SOLUTION HOMEWORK 2
GAME THEORY Ph.D 2023

December 5, 2022

1 Exercise 1

Consider the extensive form games in figure 1

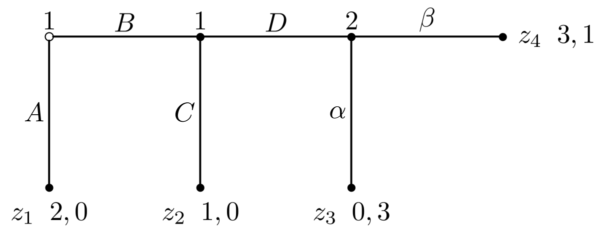

Figure 1

Calculate

1. the set of all Nash equilibria and the probabilities of outcomes in each of the equilibria;
2. the set of all subgame perfect equilibria, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria;
3. the set of weak Perfect Bayesian equilibria in pure strategies, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria;
4. the set of sequential equilibria in pure strategies, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria.

2 Solution Exercise 1

1. First we construct the strategic and the reduced strategic form are in figure 2 and 3, respectively:

	α	β
AD	2, 0	2, 0
AC	2, 0	2, 0
BC	1, 0	1, 0
BD	0, 3	3, 1

Figure 2

	α	β
$A\cdot$	2, 0	2, 0
BC	1, 0	1, 0
BD	0, 3	3, 1

Figure 3

To calculate the set of all Nash equilibria consider the reduced strategic form and eliminate the strictly dominated strategy BC, so to end up with the following 2×2 game.

	α	β
$A\cdot$	2, 0	2, 0
BD	0, 3	3, 1

1. (a) Then we calculate the players best reply correspondences.
For player 1:

$$u_1(A, \sigma_2) = 2\sigma_2(\alpha) + 2(1 - \sigma_2(\alpha)) = 2 \quad u_1(BD, \sigma_2) = 3(1 - \sigma_2(\alpha))$$

thus

- if $\sigma_2(\alpha) \in [0, 1/3)$ then 1's unique best reply is $\sigma_1(A) = 0$,
- if $\sigma_2(\alpha) \in (1/3, 1]$ then 1's unique best reply is è $\sigma_1(A) = 1$,
- if $\sigma_2(\alpha) = 1/3$ then 1's best reply is $\sigma_1(A) \in [0, 1]$.

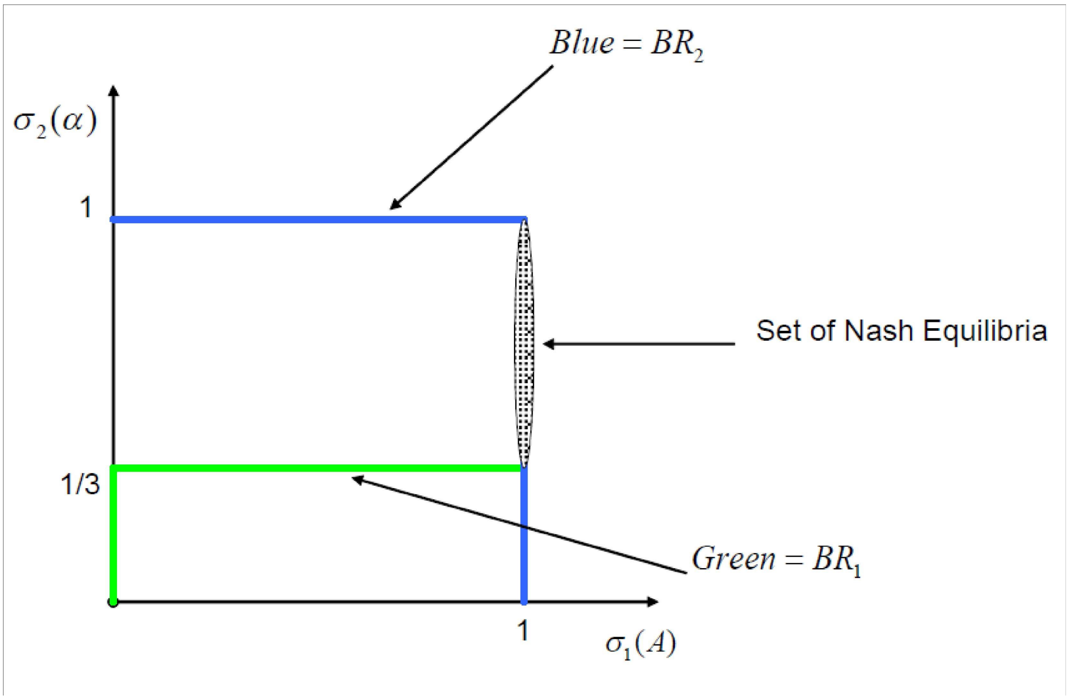

Figure 1: Figure 4

Similarly, it is possible to calculate player 2' best reply correspondence:

$$u_2(\sigma_1, \alpha) = 0\sigma_1(A) + 3(1 - \sigma_1(A)) \quad u_2(\sigma_1, \beta) = 0\sigma_1(A) + 1(1 - \sigma_1(A))$$

Then

- if $\sigma_1(A) \in [0, 1)$ then 2's unique best reply is $\sigma_2(\alpha) = 1$,
- if $\sigma_1(A) = 1$ then 2's best reply is $\sigma_2(\alpha) \in [0, 1]$.

Therefore the two best reply correspondences are:

$$BR_1(\sigma_2) = \begin{cases} \sigma_1(A) = 0 & \text{if } \sigma_2(\alpha) \in [0, 1/3] \\ \sigma_1(A) \in [0, 1] & \text{if } \sigma_2(\alpha) = 1/3 \\ \sigma_1(A) = 1 & \text{if } \sigma_2(\alpha) \in [1/3, 1]. \end{cases}$$

$$BR_2(\sigma_1) = \begin{cases} \sigma_2(\alpha) = 1 & \text{if } \sigma_1(A) \in [0, 1] \\ \sigma_2(\alpha) \in [0, 1] & \text{if } \sigma_1(A) = 1. \end{cases}$$

Graphically:

From the picture or using a system of simultaneous equations it is possible to find the intersection between the best reply correspondences and thus the set of Nash equilibria:

$$\sigma_1^*(A) = \sigma_1^*(AC) + \sigma_1^*(AD) = 1, \;\; \sigma_2^*(\alpha) \in [1/3, 1].$$

From the extensive form, it is immediate to derive

$$\zeta(s) = \begin{cases} z_1 & \text{if } s = (AC, \alpha), (AC, \beta), (AD, \alpha), (AD, \beta) \\ z_2 & \text{if } s = (BC, \alpha), (BC, \beta) \\ z_3 & \text{if } s = (BD, \alpha) \\ z_4 & \text{if } s = (BD, \beta). \end{cases}$$

Therefore if $\sigma_1^*(A) = \sigma_1^*(AC) + \sigma_1^*(AD) = 1$ *and* $\sigma_2^*(\alpha) \in [1/3, 1]$, then

$$\mathbf{P}(z_1|\sigma^*) = 1, \quad \mathbf{P}(z_2|\sigma^*) = 0 \quad \mathbf{P}(z_3|\sigma^*) = 0, \quad \mathbf{P}(z_4|\sigma^*) = 0.$$

2. Since the games has perfect information, subgame perfection is equivalent to backward induction. Therefore the set of subgame perfect equilibria is a singleton:

$$SPE = \{(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 | \sigma_1(\{AC\}) = 1 \& \sigma_2(\{\alpha\}) = 1\}.$$

From the extensive form and from the outcome function, previously derived, it is immediate that

$$\mathbf{P}(z_1|\sigma^{SPE}) = 1, \quad \mathbf{P}(z_2|\sigma^{SPE}) = 0 \quad \mathbf{P}(z_3|\sigma^{SPE}) = 0, \quad \mathbf{P}(z_4|\sigma^{SPE}) = 0.$$

Moreover A is the equilibrium path, while C and α are out of the equilibrium path.

3. In perfect information game,

$$SPE = WPBE$$

hence we have the same result of point 2.

