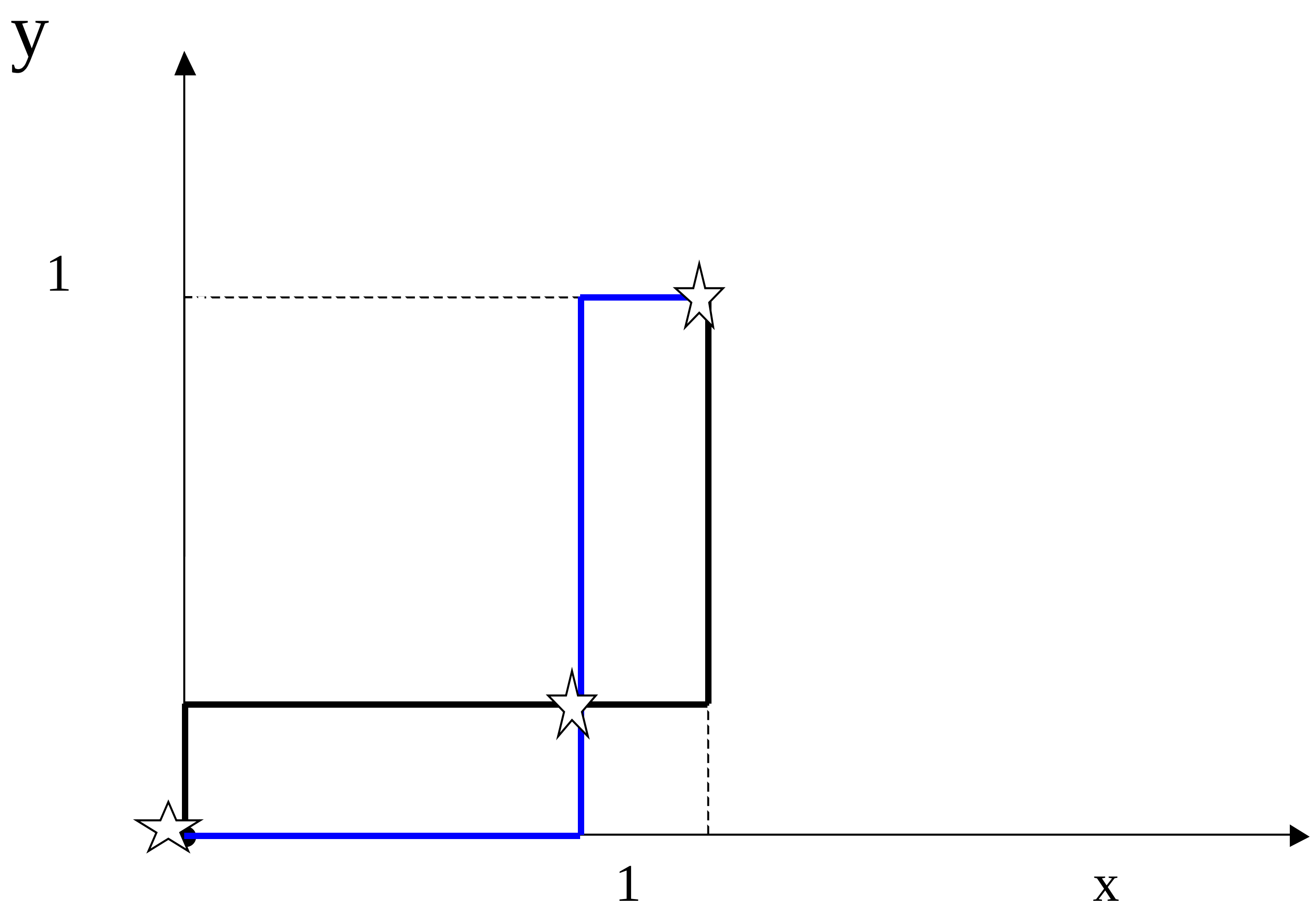
and

$$y = \begin{cases} 1 & if \quad x \geq 3/4 \\ \in [0,1] & if \quad x = 3/4 \\ 0 & if \quad x \leq 3/4. \end{cases}$$

The set of Nash equilibria using best reply correspondences

Pl. 1= black

Pl.2=blue



The set of Nash Equilibria in the battle of sexes

$$NE = \{\sigma_1(H) = 1, \sigma_2(h) = 1\} \cup$$
$$\cup \{\sigma_1(H) = 3/4, \sigma_2(h) = 1/4\} \cup$$
$$\cup \{\sigma_1(H) = 0, \sigma_2(h) = 0\}.$$

APPLYING THESE SOLUTION CONCEPTS TO BAYESIAN GAMES

Example 1: a modified prisoner's dilemma with different possible payoffs

- Prisoner 2 has two possible different payoffs:
  - With probability m the players' payoffs are that of figure 1
  - With probability 1-m the players' payoffs are that of figure 2
  - Player 2 knows his own payoffs
- Thus the players are possibly playing two different games, with player 2 informed of the true game (asymmetric information).