4. In perfect information game,

$$SPE = SE$$

hence we have the same result of point 2.

3 Exercise 2

Consider the extensive form game pictured in figure 2

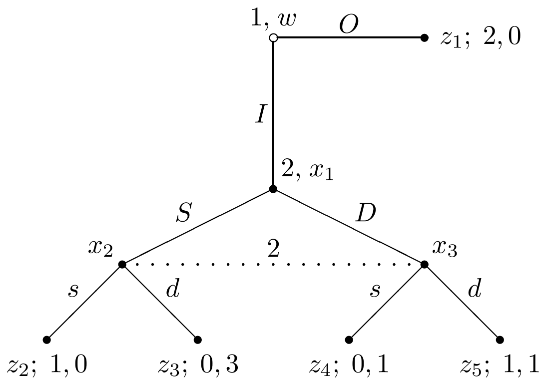

Figure 2

Calculate

1. the set of all subgame perfect equilibria, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria;
2. the set of Weak Perfect Bayesian equilibria in pure strategies, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria;
3. the set of Sequential equilibria in pure strategies, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria.

4 Solution to exercise 2

1. In the game of figure 2 we have a proper subgame, which is a one player game. The set of equilibria of this game will depend on whether we consider the strategic form or the agent normal form.

If we consider the strategic form of this subgame as in figure 4, then there is a unique Nash equilibrium in the subgame, i.e. (Sd)

	Ss	Sd	Ds	Dd
I	$1,0$	$0,3^*$	$0,1$	$1,1$

Figure 4

Clearly player 1 best response to this Nash equilibrium of the sub game is O. Therefore the set of subgame perfect equilibria is a singleton:

$$SPE = \{(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 | \sigma_1(\{O\}) = 1 \& \sigma_2(\{Sd\}) = 1\}$$

and it will induce a degenerate probability distribution on outcomes such that $P(z_1|\sigma^{SPE}) = 1 \quad \sigma^{SPE} \in SPE$, with S and d out of equilibrium actions.

2. Since

$$WPBE \subseteq NE$$

as a preliminary analysis we look for the set of Nash Equilibria. Consider the strategic form game associated to the extensive form

	Ss	Sd	Ds	Dd
O	$2,0$	$2,0$	$2,0$	$2,0$
I	$1,0$	$0,3$	$0,1$	$1,1$

Figure 7

From the simple inspection of the game of figure 7, it is clear that the strategy I is strictly dominated. After deleting this strategy, all remaining strategies for player 2 are equivalent, consequently the set of Nash equilibria is:

$$NE = \{(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 | \sigma_1(\{O\}) = 1\} = \{O\} \times \Sigma_2.$$

Thus in any $WPBE$, $\pi_1(O|\{w\}) = 1$.

Now, consider 2 sequential rationality in $\{x_2, x_3\}$:

$$s \text{ is sequentially rational if and only if}$$

$$u_2(s, \mu(x_2|\{x_2,x_3\})) = (1-\mu(x_2|\{x_2,x_3\})) \geq$$
$$\geq u_2(d, \mu(x_2|\{x_2,x_3\})) = 3\mu(x_2|\{x_2,x_3\}) + (1-\mu(x_2|\{x_2,x_3\})) \Leftrightarrow$$
$$\Leftrightarrow 0 \geq 3\mu(x_2|\{x_2,x_3\}) \Leftrightarrow \mu(x_2|\{x_2,x_3\}) \leq 0.$$

Thus

$$\pi_2^{SR}(d|\{x_2,x_3\}) \in \begin{cases} [0,1] & \mu(x_2|\{x_2,x_3\}) = 0 \\ \{1\} & \mu(x_2|\{x_2,x_3\}) \in (0,1]. \end{cases}$$

Working backward

i. if $\mu(x_2|\{x_2,x_3\}) = 0$, then from the calculations at point 1, $\pi_2^{SR}(d|\{x_2,x_3\}) \in [0,1]$,which in turn means that

$$\begin{cases} \pi_2^{SR}(D|\{x_1\}) = 1 \Leftrightarrow u_2(D|\{x_1\}) = 1 \geq u_2(S|\{x_1\}) = 3\pi_2(d|\{x_2,x_3\}) \\ \pi_2^{SR}(D|\{x_1\}) \in [0,1]\, 1 \Leftrightarrow u_2(D|\{x_1\}) = 1 = u_2(S|\{x_1\}) = 3\pi_2(d|\{x_2,x_3\}) \\ \pi_2^{SR}(D|\{x_1\}) = 0 \Leftrightarrow u_2(D|\{x_1\}) = 1 \leq u_2(S|\{x_1\}) = 3\pi_2(d|\{x_2,x_3\}) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \pi_2^{SR}(D|\{x_1\}) = 1 \Leftrightarrow \pi_2(d|\{x_2,x_3\}) \in \left[0, \frac{1}{3}\right] \\ \pi_2^{SR}(D|\{x_1\}) \in [0,1]\, 1 \Leftrightarrow \pi_2(d|\{x_2,x_3\}) = \frac{1}{3} \\ \pi_2^{SR}(D|\{x_1\}) = 0 \Leftrightarrow \pi_2(d|\{x_2,x_3\}) \in \left[\frac{1}{3}, 1\right] \end{cases}$$

ii. if $\mu(x_2|\{x_2,x_3\}) \in (0,1]$, then $\pi_2^{WPBE}(d|\{x_2,x_3\}) = 1$ and $\pi_2^{SR}(S|\{x_1\}) = 1$

Thus, we might conclude the set of $WPBE$ has the following structure:

i.

$$\pi_1^{WPBE}(O|\{w\}) = 1, \quad \mu(x_2|\{x_2,x_3\}) = 0, \ \pi_2^{WPE}(D|\{x_1\}) = 1, \ \pi_2^{WPBE}(d|\{x_2,x_3\}) \in \left[0, \frac{1}{3}\right]$$

$$\pi_1^{WPBE}(O|\{w\}) = 1, \quad \mu(x_2|\{x_2,x_3\}) = 0, \ \pi_2^{WPE}(D|\{x_1\}) \in [0,1], \ \pi_2^{WPBE}(d|\{x_2,x_3\}) = \frac{1}{3}$$

$$\pi_1^{WPBE}(O|\{w\}) = 1, \quad \mu(x_2|\{x_2,x_3\}) = 0, \ \pi_2^{WPE}(D|\{x_1\}) = 0, \ \pi_2^{WPBE}(d|\{x_2,x_3\}) \in \left[\frac{1}{3}, 1\right]$$

so that $\mathbf{P}(z_1|\pi^{WPBE}) = 1 \quad \forall \pi^{WPBE} \in WPBE$, with S, D, s and d out of equilibrium actions

ii.

$$\pi_1^{WPBE}(O|\{w\}) = 1, \quad \mu(x_2|\{x_2,x_3\}) \in (0,1], \ \pi_2^{WPBE}(S|\{x_1\}) = 1, \ \pi_2^{WPBE}(d|\{x_2,x_3\}) = 1$$

so that $\mathbf{P}(z_1|\pi^{WPBE}) = 1 \quad \forall \pi^{WPBE} \in WPBE$, with S and d out of equilibrium actions

3. Since Sequential equilibria are a refinement of subgame perfection, i.e.

$$SE \subseteq SPE$$

we first consider the set of Subgame perfect pure strategy equilibria.

From previous calculations, the set of subgame perfect equilibria in pure strategies is:

$$SPE = \{O, Sd\}.$$

Now, let consider the strategy profile and whether it is consistent with consistency and sequential rationality.

Suppose $\{O, Sd\}$ is played. Then, 2 beliefs are

$$\mu(x_2 | \{x_2, x_3\}) = \frac{\pi_1(I | \{w\}) \times \pi_2(S | \{x_1\})}{\pi_1(I | \{w\}) \times \pi_2(S | \{x_1\}) + \pi_1(I | \{w\}) \times \pi_2(D | \{x_1\})} =$$

$$= \frac{\pi_2(S | \{x_1\})}{\pi_2(S | \{x_1\}) + \pi_2(D | \{x_1\})} = \frac{1}{1+0} = 1$$

because by consistency

$$\pi_1(I | \{w\}) > 0$$

and by equilibrium conditions

$$\pi_2(S | \{x_1\}) = 1.$$

Then

$$u_2(s | \{x_2, x_3\}) = 0 < u_2(d | \{x_2, x_3\}) = 3$$

hence d is the sequential best reply of player 2 and we get the following SE:

$$\pi_1^{SE}(O | \{w\}) = 1,\ \pi_2^{SE}(S | \{x_1\}) = 1,\ \pi_2^{SE}(d | \{x_2, x_3\}) = 1,\ \mu(x_2 | \{x_2, x_3\}) = 1$$

so that $\mathbf{P}(z_1 | \pi^{SE}) = 1 \quad \forall \pi^{SE} \in SE$, with D and s out of equilibrium actions

5 Exercise 3

Consider the game of figure 3

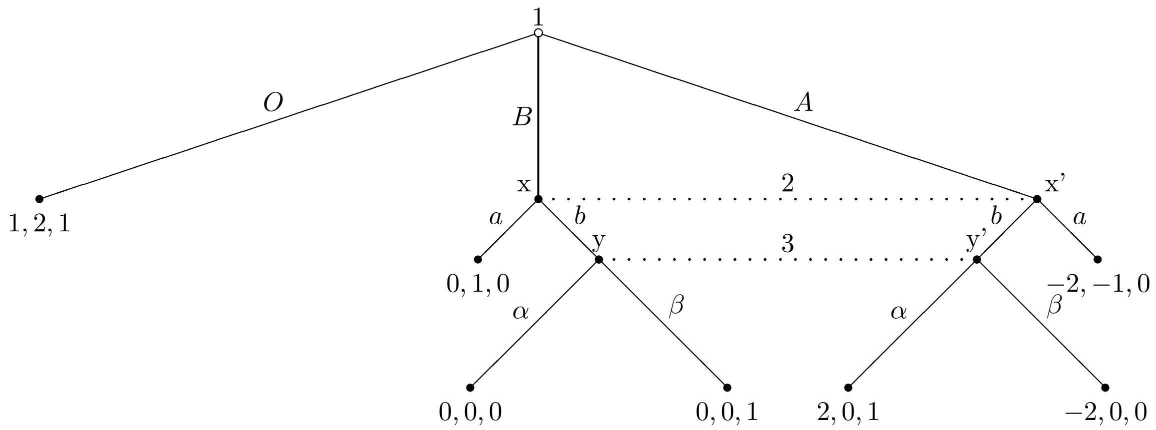

Figure 3

1. Construct the strategic form and calculate the set of Nash equilibria in pure strategies;
2. Calculate the set of Subgame Perfect Equilibria in pure strategies;
3. Calculate the set of Weak Perfect Bayesian Equilibria in pure strategies;
4. Find the set of sequential equilibria in pure strategies, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria;
5. Discuss the beliefs associated to each Sequential Equilibrium you find.

6 Solution to exercise 3

1. In figure 5 there is the strategic form of the game of figure 2

	α		β	
	a	b	a	b
O	**1**, **2**, **1**	1, **2**, **1**	**1**, **2**, **1**	**1**, **2**, **1**
A	−2, −1, **0**	**2**, **0**, **1**	−2, −1, **0**	−2, **0**, 0
B	0, **1**, **0**	0, 0, 0	0, **1**, **0**	0, 0, **1**

Figure 5

1. From figure 5, it is immediate that the Nash equilibria in pure strategies are

$$(O, a, \alpha) \qquad (A, b, \alpha) \qquad (O, a, \beta) \qquad (O, b, \beta).$$

2. The game has no proper subgame, hence the set of Subgame Perfect equilibria coincides with the set of Nash equilibria;

3. First note that

   - α is sequentially rational if and only if $\mu(y|\{y, y'\}) \leq 1/2$, and
   - a is sequentially rational if and only if $\mu(x|\{x, x'\}) \geq 1/2$.

   Now since the Weak Perfect Bayesian equilibria are refinements of Nash equilibria, consider the above Nash equilibria and check whether they satisfy sequential rationality and Bayes rule whenever possible:

   - Consider (A, b, α). Then by Bayes rule $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\}) = 0$ and thus for players 2 and 3, b and α are sequentially rational, and A is a best reply to these strategies. It is a WPBE.
   - Consider (O, a, α). Then Bayes rule does not restrict $\mu(x|\{x, x'\})$ and $\mu(y|\{y, y'\})$. Thus if $\mu(x|\{x, x'\}) \geq 1/2$ and $\mu(y|\{y, y'\}) \leq 1/2$, a and α are sequentially rational, and O is a best reply to these strategies. It is a WPBE.
   - Consider (O, a, β). Then Bayes rule does not restrict $\mu(x|\{x, x'\})$ and $\mu(y|\{y, y'\})$. Thus if $\mu(x|\{x, x'\}) \geq 1/2$ and $\mu(y|\{y, y'\}) \geq 1/2$, a and β are sequentially rational, and O is a best reply to these strategies. It is a WPBE.
   - Consider (O, b, β). Then Bayes rule does not restrict $\mu(x|\{x, x'\})$ and $\mu(y|\{y, y'\})$. Thus if $\mu(x|\{x, x'\}) \leq 1/2$ and $\mu(y|\{y, y'\}) \geq 1/2$, b and β are sequentially rational, and O is a best reply to these strategies. It is a WPBE.

Thus the set of Weak Perfect Bayesian equilibria in pure strategies is:

(a) (A, b, α) with beliefs $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\}) = 0$
(b) (O, a, α) with beliefs $\mu(x|\{x, x'\}) \geq 1/2$ and $\mu(y|\{y, y'\}) \leq 1/2$
(c) (O, a, β) with beliefs $\mu(x|\{x, x'\}) \geq 1/2$ and $\mu(y|\{y, y'\}) \geq 1/2$
(d) (O, b, β) with beliefs $\mu(x|\{x, x'\}) \leq 1/2$ and $\mu(y|\{y, y'\}) \geq 1/2$.

2. The set of Sequential equilibria is a refinement of the set of WPBE. In the previous point of the homework, we have derived the set of Weak Perfect Bayesian equilibria in pure strategies:

(a) (A, b, α) with beliefs $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\}) = 0$
(b) (O, a, α) with beliefs $\mu(x|\{x, x'\}) \geq 1/2$ and $\mu(y|\{y, y'\}) \leq 1/2$
(c) (O, a, β) with beliefs $\mu(x|\{x, x'\}) \geq 1/2$ and $\mu(y|\{y, y'\}) \geq 1/2$
(d) (O, b, β) with beliefs $\mu(x|\{x, x'\}) \leq 1/2$ and $\mu(y|\{y, y'\}) \geq 1/2$.

Note that in the first WPBE there are no out-of-equilibrium information set, thus is a Sequential equilibrium. When there are out-of-equilibrium nodes, the beliefs are derived from Bayes rule assuming strictly mixed behavioral strategies. Then applying Bayes rule, the beliefs are

$$\mu(y'|\{y, y'\}) = \frac{\pi_1(A) \times \pi_2(b)}{\pi_1(A) \times \pi_2(b) + \pi_1(B) \times \pi_2(b)} = \frac{\pi_1(A)}{\pi_1(A) + \pi_1(B)} = \mu(x'|\{x, x'\})$$

where the fractions are well defined because by assumption $\pi_i(\cdot) > 0$. Hence in any sequential equilibrium $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\})$. Thus the set of Sequential equilibria in pure strategies is:

i. (A, b, α) with beliefs $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\}) = 0$

ii. (O, a, α) with beliefs $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\}) = 1/2$

iii. (O, a, β) with beliefs $\mu(x|\{x, x'\}) = \mu(y) \geq 1/2$

iv. (O, b, β) with beliefs $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\}) = 1/2$.

5. Since action B is strictly dominated by O for player 1, if players 2 and 3 are called to play, then both should infer that player 1 has chosen A. Thus the unique Sequential Equilibrium consistent with this reasoning and the consequent beliefs is (A, b, α) with beliefs $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\}) = 0$.