The possible payoffs of player 2

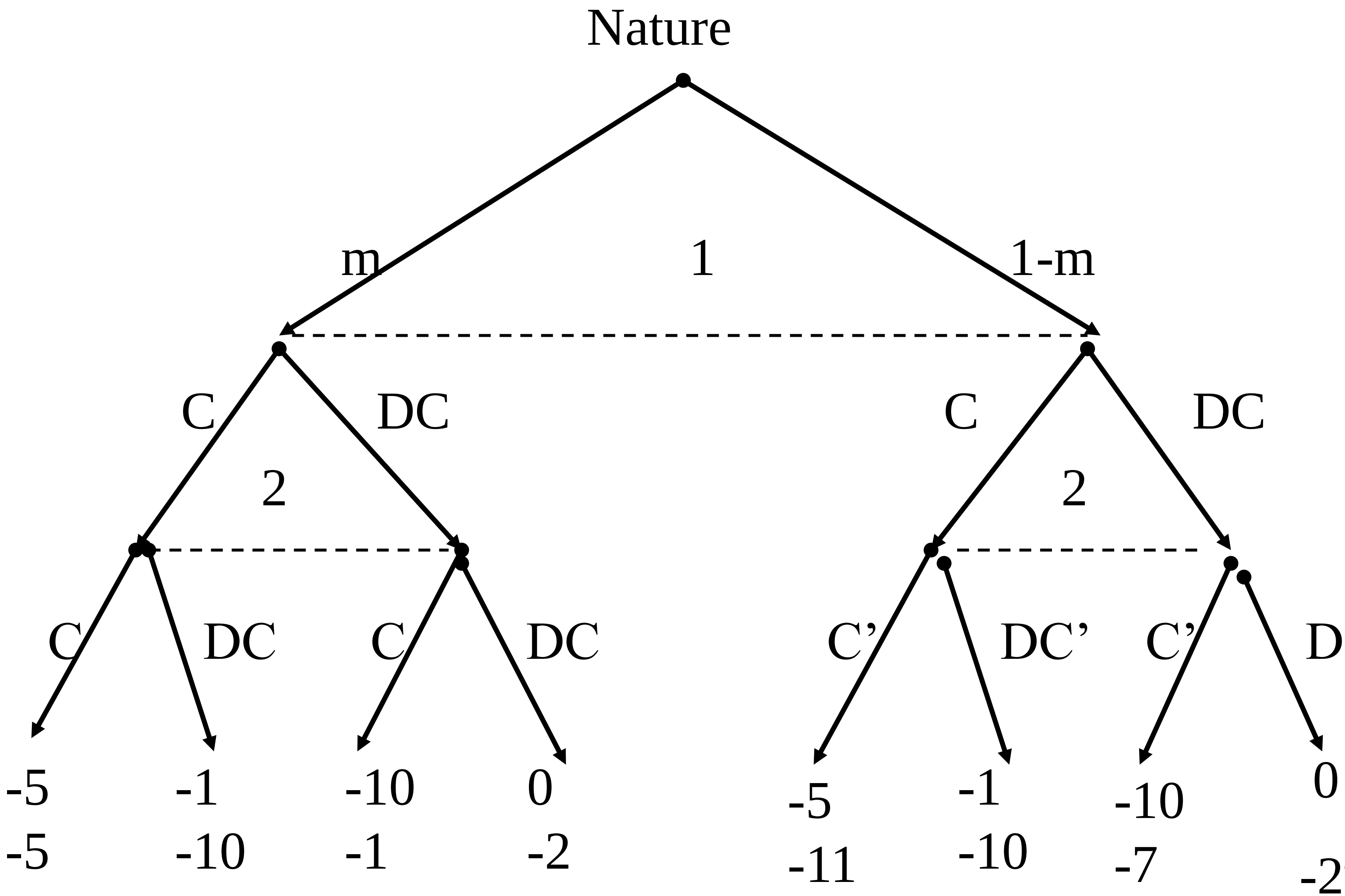
Figure 1

Player 1 \ Player 2	DC	C
DC	0, -2	-10, -1
C	-1, -10	-5, -5

Figure 2

Player 1 \ Player 2	DC	C
DC	0, -2	-10, -7
C	-1, -10	-5, -11

The Extensive Form of example 1



The Bayesian strategic form of example 1

1 \ 2	C-C'	C-DC'	DC-C'	DC-DC'
C	-5, -5m-11(1-m)	-5m-1(1-m), -5m-10(1-m)	-m-5(1-m), -10m-11(1-m)	-1, -10
DC	-10, -1m-7(1-m)	-10m+0(1-m), -1m-2(1-m)	0m-10(1-m), -2m-7(1-m)	0, -2

Definitions

- A ***strategy in a Bayesian game*** for i is a plan of action for each of i's possible types

  d_i:$T_i \rightarrow A_i$ it says what to do in every possible contingency (each of the possible types).
- A strategy profile $d = (d_1,...,d_n)$ is a ***Bayes-Nash Equilibrium*** of Γ if

$$\forall i, \ \ \forall d_i'$$

$$E_t\left[u_i(d_i(t_i),d_{-i}(t_{-i});t_i)\right] \geq E_t\left[u_i(d_i'(t_i),d_{-i}(t_{-i});t_i)\right]$$

Bayes-Nash Equilibrium

- Existence of a Bayes-Nash Equilibrium when the type sets and pure-strategy spaces are finite follows from the standard existence theorem for finite games.
- **Given consistent beliefs, a Bayes-Nash Equilibrium of Γ is simply a Nash equilibrium of the game with imperfect information in which nature moves first.**
- Any game of incomplete information *with consistent beliefs* can be transformed into a standard normal form game.

The Bayes-Nash equilibria of the game of example 1 when m=0.5

1 \ 2	C-C’	C-DC’	DC-C’	DC-DC’
C	-5, -8	-3, -7.5	-3, -8	-1, -10
DC	-10, -4	-5, -1.5	-5, -4.5	0, -2

- **GENERAL WAY OF CALCULATING THE SET OF NASH EQUILIBRIA**
- **THE USE OF BEST REPLY CORRESPONDENCES**

Further example

- Consider a partnership between two people:
  - They share a profit

$$P = 4(x + y + 0.25xy)$$

  - that depends on their effort, x and y
  - The effort is any real number in [0,4] and cost to each player respectively x^2 and y^2
  - The players choose the effort simultaneously and indipendently.
  - The game in strategic form is:

$$N = \{1,2\}, \qquad S_i = [0,4],$$

$$v_1(x,y) = 2(x + y + 0.25xy) - x^2$$

$$v_2(x,y) = 2(x + y + 0.25xy) - y^2$$

First best: Pareto efficient efforts

- Find x, y to maximize the joint profit

$$4(x+y+0.25xy)-x^2-y^2$$

$$FOC: \quad 4+\text{y-2x}=0 \ \text{ and } \ 4+x-2y=0$$

$$x^{FB}=4 \quad y^{FB}=4.$$

Non cooperative solution

- Find the best reply function:

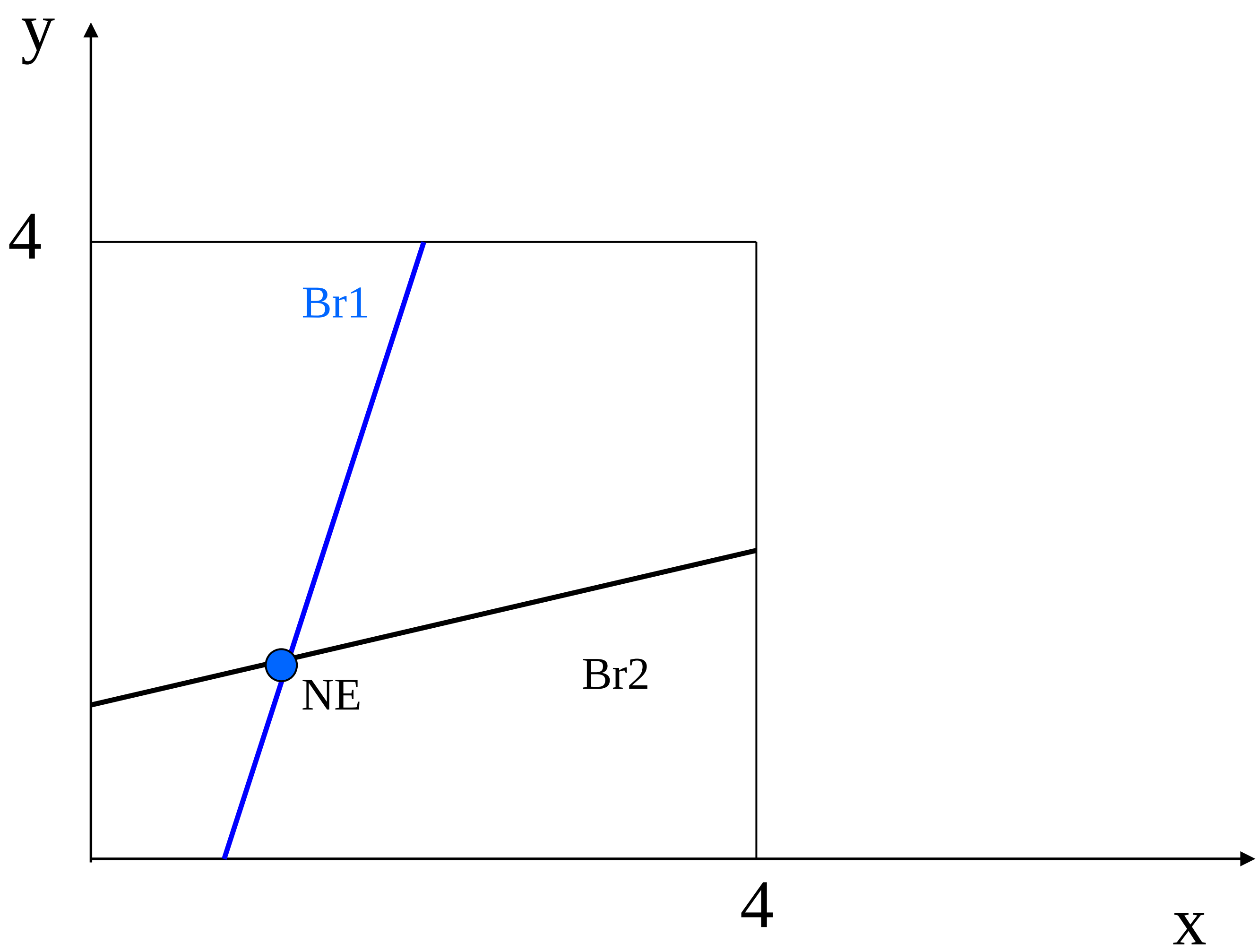
$$\frac{\partial v_1}{\partial x} = 2 + 0.5y - 2x = 0 \Rightarrow x = BR_1(y) = 0.25y + 1$$

$$\frac{\partial v_2}{\partial y} = 2 + 0.5x - 2y = 0 \Rightarrow y = BR_2(x) = 0.25x + 1$$

- Find the Nash equilibria:

$$\begin{cases} x = BR_1(y) = 0.25y + 1 \\ y = BR_2(x) = 0.25x + 1 \end{cases} = \begin{cases} x = \frac{1}{16}x + 1.25 \\ y = 0.25x + 1 \end{cases} = \begin{cases} x = \frac{4}{3} \\ y = \frac{4}{3} \end{cases}$$

Graphically



Applications

Electoral Competition

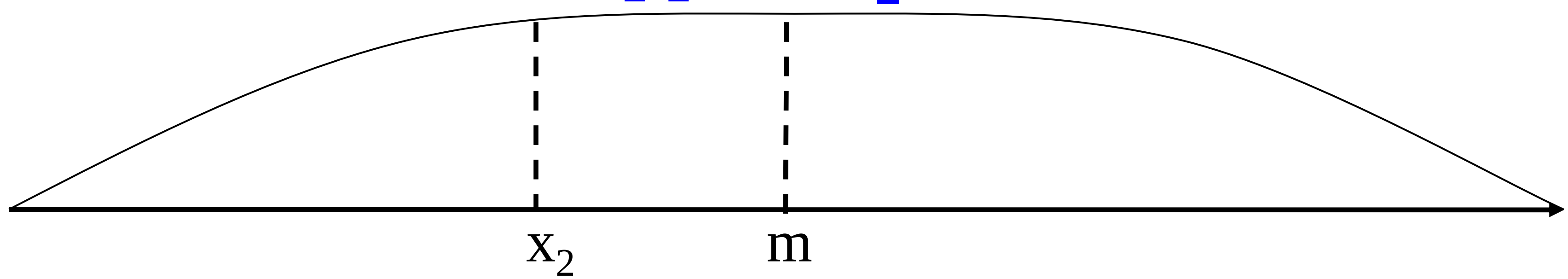
Electoral Competition

- The players are 2 candidates.
- A policy is a real number k, referred to as a "position."
- After candidates choose positions, each citizen votes for candidate with the policy she prefers.
- The candidate who obtains the most votes wins. Candidates care only about winning.
- Voters are a continuum with diverse ideologies y, with cumulative distribution F. For any k, a voter with ideology y is indifferent policies y - k and y + k.
- Median m is such that 1/2 of voters' ideologies $y \geq m$ & 1/2 of ideologies $y \leq m$: $F(m) = 1/2$.

Best Response Functions - 1

- Fix x_2 of candidate 2 and consider 1's choice.

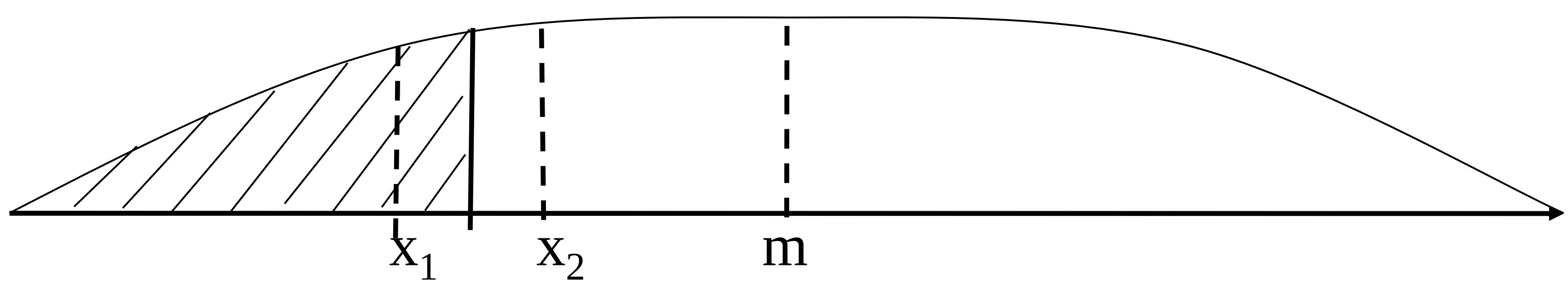
  - **Suppose $x_2 < m$**



- If candidate 1 chooses $x_1 < x_2$ then she wins the votes with ideology $y < ½ (x_1 + x_2)$.