7 Exercise 4

Consider the game of figure 4

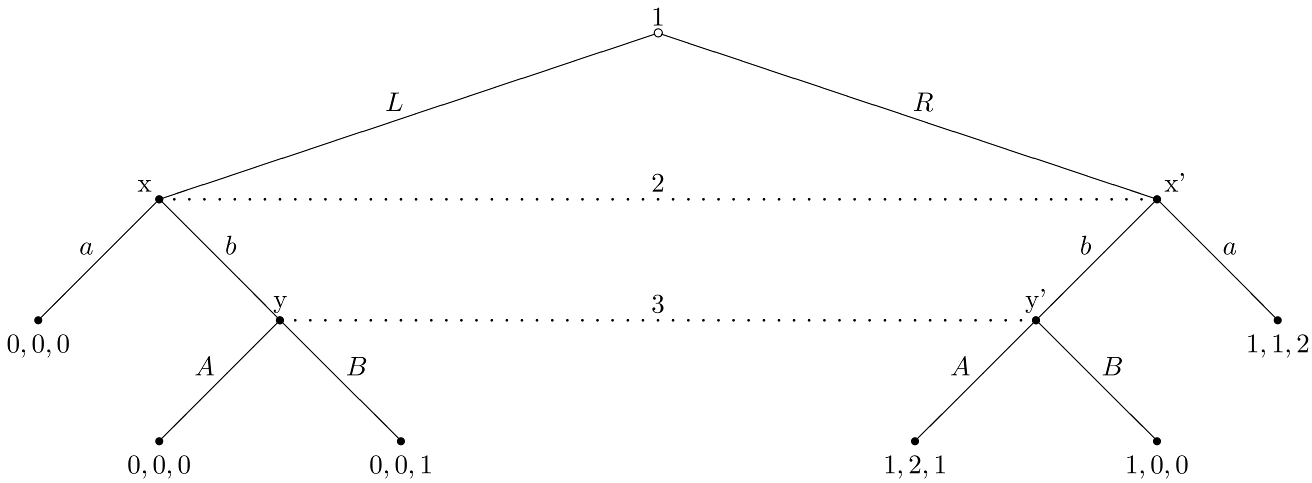

Figure 4

1. Construct the strategic form and calculate the set of Nash equilibria, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria;

2. Calculate the set of Subgame Perfect Equilibria, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria;

3. Calculate the set of Weak Perfect Bayesian Equilibria in pure strategies, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria;

4. Find the set of Sequential Equilibria in pure strategies, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria;

5. Discuss the beliefs associated to each WPBE.

8 Solution to exercise 4

1. In figure 6 there is the strategic form of the game of figure 4

A

	a	b
L	$0,0,0$	$0,0,0$
R	$1,1,2$	$1,2,1$

B

	a	b
L	$0,0,0$	$0,0,1$
R	$1,1,2$	$1,0,0$

Figure 6

From figure 4 or from figure 6, it is immediate that the pure strategy L of player 1 is strictly dominated so that player 1 will play R with probability 1 in any Nash equilibrium of this game. Then we can consider the game between player 2 and 3 pictured in figure 6

	A	B
a	$1,2$	$1,2$
b	$2,1$	$0,0$

Figure 7

To find the set of Nash equilibria of the game of figure 6, we need to calculate the players best reply correspondences.

1. For player 2:

   $u_2(a,\sigma_3) = 1\times\sigma_3(A)+1\times(1-\sigma_3(A)) \quad u_2(b,\sigma_3) = 2\times\sigma_3(A)+0\times(1-\sigma_3(A))$

   thus

   - if $\sigma_3(A) \in [0,1/2]$ then 2's best reply is $\sigma_2(a) = 1$,
   - if $\sigma_3(A) \in [1/2,1]$ then 2's best reply is $\sigma_2(a) = 0$,
   - if $\sigma_3(A) = 1/2$ then 2's best reply is $\sigma_2(a) \in [0,1]$.

   Similarly, it is possible to calculate player 3' best reply correspondence:

   $u_3(\sigma_2,A) = 2\times\sigma_2(a)+1\times(1-\sigma_2(a)) \quad u_3(\sigma_2,B) = 2\times\sigma_2(a)+0\times(1-\sigma_2(a))$.

   Then

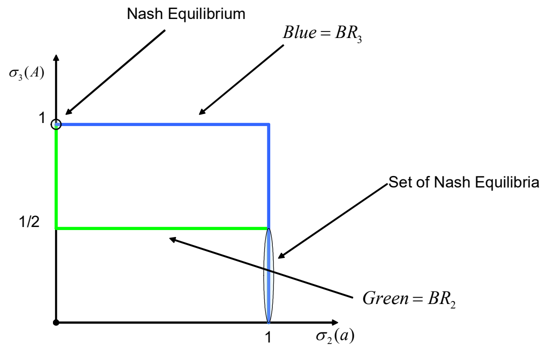

Figure 2:

- if $\sigma_2(a) \in [0,1]$ then 3's best reply is $\sigma_3(A) = 1$,
- if $\sigma_2(a) = 1$ then 3's best reply is $\sigma_3(A) \in [0,1]$.

Therefore the two best reply correspondences are:

$$BR_2(\sigma_3) = \begin{cases} \sigma_2(a) = 1 & \text{if } \sigma_3(A) \in [0, 1/2] \\ \sigma_2(a) \in [0,1] & \text{if } \sigma_3(A) = 1/2 \\ \sigma_2(a) = 0 & \text{if } \sigma_3(A) \in [1/2, 1]. \end{cases}$$

$$BR_3(\sigma_2) = \begin{cases} \sigma_3(A) = 1 & \text{if } \sigma_2(a) \in [0,1] \\ \sigma_3(A) \in [0,1] & \text{if } \sigma_2(a) = 1. \end{cases}$$

Graphically:

From the picture or using a system of simultaneous equations it is possible to find the intersection between the best reply correspondences and thus, remembering that player 1 is always choosing R with probability 1, the set of Nash equilibria of the game:

$$NE = \{R, b, A)\} \cup \{(R, a, \sigma_3(A) \in [0, 1/2]\}.$$

2. Since there are no proper subgame, the set of Subgame Perfect equilibria and the set of Nash equilibria coincide

$$SPE = \{R, b, A)\} \cup \{(R, a, \sigma_3(A) \in [0, 1/2]\}.$$

3. The set of Weak Perfect Bayesian equilibria is a subset of Nash equilibria, thus we can start from the Nash equilibria in pure strategies, which are

$$\{R, b, A)\} \cup \{(R, a, B\}.$$

Now we have to compute players' beliefs that sustain such equilibria, if they exists.

(a) Consider (R, b, A). Then by Bayes rule

$$\mu(x) = \frac{\pi_1(L)}{\pi_1(L) + \pi_1(R)} = \frac{0}{0+1} = 0$$

Therefore $\mu(x') = 1$
Moreover

$$\mu(y) = \frac{\pi_1(L) \times \pi_2(b)}{\pi_1(L) \times \pi_2(b) + \pi_1(R) \times \pi_2(b)} = \frac{\pi_1(L)}{\pi_1(L) + \pi_1(R)} = \mu(x) = 0.$$

Therefore $\mu(y') = \mu(x') = 1$ and thus for player 3 is sequentially rational to play A. Moreover also b for player 2 is sequentially rational given previous beliefs and given A by player 3.
Therefore

$$\mu(x) = 0, \mu(y) = 0 \quad and \quad R, b, A$$

it is a WPBE in pure strategies.

(b) Consider (R, a, B). Then by Bayes rule

$$\mu(x) = \frac{\pi_1(L)}{\pi_1(L) + \pi_1(R)} = \frac{0}{0+1} = 0$$

Therefore $\mu(x') = 1$
Moreover

$$\mu(y) = \frac{\pi_1(L) \times \pi_2(b)}{\pi_1(L) \times \pi_2(b) + \pi_1(R) \times \pi_2(b)} = \frac{0 \times 0}{0 \times 0 + 0 \times 0} = \frac{0}{0} \in [0, 1].$$

Then Bayes rule does not restrict $\mu(y)$. But B is sequentially rational only if

$$Eu_3(B|\mu(y)) \geq Eu_3(A|\mu(y))$$

i.e.

$$1 \times \mu(y) + 0 \times (1 - \mu(y)) \geq 0 \times \mu(y) + 1 \times (1 - \mu(y))$$

which means $\mu(y) \geq 1/2$.
Moreover player 2 sequential rational strategy given R and B is a, which in turn implies that R is sequentially rational for player 1.
Therefore

$$\mu(x) = 0, \mu(y) \geq 1/2 \quad and \quad R, a, B$$

it is a (continuum of) WPBE in pure strategy.

4. The set of Sequential equilibria is a refinement of the set of WPBE. In the previous point of the homework, we have derived the set of Weak Perfect Bayesian equilibria in pure strategies:

(a) (R, b, A) with beliefs $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\}) = 0$

(b) (R, a, B) with beliefs $\mu(x|\{x, x'\}) = 0$ and $\mu(y|\{y, y'\}) \geq 1/2$.

Note that in the first WPBE there are no out-of-equilibrium information set, thus beliefs satisfy consistency and it is a Sequential equilibrium.

Consider the second (set of) WPBE. When there are out-of-equilibrium nodes, the beliefs are derived from Bayes rule assuming strictly mixed behavioral strategies. Then applying Bayes rule, the beliefs are

$$\mu(y|\{y, y'\}) = \frac{\pi_1(L) \times \pi_2(b)}{\pi_1(R) \times \pi_2(b) + \pi_1(L) \times \pi_2(b)} = \frac{\pi_1(L)}{\pi_1(R) + \pi_1(L)} = \mu(x|\{x, x'\})$$

where the fractions are well defined because by assumption $\pi_i(\cdot) > 0$.

Hence in any sequential equilibrium $\mu(x|\{x, x'\}) = \mu(y|\{y, y'\})$ and consequently the second (set of) WPBE is not a sequential equilibrium.

Thus the set of Sequential equilibria in pure strategies is a singleton:

$$(R, b, A), \quad \mu(x|\{x, x'\}) = \mu(y|\{y, y'\}) = 0.$$

5. In the first WPBE there are no out-of-equilibrium information set, thus there is no room for discussion of players' beliefs.

Let consider the second (set of) WPBE, which are not SE and where $\mu(x|\{x, x'\}) = 0$ and $\mu(y|\{y, y'\}) \geq 1/2$: in this case the beliefs $\mu(y|\{y, y'\}) \geq 1/2$ are not restricted by Bayes rules, so in this sense they are arbitrary.

From the game tree, node y is reached with probability $\pi_1(L) \times \pi_2(b)$, while y' is reached with probability $\pi_1(R) \times \pi_2(b)$. Hence:

$$\mu(y|\{y, y'\}) \geq 1/2 \Leftrightarrow \pi_1(L) \geq \pi_1(R)$$

however using dominance is not possible to conclude that L is a more costly "mistake" than R. Thus, it is not easy to exclude $\mu(y|\{y, y'\}) \geq 1/2$ using FI arguments, while it is immediate using independent vanishing mistakes because node y is reached with two independent infinitesimal mistakes, i.e. with probability $\pi_1(L) \times \pi_2(b) = \varepsilon \times \delta$, while y' is reached with probability $\pi_1(R) \times \pi_2(b) = (1 - \varepsilon) \times \delta$.

9 Exercise 5

Before doing economics at the PhD in Economics at Bicocca you were a famous pastry cook. Mario knows this and regularly asks you to make cakes for him in exchange for good grades.

- This time, Mario needs **two cakes**;
- He will give you **one extra point on your final grade for each cake that he judges delicious**.

As usual, Mario made his request too late and you will not have time to make both cakes. However, you know a friend you can ask and who actually already helped you before to make cakes for Mario. Of course, this friend's cakes are usually less good than yours. In particular, you and your friend know that in the past,

- Mario judged your cakes delicious 2/3 of the time while he judged your friend's cakes delicious only 1/3 of the time;
- your abilities are independent and you have the same production cost, normalized to 0;
- the two cakes are delivered in time but **Mario only gives you 1 extra point to share with your friend**;
- you set up a bargaining game to share this point.

The rules are the following:

1. you make an offer $s_Y \in [0; 1]$ which states that you want s_Y point for yourself and that your friend can keep $1 - s_Y$ point for himself;
2. your friend can then accept or reject this offer
   (a) if (s)he accepts it, (s)he gets $1 - s_Y$ point and you get s_Y point;
   (b) if (s)he rejects, then you have to bother Mario in his office and ask him directly. Mario will then give the point to the one that indeed cooked the delicious cake but will also remove $c = 1/5 + \varepsilon$ point to each of you because he does not like to be disturbed (with $\varepsilon > 0$ very small). Thus, if you end up going to Mario, the one that indeed made the cake will get $1 - c$ extra point on his grade and the other will actually loose c point;
3. finally, note that you and your friend have the same very simple utility function, defined over point won: $u(Points) = Points$.

QUESTIONS

1. Construct the extensive form game corresponding to this strategic situation;
2. show that the probability that you cooked the delicious cake is 4/5;
3. what are your expected utilities if your friend rejects your offer and you both end up going to Mario?
4. Is it possible to find Nash equilibria where you end up going to Mario's office?
5. Show that

$$\forall \alpha \in \left[\frac{3}{5}, 1\right] \quad \left(s_Y = \alpha, s_F(s_Y) = \begin{cases} yes & \text{if} \quad s_Y \leq \alpha \\ No & \text{if} \quad s_Y > \alpha \end{cases}\right)$$

   is a Nash equilibrium. Note that, s_Y denotes your strategy and $s_F(s_Y)$ denotes your friend's strategy. What are the corresponding payoffs?
6. What are the subgame perfect Nash equilibria of the game?
7. Consider what happens if instead of the game ending after the decision of your friend, it is extended to two periods: if (s)he rejects then the game moves to period $t = 2$, and (s)he has to make an offer $s_F \in [0; 1]$, which specifies that (s)he keeps s_F point for himself and let you $1 - s_F$ point. Then, you can accept or reject this offer. If you reject, you both go to Mario and your payoffs are defined as before and the game ends. Otherwise, as before you get $1 - s_F$ and your friend gets s_F. You have the same discount factor given by $\delta \in (0, 1)$. Find the subgame perfect Nash equilibria for this extension of the game.

10 Solution to exercise 5

1. Note that

    - Mario needs two cakes;
    - Mario judged your cakes delicious 2/3 of the time while he judged your friend's cakes delicious only 1/3 of the time

This means that ex ante there are four possible state of nature: (y, f) with $y, f \in \{D, N\}$ and D means delicious, N not delicious, y means your cake, f your friend cake. Thus the entire set of state of nature is

$$\{(D, D), (D, N), (N, D), (N, N)\}.$$

Moreover, Mario

- will give you one extra point on your final grade for each cake that he judges delicious, and
- the two cakes are delivered in time but Mario only gives you 1 extra point to share with your friend.