  Because $½ (x_1 + x_2) < x_2 < m$, then

  $F(½ (x_1 + x_2)) < F(m)= ½$, so that candidate 1 wins less than ½ of the votes and loses the election.



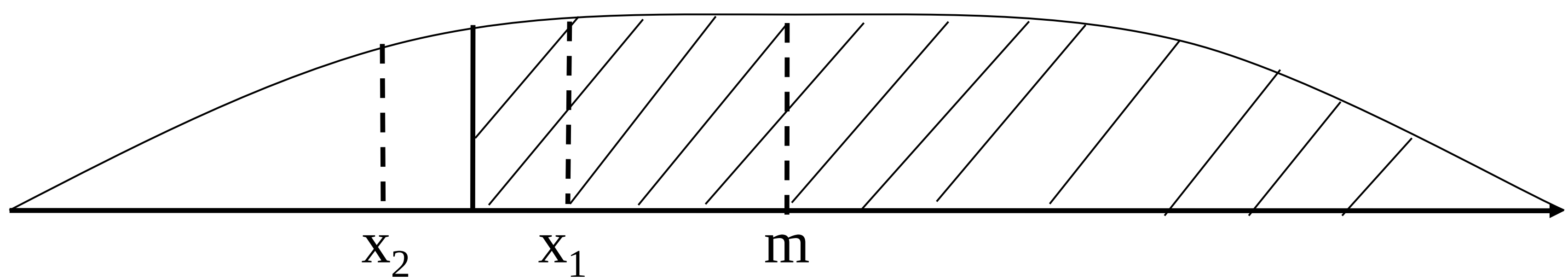
Best Response Functions - 2

- If $x_1 > x_2$, then 1 wins the votes of ideology $y > ½ (x_1 + x_2)$.
- She wins the election if she get more than ½ of the votes, i.e. if and only if

$$1 - F(½ (x_1 + x_2)) > ½ \Leftrightarrow F(½ (x_1 + x_2)) < ½ = F(m)$$

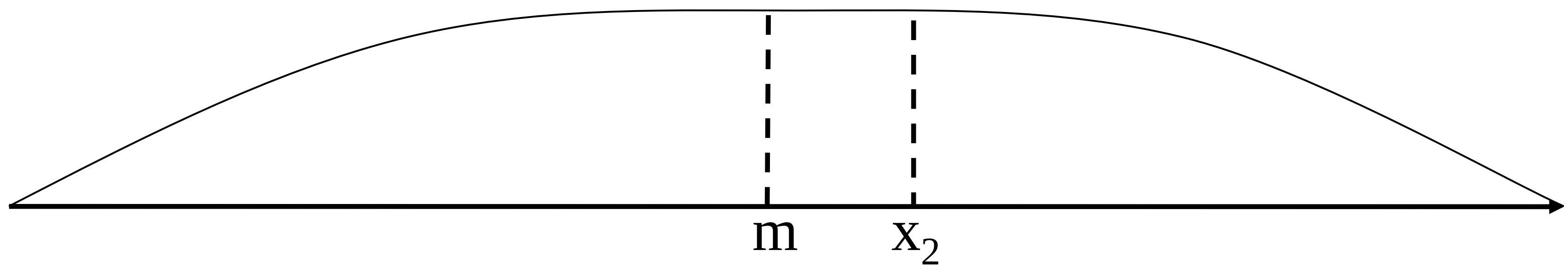
$$\Leftrightarrow ½ (x_1 + x_2) < m \Leftrightarrow x_1 < 2m - x_2.$$

- So, $BR_1 (x_2) = \{x_1 : x_2 < x_1 < 2m - x_2 \}$ for $x_2 < m$.



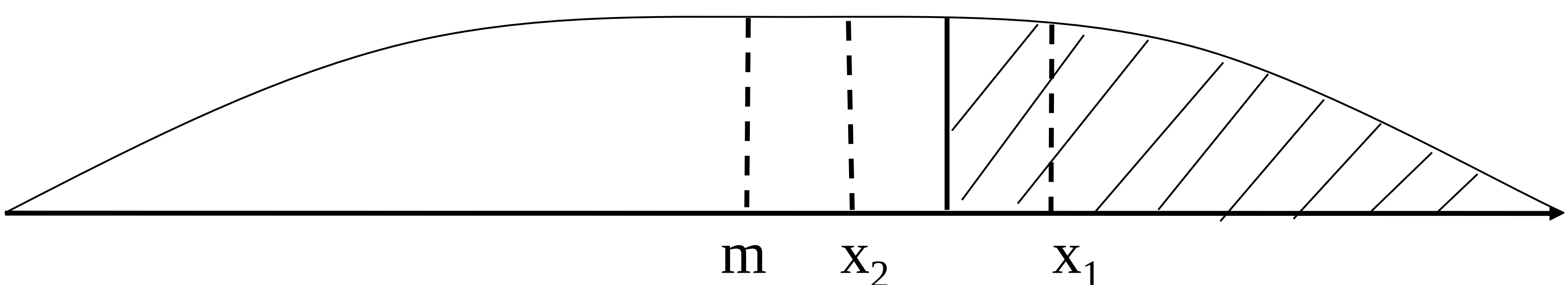
Best Response Functions - 3

- **Suppose $x_2 > m$**



- If candidate 1 chooses $x_1 > x_2$ then she wins the votes with ideology $y > ½ (x_1 + x_2)$.

  Because $½ (x_1 + x_2) > x_2 > m$, then $1- F(½ (x_1 + x_2)) < 1- F(m) = ½$, so candidate 1 wins less than ½ of the votes and loses the election.



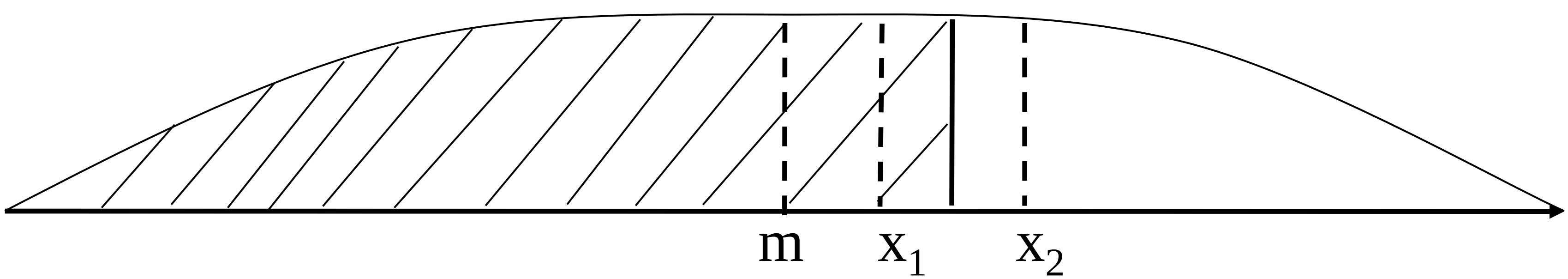
Best Response Functions - 4

- If $x_1 < x_2$, then 1 wins the votes of ideology
  $y < ½ (x_1 + x_2)$.
- She wins the election if she get more than ½ of the votes, i.e. if and only if

$$F(½ (x_1 + x_2)) > ½ = F(m) \Leftrightarrow ½ (x_1 + x_2) > m \Leftrightarrow$$

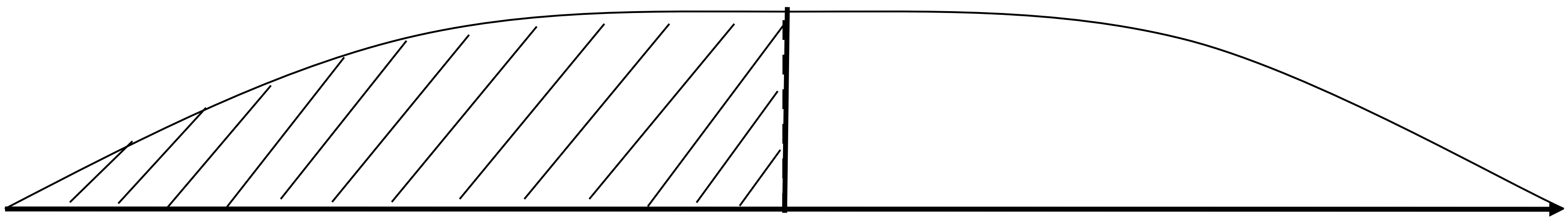
$$\Leftrightarrow x_1 > 2m - x_2.$$

- So, $BR_1 (x_2) = \{x_1 : 2m - x_2 < x_1 < x_2 \}$ for $x_2 > m$.



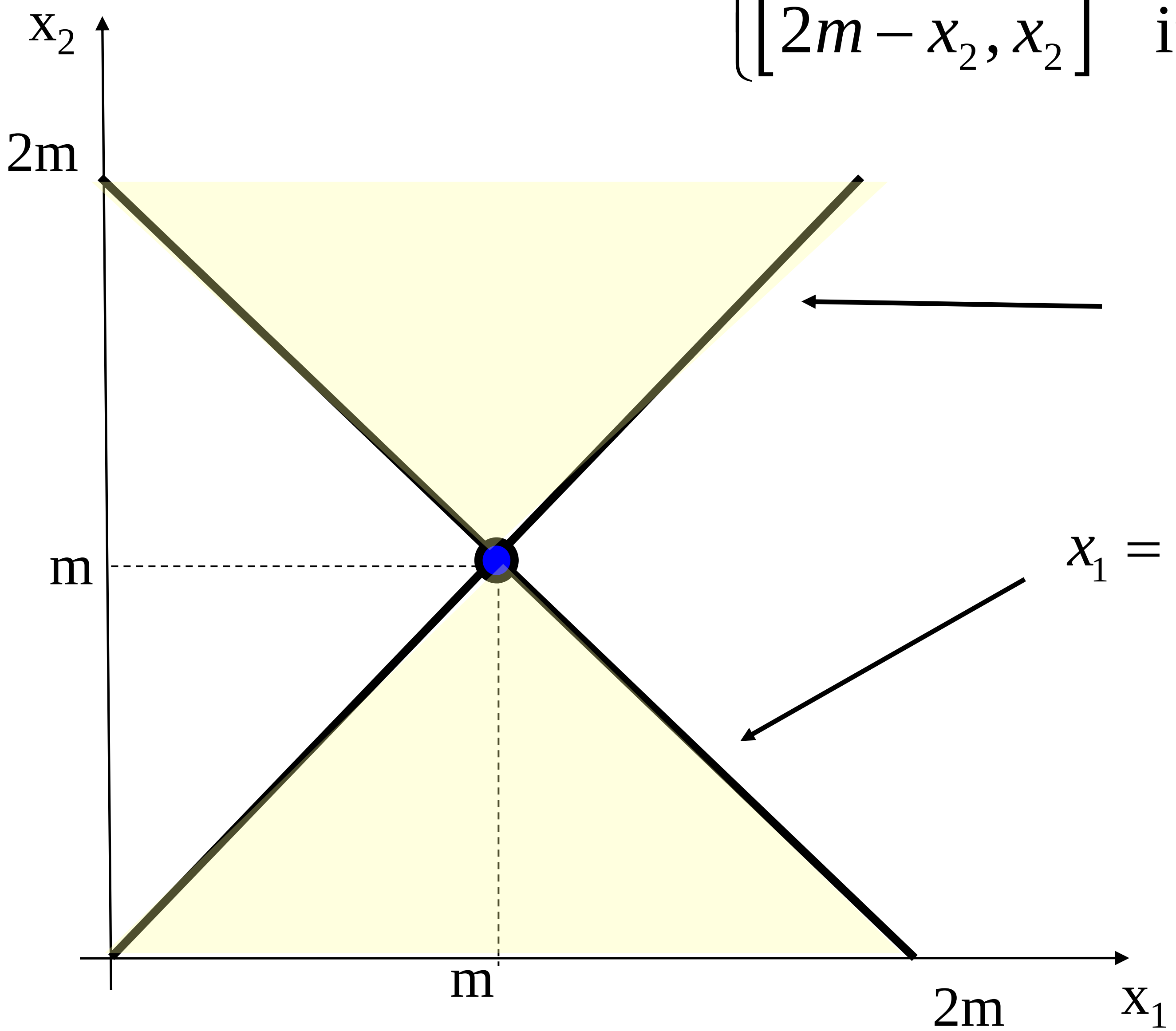
Best Response Functions - 5

- If $x_2 = m$, then player 1 loses the election unless she plays $x_1 = m$. So $BR_1(m) = \{m\}$.



Player 1's best response correspondence.

$$x_1 = BR_1(x_2) \in \begin{cases} [x_2, 2m - x_2] & \text{if } x_2 < m \\ \{m\} & \text{if } x_2 = m \\ [2m - x_2, x_2] & \text{if } x_2 > m \end{cases}$$



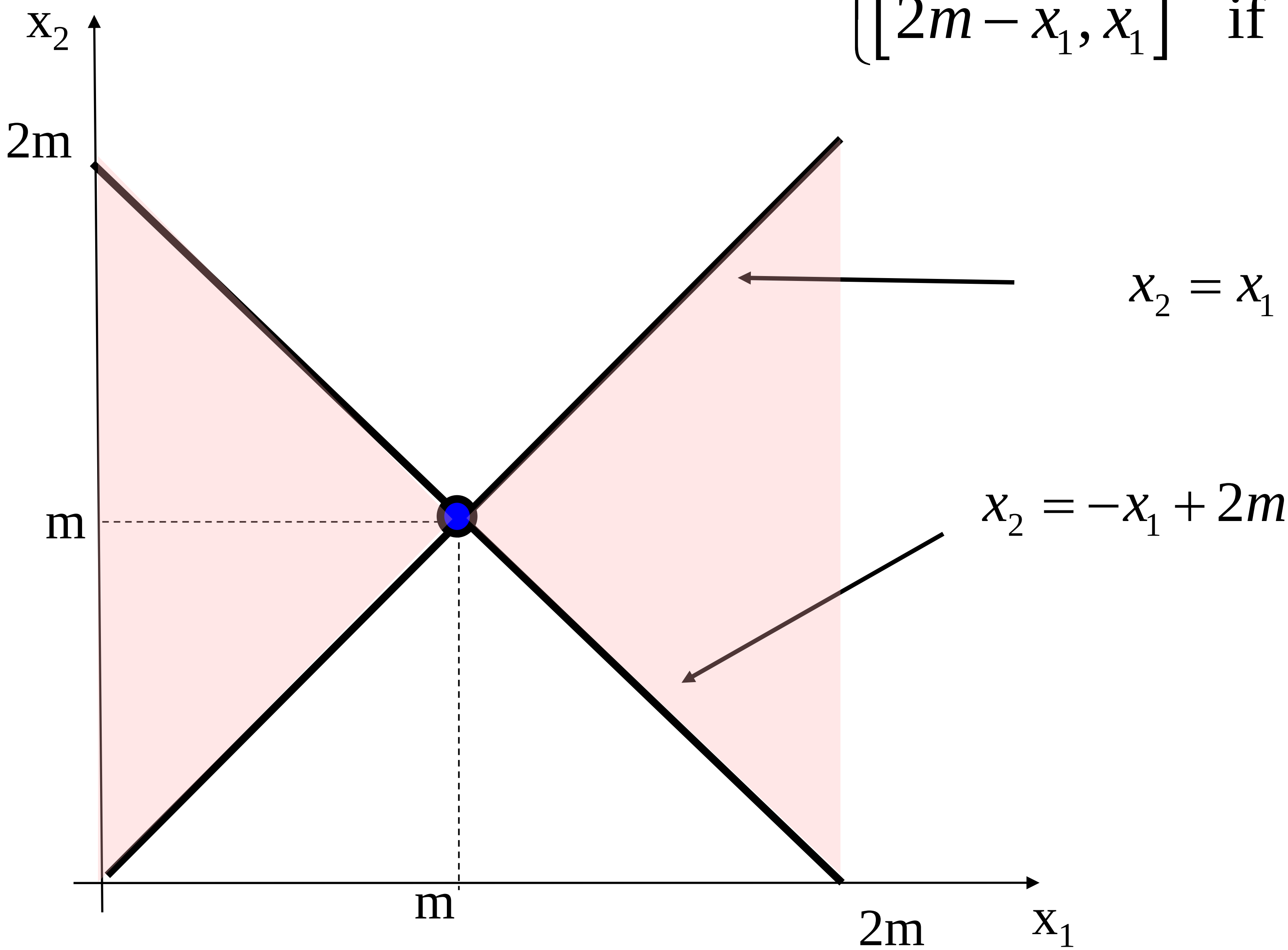
$x_1 = x_2$

$x_1 = -x_2 + 2m$

Player 2's best response correspondence

By symmetry

$$x_2 = BR_2(x_1) \in \begin{cases} [x_1, 2m - x_1] & \text{if } x_1 < m \\ \{m\} & \text{if } x_1 = m \\ [2m - x_1, x_1] & \text{if } x_1 > m \end{cases}$$

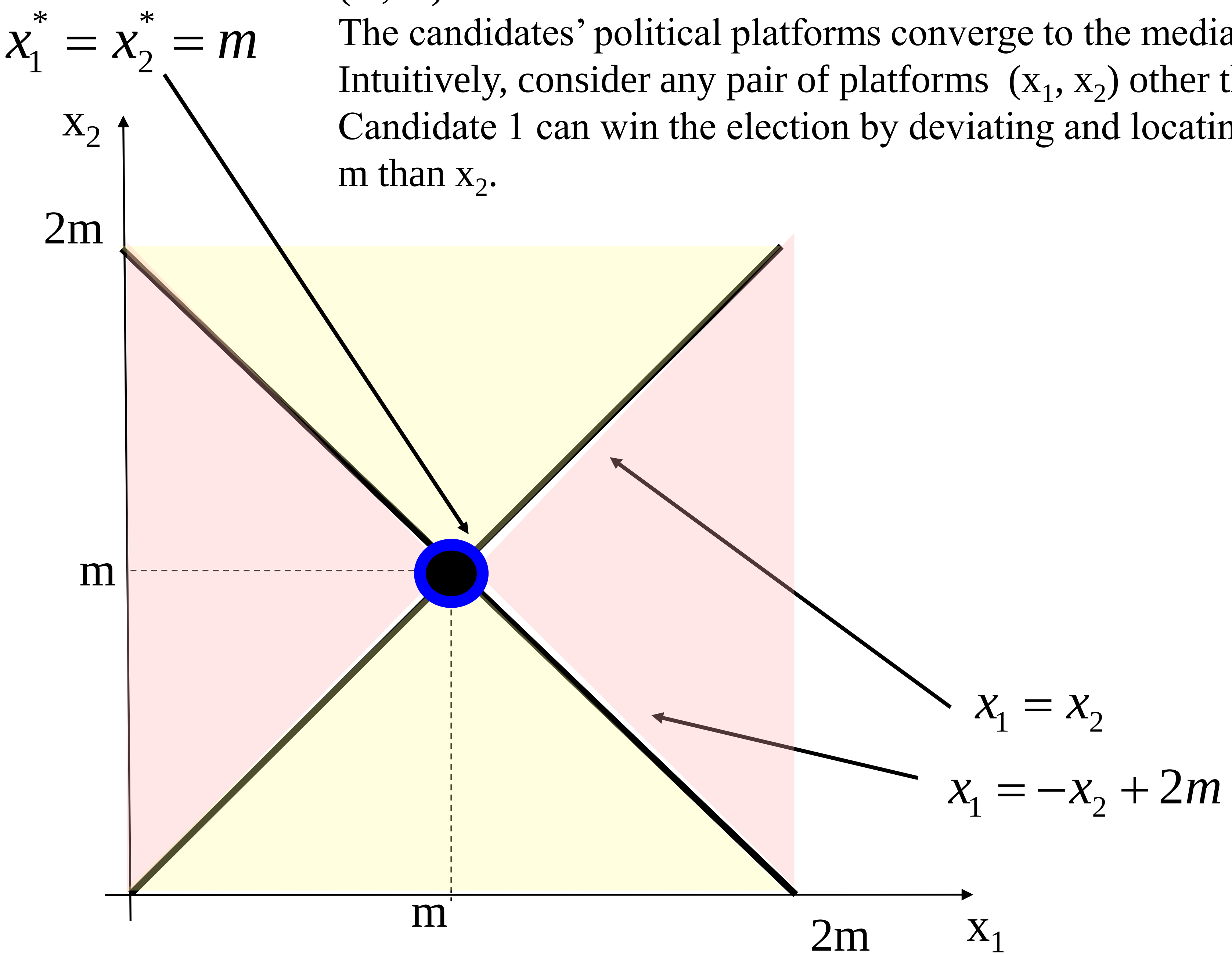


The unique Nash equilibrium.

Overlapping the best-response correspondences, the Nash Equilibrium is (m, m).

The candidates' political platforms converge to the median policy.

Intuitively, consider any pair of platforms (x_1, x_2) other than (m, m). Candidate 1 can win the election by deviating and locating x'_1 closer to m than x_2.



Providing a Public Good

Providing a public good

- **A public good is provided to a group of people if at least one person is willing to pay the cost of the good.**
- Assume that
  1. the people differ in their valuations of the good, and
  2. each person knows only her own valuation.
  3. the number of individuals be n,
  4. the cost of the good be $c > 0$,
  5. individual i's payoff if the good is provided be v_i. If the good is not provided then each individual's payoff is 0.
- Each individual i knows her own valuation v_i. She does not know anyone else's valuation, but knows that all valuations are at least 0 and at most 1, and that $0 < c < 1$.

- The probability that any one individual's valuation is at most v is $F(v)$, independent of all other individuals' valuations, where the cumulative distribution function F is continuous and increasing.
- All n individuals simultaneously submit contributions of either c or 0 (no intermediate contributions are allowed).
- If all individuals submit 0 then the good is not provided and each individual's payoff is 0.
- If at least one individual submits c then the good is provided, each individual i who submits c obtains the payoff $v_i - c$, and each individual i who submits 0 obtains the payoff v_i.

Bayesian Game Representation

- ***Players***: The set of n individuals.
- ***States***: The set of all profiles $(v_1, \ldots, v_n)$ of valuations, where $0 < v_i < 1$ for all i.
- ***Actions***: Each player's set of actions is $\{0, c\}$.
- ***Types***: The set of types T_i of each player i is given by $T_i(v_1, \ldots, v_n) = v_i$.
- ***Beliefs***: Each type of player i assigns probability

$$F(v_1)F(v_2) \cdots F(v_i - 1)F(v_i + 1) \cdots F(v_n)$$

to the event that the valuation of every other player j is at most v_j.

- ***Payoff functions***: Player *i*'s payoff in state $(v_1, \ldots, v_n)$ is
  - $u_i(v_1, \ldots, v_n) = 0$ if no one contributes,
  - $u_i(v_1, \ldots, v_n) = v_i$ if *i* does not contribute but some other player does,
  - $u_i(v_1, \ldots, v_n) = v_i - c$ if *i* contributes.
- ***Strategies: $f_i : v_i \rightarrow \{C, NC\}$***
- **Result 1:**
  - This game has a pure strategy Bayes-Nash equilibrium such that
    - each type v_j of player *i* with $v_j \geq c$ contributes,
    - whereas every other type of player *i*, and all types of every other player, do not contribute.
  - **Proof:** no player has an incentive to deviate.

Indeed, if player *i* does not contributes goes from a positive payoff to 0, while if the other players decide to contribute would incur in a cost without increasing the gross payoff *v*.

- **Result 2: The game has a symmetric Bayes-Nash equilibrium in which every player i contributes if and only if $v_i \geq v*$.**
  - **Proof:**
  - Consider player *i*. Suppose that every other player j contributes if and only if $v_j \geq v*$.
  - The probability that at least one of the other players contributes is $1-(F(v*))^{n-1}$.
  - Player *i*'s tipe v_j expected payoff is $[1-(F(v*))^{n-1}]v_j$ if she does not contribute and $v_j - c$ if she does contribute.
  - The conditions for player *i* tipe v_j to not contribute when $v_j < v*$ and contribute when $v_j \geq v*$ are:
    1. $(1-(F(v*))^{n-1})v_j \geq v_j-c$ if $v_j < v*$,
    2. $(1-(F(v*))^{n-1})v_j \leq v_j-c$ if $v_j \geq v*$:

    i.e.

    1. $v_j(F(v*))^{n-1} \leq c$ if $v_j < v*$,
    2. $v_j(F(v*))^{n-1} \geq c$ if $v_j \geq v*$.
  - Hence, in equilibrium, $\boldsymbol{v*(F(v*))^{n-1} = c}$.
  - The equilibrium $v*$ is the solution to this equation, and it is easy to show that it exists under general conditions

- **Properties of the symmetric equilibrium:**
- As the number of individuals increases, is the good more or less likely to be provided in this equilibrium?
- The probability that the good is provided is the probability that at least for one i, $v_j \geq v*$, which is equal to $1 - (F(v*))^n$.
- In equilibrium, this probability is equal to $1 - cF(v*)/v*$.
- **Properties of the equilibrium $v*$:**
- As n increases, for any $v*$ the value of $(F(v*))^{n-1}$ decreases, and thus $v*(F(v*))^{n-1}$ decreases.
- The value of $v*$ then should increases as n increases.
- As n increases the change in the probability that the good is provided increases if $F(v*)/v*$ decreases in $v*$, whereas it decreases if $F(v*)/v*$ increases in $v*$.
- If F is uniform, $F(v*)/v*$ decreases in $v*$, so that the probability increases as n increases.

Juries

Juries

- In a trial, jurors are presented with evidence on the guilt or innocence of a defendant. They may interpret the evidence differently.
- Each juror votes either to convict or acquit the defendant.
- A unanimous verdict is required for conviction: the defendant is convicted if and only if every juror votes to convict her.
- In deciding how to vote, each juror considers the costs of convicting an innocent person and of acquitting a guilty person. She must consider also the likely effect of her vote on the outcome, which depends on the other jurors' votes.

- We model the strategic interaction between the jurors as a Bayesian game.
- Each juror comes to the trial with the belief that the defendant is guilty with probability π .
- Given the defendant's true statuses (guilty and innocent), each juror receives a signal on the defendant guilty.
- Denote the probability of any given juror's to receive
  - the **signal "guilty" when the defendant is guilty** by $\boldsymbol{p}$, and the probability that
  - the **signal is "innocent" when the defendant is innocent** by $\boldsymbol{q}$.
- Jurors are more likely than not to interpret the evidence correctly:
  - $\boldsymbol{p > 1/2}$ **and** $\boldsymbol{q > 1/2}$, and hence $p > 1 - q$.

- Each juror wishes to convict a guilty defendant and acquit an innocent one.
- Each **juror's payoffs** are:
  - 0 if guilty defendant convicted or innocent defendant acquitted
  - $-z$ if innocent defendant convicted
  - $-(1-z)$ if guilty defendant acquitted.
- Let ***r*** **be the probability of the defendant's guilt**, given a juror's information.
- Her **expected payoff** if the **defendant is acquitted** is

  $$-r(1-z) + (1-r)\cdot 0 = -r(1-z)$$

  and her **expected payoff** if the **defendant is convicted** is

  $$r \cdot 0 - (1-r)z = -(1-r)z.$$

- She prefers the defendant to be acquitted if

$$-r(1-z) > -(1-r)z \Leftrightarrow r(1-z) < (1-r)z \Leftrightarrow$$

$$\Leftrightarrow r - rz < z - rz \Leftrightarrow r < z.$$

Conclusion:

- **defendant acquitted** if $r \leq z$
- **defendant is convicted** if $r \geq z$

We may now formulate the trial as a **Bayesian game**.

- ***Players*** A set of *n* jurors.
- ***States*** The set of states is the set of all lists
  - $(X, s_1, . . . , s_n)$ where
  - $X \in \{G, I\}$ $X = G$ if the defendant is guilty, $X = I$ if she is innocent,
  - $s_j \in \{g, ng\}$ for every juror *j*, $s_j = g$ if player *j* receives the signal "guilty," and $s_j = ng$ if player *j* receives the signal "innocent".
- ***Actions*** The set of actions of each player is $\{C, Q\}$,
  - *C* is voting to convict, and
  - *Q* is voting to acquit.
- ***Types*** The set of types for each player j is each player's signal $s_j \in \{g, ng\}$: $\tau_j(X, s_1, . . . , s_n) = s_j$ (each juror is informed only of her own signal).

- ***Beliefs***
  - **Type *g*** of a player *i* believes that the state is
    - (G, $s_1, \ldots, s_n$) with probability $\pi p^{k-1}(1-p)^{n-k}$ and
    - (I, $s_1, \ldots, s_n$) with probability $(1-\pi)q^{k-1}(1-q)^{n-k}$,
      - **where *k* is the number of players *j* (including *i*) for whom $s_j = g$.**
  - **Type *ng*** believes that the state is
    - (G, $s_1, \ldots, s_n$) with probability $\pi p^{k}(1-p)^{n-k-1}$ and
    - (I, $s_1, \ldots, s_n$) with probability $(1-\pi)q^{k}(1-q)^{n-k-1}$,
      - **where *k* is the number of players *j* (including *i*) for whom $s_j = g$.**

- ***Payoff functions***

- The payoff function of each player i is:
  - $u_i(a, \omega) = 0$ if $a = (A)$, $\omega_1 = I$ or if $a = (C)$, $\omega_1 = G$,
  - $u_i(a, \omega) = -z$ if $a = (C)$ and $\omega_1 = I$
  - $u_i(a, \omega) = -(1 - z)$ if $a = (A)$ and $\omega_1 = G$,

  where ω_1 is the first component of the state, giving the defendant's true status.

Nash Equilibrium

- *One juror* Suppose there is a single juror with signal (type) *ng*.

  To determine whether she prefers conviction or acquittal find the probability $\Pr(G|ng)$, that the defendant's is guilty. By the Bayes' Rule:

$$\Pr(G|ng) =$$

$$\Pr(ng|G)\Pr(G)/[\Pr(ng|G)\Pr(G)+\Pr(ng|I)\Pr(I)]$$

$$= (1-p)\pi\,/[(1-p)\pi + q(1-\pi)].$$

- The juror votes Acquittal if and only if

$$z \geq r = (1-p)\pi\,/\,[(1-p)\pi + q(1-\pi)].$$

- Suppose there are ***n* jurors**.
- Suppose that in equilibrium every juror other than juror 1 votes acquit if her signal is *ng* and convict if it is *g*.
- Consider type *ng* of juror 1. Her vote has no effect on the outcome unless every other juror's signal is *g*.
- Hence, she votes Acquittal if the probability that the defendant is guilty, given juror 1's signal is *ng* and every other juror's signal is *g*, is greater than *z:*

$$\Pr(G|ng,g,\ldots, g) \geq z.$$

- where $\Pr(G|ng,g,\ldots,g) =$

$$=\Pr(ng,g,\ldots,g|G)\Pr(G)/[\Pr(ng,g,\ldots,g|G)\Pr(G)+\Pr(ng,g,\ldots,g|I)\Pr(I)]=$$

$$= (1-p)p^{n-1}\pi/[(1-p)\,p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)].$$

- ***Type ng*** of juror 1 optimally votes for acquittal if

$$z \geq (1-p)p^{n-1}\pi/[(1-p)p^{n-1}\pi+q(1-q)^{n-1}(1-\pi)]$$

$$= 1/[1+q/(1-p)\,[(1-q)/p]^{n-1}\,(1-\pi)/\pi].$$

- **Under some conditions there is a symmetric mixed strategy equilibrium in which each type *g* juror votes for conviction, and each type *ng* juror randomizes.**
- Denote by β such mixed strategy of each juror.
- Each type *ng* juror is indifferent between voting conviction and acquittal. Hence the mixed strategy β is such that:

$$z = \Pr(G|\text{signal } b, n-1 \text{ votes for } C)$$

$$= \Pr(ng|G)(\Pr(\text{vote } C|\, G))^{n-1}\Pr(G) \,/$$
$$[\Pr(ng|G)(\Pr(\text{voteC}|G))^{n-1}\Pr(G)+\Pr(ng|I)(\Pr(\text{vote C}|I))^{n-1}\Pr(I)] =$$

$$= (1-p)(p+(1-p)\beta(C))^{n-1}\pi/$$
$$[(1-p)(p+(1-p)\beta(C))^{n-1}\pi + q(1-q+q\beta(C))^{n-1}(1-\pi)].$$

- The condition that this probability equals *z* implies

$$(1-p)(p+(1-p)\beta(C))^{n-1}\pi(1-z) = q(1-q+q\beta(C))^{n-1}(1-\pi)z$$

hence

$$\beta(C) = [pX - (1-q)] / [q - (1-p)X],$$

where

$$X = [\pi(1-p)(1-z)/((1-\pi)qz)]^{1/(n-1)}$$

- When n is large, X is close to 1, and hence $\beta(C)$ is close to 1: **a juror who interprets the evidence as pointing to innocence very likely nonetheless votes for conviction.**
- An interesting property of this equilibrium is that the probability that **an innocent defendant is convicted *increases* as n increases: the larger the jury, the *more* likely an innocent defendant is to be convicted**.