This means that only one of the two cakes is delicious, hence the only two possible state of nature are

$$\{(D, N), (N, D)\}.$$

So, the game tree is the following:

Thus the basic elements of the game are:

- **Players**: Y (you), F (friend)
- **Set of strategies**: for You $S_Y = [0, 1]$, for Friend $S_F : [0, 1] \to \{Yes, No\}$
- **Payoffs**:

$$u_Y(s_Y, s_F; (y, f)) = \begin{cases} \alpha & \text{if} \quad (s_Y, s_F; (y, f)) = (\alpha, Yes, (D, N)) \\ 1 - c & \text{if} \quad (s_Y, s_F; (y, f)) = (\alpha, No, (D, N)) \\ \alpha & \text{if} \quad (s_Y, s_F; (y, f)) = (\alpha, Yes, (N, D)) \\ -c & \text{if} \quad (s_Y, s_F; (y, f)) = (\alpha, No, (N, D)) \end{cases}$$

$$u_F(s_Y, s_F; (y, f)) = \begin{cases} 1 - \alpha & \text{if} \quad (s_Y, s_F; (y, f)) = (\alpha, Yes, (D, N)) \\ -c & \text{if} \quad (s_Y, s_F; (y, f)) = (\alpha, No, (D, N)) \\ 1 - \alpha & \text{if} \quad (s_Y, s_F; (y, f)) = (\alpha, Yes, (N, D)) \\ 1 - c & \text{if} \quad (s_Y, s_F; (y, f)) = (\alpha, No, (N, D)) . \end{cases}$$

2. Because of the independence assumption, we have the following joint probabilities on the 4 possible states of natures:

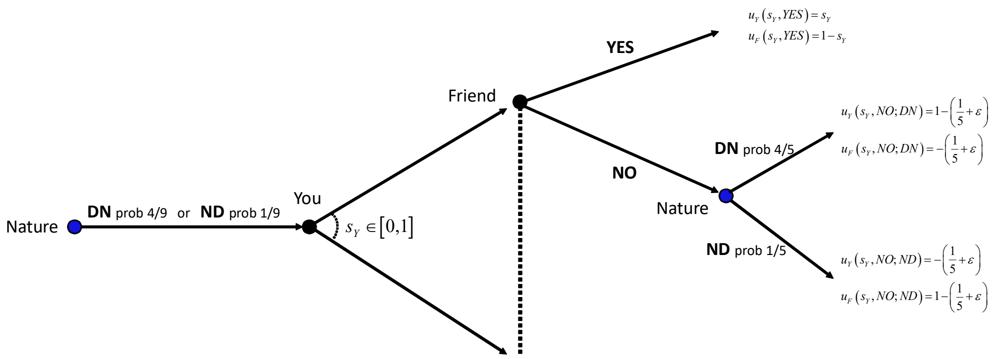

Figure 3:

	Cake of the Friend			
Your cake		D	N	Marginal prob
	D	2/9	4/9	2/3
	N	1/9	2/9	1/3
	Marginal prob	1/3	2/3	1

Figure 4:

The question requires to calculate the probability of (D, N) given the two mutually exclusive possible state of nature (D, N) or (N, D), i.e.

$$\Pr\{(D,N) \mid (D,N) \cup (N,D)\} = \frac{\Pr\{(D,N)\}}{\Pr\{(D,N) \cup (N,D)\}} = \frac{\frac{4}{9}}{\frac{4}{9}+\frac{1}{9}} = \frac{4}{5}.$$

3. From the expression of the players' payoff, we get

$$E\left[u_Y\left(s_Y, No; (y,f)\right)\right] = \frac{4}{5}(1-c) + \frac{1}{5}(-c) = \frac{4}{5} - \frac{1}{5} - \varepsilon = \frac{3}{5} - \varepsilon > 0$$

$$E\left[u_F\left(s_Y, No; (y,f)\right)\right] = \frac{1}{5}(1-c) + \frac{4}{5}(-c) = \frac{1}{5} - \frac{1}{5} - \varepsilon = -\varepsilon < 0.$$

4. No, since to end up in Oliver office, your friend should reject your offer, however

$$E\left[u_F\left(s_Y, No; (y,f)\right)\right] = -\varepsilon < E\left[u_F\left(s_Y, Yes; (y,f)\right)\right] = 1 - s_Y \quad \forall s_Y \in [0,1].$$

Hence, No is a strictly dominated action whatever s_Y, and there are no Nash equilibria with strictly dominated actions on the equilibrium path.

5. To show that this strategy profile is a Nash equilibrium, we need to calculate the players expected payoff in equilibrium, showing that no player has an incentive to deviate. Since

$$E\left[u_Y^*\left(s_Y^*, s_F^*; (y,f)\right)\right] = \alpha \geq E\left[u_Y\left(s_Y', s_F^*; (y,f)\right)\right] =$$

$$= \begin{cases} \alpha' < \alpha & \text{if} \quad s_Y' = \alpha' < \alpha \\ \frac{3}{5} - \varepsilon & \text{if} \quad s_Y' = \alpha' > \alpha \end{cases} \Leftrightarrow \alpha \geq \frac{3}{5} - \varepsilon$$

Y has no incentive to deviate. Then, let consider a generic deviation of F to

$$s_F' = \begin{cases} Yes & \text{if} \quad s_Y \in A \\ No & \text{if} \quad s_Y \notin A. \end{cases}$$

Then

$$E\left[u_F^*\left(s_Y^*, s_F^*; (y,f)\right)\right] = 1 - \alpha \geq E\left[u_Y\left(s_Y^*, s_F'; (y,f)\right)\right] = \begin{cases} 1-\alpha & \text{if} \quad s_F' = Yes \\ -\varepsilon & \text{if} \quad s_F' = No \end{cases}$$

so that F too has no incentive to deviate.

6. Since it is a game with perfect information, subgame perfection and backward induction, coincide. Thus, let we start from the end:

   (a) after No, Nature chooses randomly, so that the players expected payoff are

$$E\left[u_Y\left(s_Y, No; (D,N)\right)\right] = \frac{3}{5} - \varepsilon \ \text{ and } E\left[u_Y\left(s_Y, No; (N,D)\right)\right] = -\varepsilon$$

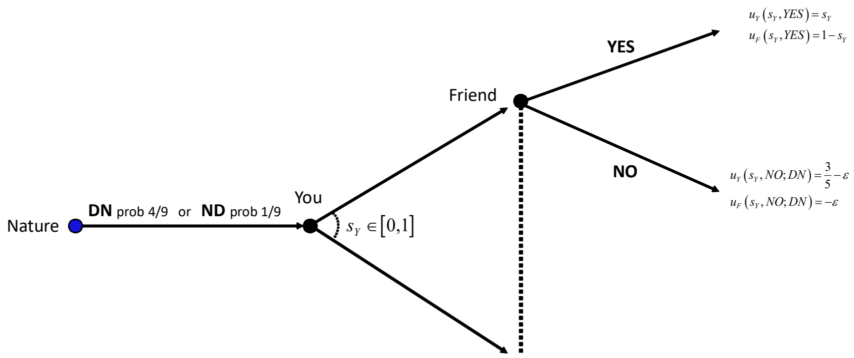

Figure 5:

(b) given F expected payoffs, after any s_Y , F sequential rational choice is Yes, since

$$E\left[u_F\left(s_Y, No; (y, f)\right)\right] = -\varepsilon < E\left[u_F\left(s_Y, Yes; (y, f)\right)\right] = 1 - s_Y \quad \forall s_Y \in [0, 1] .$$

Formally

$$s_F^{SR}(s_Y) = Yes \quad \forall s_Y \in [0, 1]$$

(c) given F sequential rational choice, then Y expected payoff at the beginning of the game is

$$E\left[u_F\left(s_Y, s_F^{SR}(s_Y); (y, f)\right)\right] = s_Y \in [0, 1]$$

which is maximized for

$$s_F^{SR}(s_Y) = s_Y = 1.$$

Hence, the unique subgame perfect equilibrium is

$$\left(s_Y = 1, s_F(s_Y) = yes \quad \forall s_Y \in [0, 1]\right).$$

This means that all the Nash equilibria with $\alpha < 1$ are based on the non credible threat that F would reject any offer $s_Y > \alpha$.

(d) Graphically:

7. Let we start from the end:

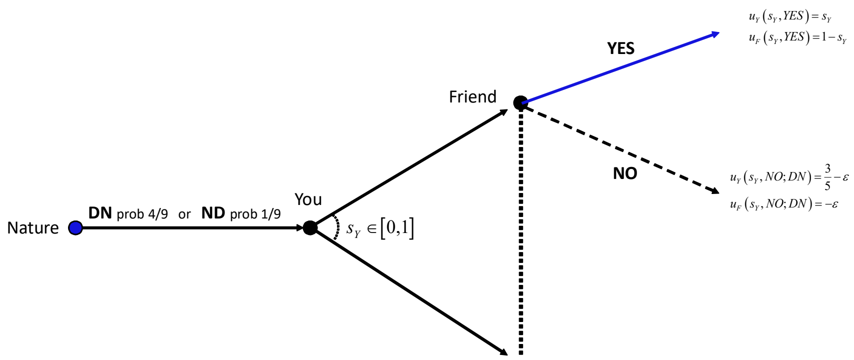

Figure 6:

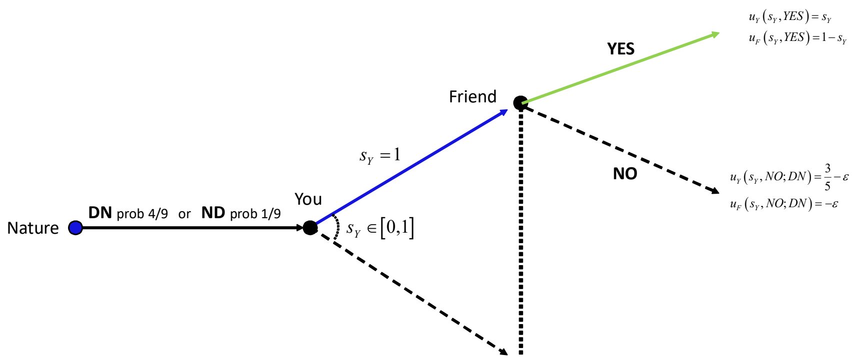

Figure 7:

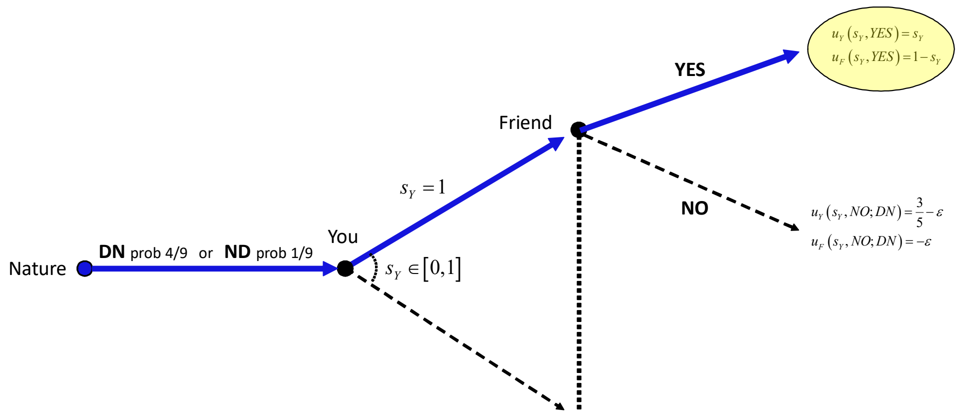

Figure 8:

(a) after No, Nature chooses randomly, so that the players expected payoff are

$$E\left[u_Y\left(s_Y, No; (y,f)\right)\right] = \delta\left(\frac{3}{5} - \varepsilon\right) > 0 \text{ and } E\left[u_F\left(s_Y, No; (y,f)\right)\right] = -\delta\varepsilon < 0;$$

(b) given Y expected payoff, after any s_F, Y sequential rational choice is Yes if and only if

$$E\left[u_Y\left(s_Y, No; (y,f)\right)\right] = \delta\left(\frac{3}{5} - \varepsilon\right) \leq E\left[u_Y\left(s_Y, Yes; (y,f)\right)\right] = \delta\left(1 - s_F\right) \Leftrightarrow s_F \in \left[0, \frac{2}{5}\right].$$

Formally

$$s_Y^{SR}\left(s_F\right) = \begin{cases} Yes & \text{if} \quad s_F \in \left[0, \frac{2}{5}\right] \\ No & \text{if} \quad s_F \notin \left[0, \frac{2}{5}\right] \end{cases};$$

(c) Given Y sequential rational choice, then F continuation expected payoff is

$$E\left[u_F\left(s_F, s_Y^{SR}; (y,f)\right)\right] = \begin{cases} s_F & \text{if} \quad s_F \in \left[0, \frac{2}{5}\right] \\ -\delta\varepsilon & \text{if} \quad s_F \notin \left[0, \frac{2}{5}\right] \end{cases}$$

which is maximized for $s_F^{SR} = \frac{2}{5}$;

(d) Given Y and F sequential rational choice, then F continuation expected payoff is

$$E\left[u_F\left(\cdot, s_Y^{SR}, s_F^{SR}; (y,f)\right)\right] = \begin{cases} \frac{2}{5}\delta & \text{if} \quad No \\ 1 - s_Y & \text{if} \quad Yes \end{cases}$$

which is maximized for

$$s_F^{SR}(s_Y) = \begin{cases} Yes & \text{if} \quad s_F \in \left[0, 1 - \frac{2}{5}\delta\right] \\ No & \text{if} \quad s_F \notin \left[0, 1 - \frac{2}{5}\delta\right]; \end{cases}$$

(e) Given Y and F sequential rational choice, then Y continuation expected payoff is

$$E\left[u_Y\left(s_F, s_Y^{SR}, s_F^{SR}; (y,f)\right)\right] = \begin{cases} \frac{2}{5}\delta & \text{if} \quad s_Y \in \left[1 - \frac{2}{5}\delta, 1\right] \\ s_Y & \text{if} \quad s_Y \notin \left[0, 1 - \frac{2}{5}\delta\right]; \end{cases}$$

which is maximized for $s_Y^{SR} = 1 - \frac{2}{5}\delta$.

Hence, the unique subgame perfect equilibrium is

$$\left(\begin{array}{l} s_Y^{SR} = 1 - \frac{2}{5}\delta, s_Y^{SR}(s_F) = \begin{cases} Yes & \text{if} \quad s_F \in \left[0, \frac{2}{5}\right] \\ No & \text{if} \quad s_F \notin \left[0, \frac{2}{5}\right] \end{cases}, \\ s_F^{SR}(s_Y) = \begin{cases} Yes & \text{if} \quad s_F \in \left[0, 1 - \frac{2}{5}\delta\right] \\ No & \text{if} \quad s_F \notin \left[0, 1 - \frac{2}{5}\delta\right] \end{cases}, s_F^{SR} = \frac{2}{5}. \end{array}\right)$$

Graphically

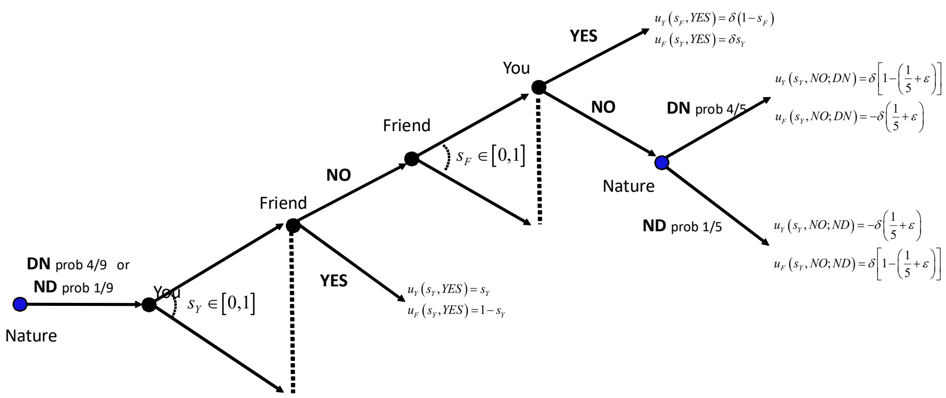

Figure 9:

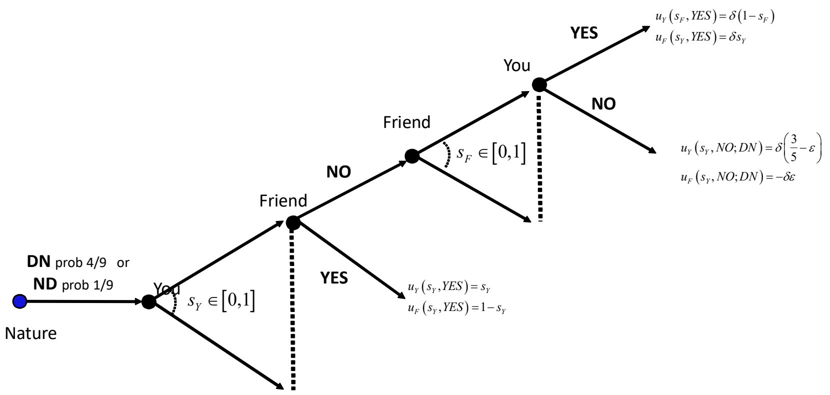

Figure 10:

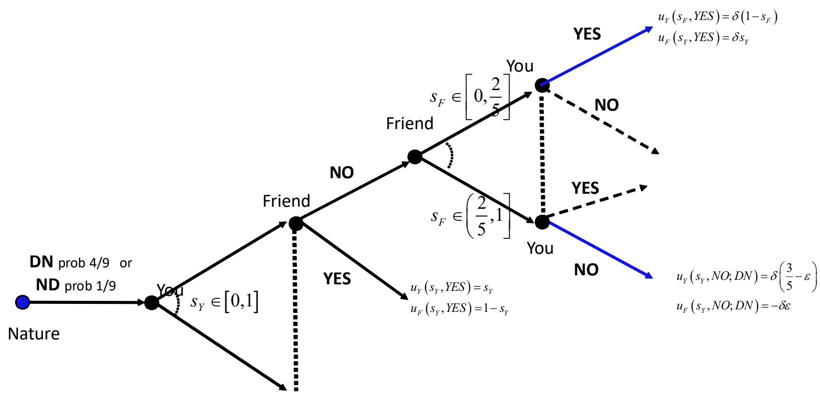

Figure 11:

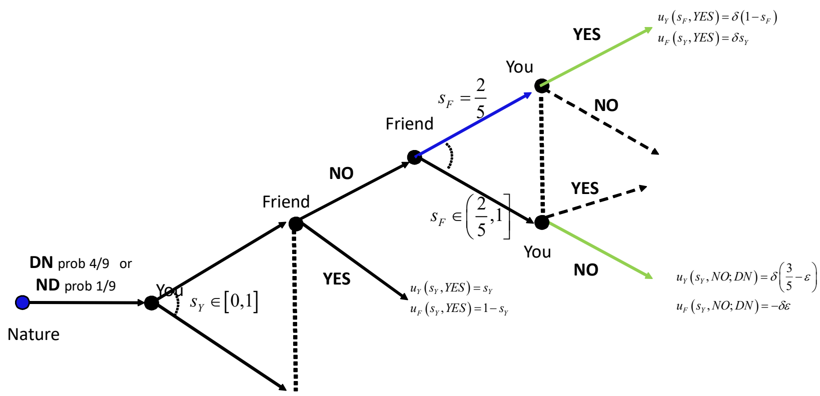

Figure 12:

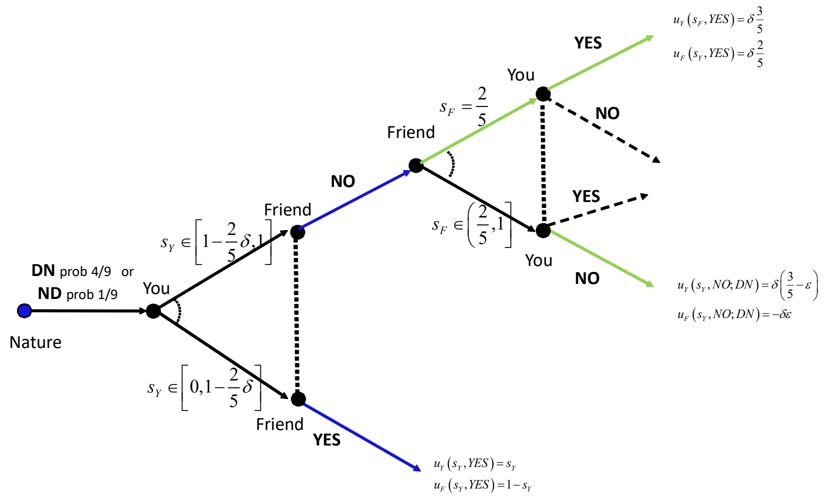

Figure 13:

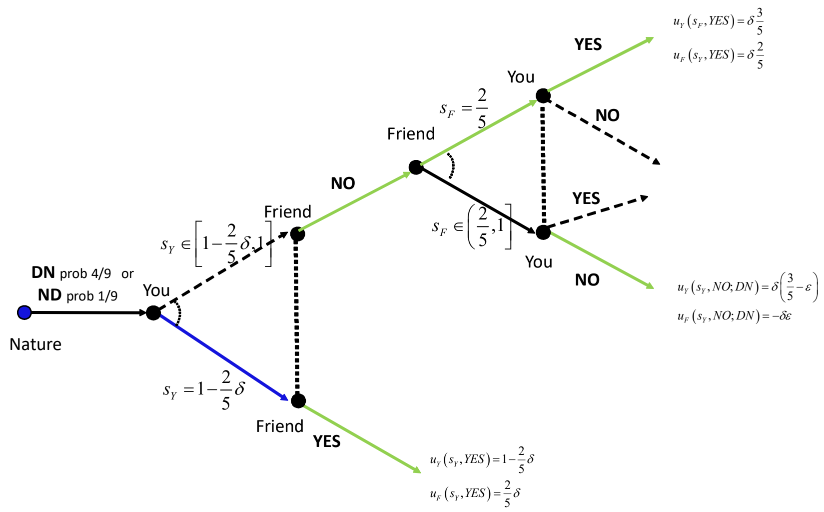

Figure 14:

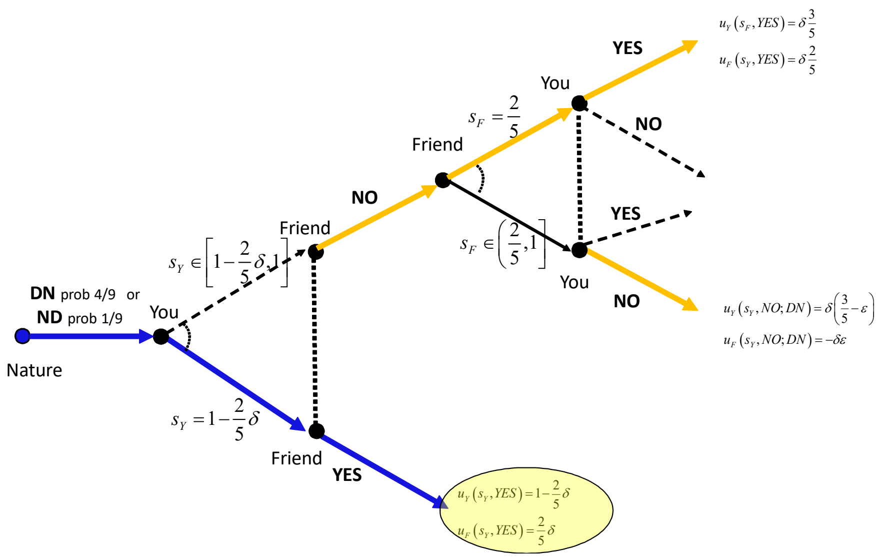
YES
You
$s_F = \frac{2}{5}$
NO
Friend
NO
YES
Friend
$s_Y \in \left[1-\frac{2}{5}\delta, 1\right]$
$s_F \in \left(\frac{2}{5}, 1\right]$
You
NO
DN prob 4/9 or
ND prob 1/9
You
Nature
$s_Y = 1-\frac{2}{5}\delta$
Friend
YES
$u_Y(s_F, YES) = \delta\frac{3}{5}$
$u_F(s_Y, YES) = \delta\frac{2}{5}$
$u_Y(s_Y, NO; DN) = \delta\left(\frac{3}{5}-\varepsilon\right)$
$u_F(s_Y, NO; DN) = -\delta\varepsilon$
$u_Y(s_Y, YES) = 1-\frac{2}{5}\delta$
$u_F(s_Y, YES) = \frac{2}{5}\delta$

Figure 15: