SOLUTION HOMEWORK 3
GAME THEORY PhD 2023

December 5, 2022

1 Exercise 1

Consider the following signalling game in extensive form.

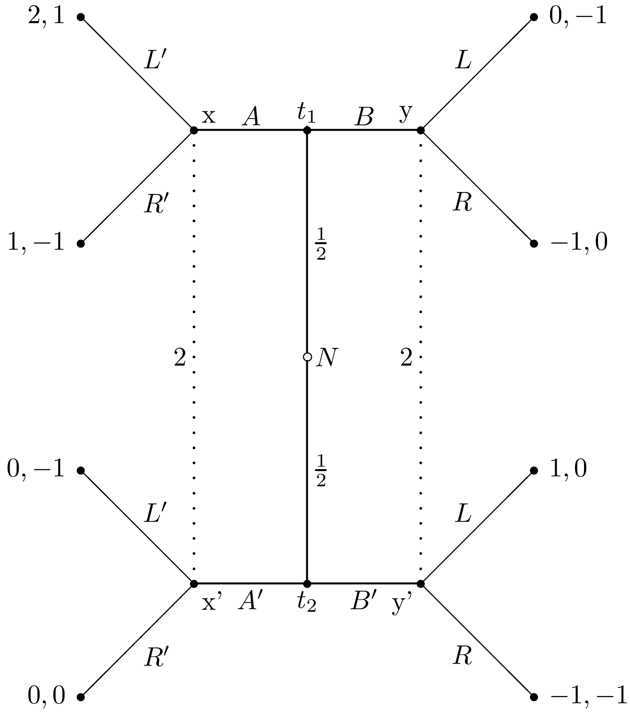

Figure 1

For this game:

1. construct the strategic form and calculate the set of Bayes-Nash Equilibria;

2. calculate the set of Sequential Equilibria in pure strategies;
3. refine the set of Sequential Equilibria in pure strategies using Forward Induction arguments.

2 Solution to exercise 1

1. In figure 4 there is the strategic form of the game of figure 1.

	LL'	LR'	RL'	RR'
AA'	1, 0	0.5, -0.5	1, 0	0.5, -0.5
AB'	1.5, 0.5	1, -0.5	0.5, 0	0, -1
BA'	0, -1	0, -0.5	-0.5, -0.5	-0.5, 0
BB'	0.5, -0.5	0.5, -0.5	-1, -0.5	-1, -0.5

Figure 4

1. Note that the strategies BA' and BB' are strictly dominated for player 1 and consequently strategies LR' and RR' are iteratively strictly dominated for player 2. Therefore these strategies can not be part of Bayes-Nash equilibria and the strategic form game to study is the following in figure 5

	LL'	RL'
AA'	1, 0	1, 0
AB'	1.5, 0.5	0.5, 0

Figure 5

Then we calculate the players best reply correspondences.

1. For player 1:

$$u_1(AA', \sigma_2) = 1 \quad u_1(AB', \sigma_2) = \frac{3}{2}\sigma_2(LL') - \frac{1}{2}(1 - \sigma_2(LL'))$$

thus

- if $\sigma_2(LL') \in [0, 1/2)$ then 1's unique best reply is $\sigma_1(AA') = 1$,
- if $\sigma_2(LL') \in (1/2, 1]$ then 1's unique best reply is $\sigma_1(AA') = 0$,
- if $\sigma_2(LL') = 1/2$ then 1's best reply is $\sigma_1(AA') \in [0, 1]$.

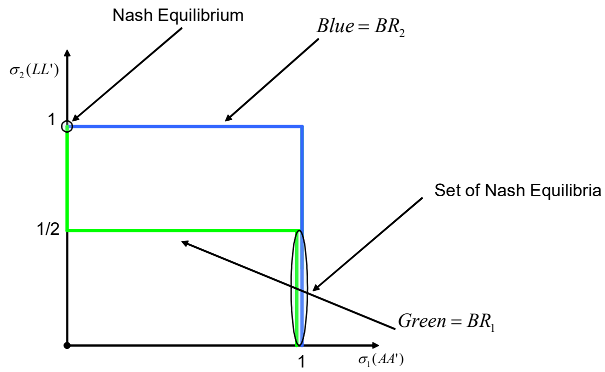

Figure 1:

Similarly, it is possible to calculate player 2' best reply correspondence:

$$u_2(\sigma_1, LL') = \frac{1}{2}(1 - \sigma_1(AA')) \quad u_2(\sigma_1, RL') = 0$$

Then

- if $\sigma_1(AA') \in [0, 1)$ then 2's unique best reply is $\sigma_2(LL') = 1$,
- if $\sigma_1(AA') = 1$ then 2's best reply is $\sigma_2(LL') \in [0, 1]$.

Therefore the two best reply correspondences are:

$$BR_1(\sigma_2) = \begin{cases} \sigma_1(AA') = 1 & \text{if } \sigma_2(LL') \in [0, 1/2] \\ \sigma_1(AA') \in [0, 1] & \text{if } \sigma_2(LL') = 1/2 \\ \sigma_1(AA') = 0 & \text{if } \sigma_2(LL') \in [1/2, 1]. \end{cases}$$

$$BR_2(\sigma_1) = \begin{cases} \sigma_2(LL') = 1 & \text{if } \sigma_1(AA') \in [0, 1] \\ \sigma_2(LL') \in [0, 1] & \text{if } \sigma_1(AA') = 1. \end{cases}$$

Graphically:

From the picture or using a system of simultaneous equations it is possible to find the intersection between the best reply correspondences and thus the set of Bayes-Nash equilibria:

$$\{(\sigma_1(AB') = 1, \sigma_2(LL')) = 1\} \cup \{(\sigma_1, \sigma_2) | \sigma_1(AA') = 1 \& \sigma_2(LL') \in [0, 1/2] \& \sigma_2(RL') = 1 - \sigma_2(LL')\}.$$

2. Because of the previous calculation of the set of Bayes-Nash equilibria, the possible strategies of a SE in pure strategies are

   (a) (AB', LL')

   (b) (AA', RL').

   Consider the first possible part of a SE: strategies are separating and thus the calculation of beliefs is easy:

$$\mu(x|\{x,x'\}) = \frac{\mu(t_1)\times\pi_1(A|t_1)}{\mu(t_1)\times\pi_1(A|t_1)+\mu(t_2)\times\pi_1(A'|t_2)} = \frac{1/2\times 1}{1/2\times 1+1/2\times 0} = 1$$

$$\mu(y|\{y,y'\}) = \frac{\mu(t_1)\times\pi_1(B|t_1)}{\mu(t_1)\times\pi_1(B|t_1)+\mu(t_2)\times\pi_1(B'|t_2)} = \frac{1/2\times 0}{1/2\times 0+1/2\times 1} = 0.$$

   Given this beliefs, strategy $\pi_2(L'|\{x,x'\}) = 1$ and $\pi_2(L|\{y,y'\}) = 1$ is sequentially rational for player 2. Similarly, given strategy $\pi_2(L'|\{x,x'\}) = 1$ and $\pi_2(L|\{y,y'\}) = 1$ the sequentially rational strategy for player 1 is $\pi_1(A|t_1) = 1$ and $\pi_1(B'|t_2) = 1$. Therefore we get a sE

$$(AB', LL') \quad \textit{with} \quad \mu(x|\{x,x'\}) = 1;\ \mu(y|\{y,y'\}) = 0.$$

   Consider the second possible part of a SE: strategies are pooling, therefore according to this strategy profile the information set $\{x,x'\}$ is reached with probability 1 while the information set $\{y,y'\}$ is never reached. Thus the calculation of beliefs is more complex:

$$\mu(x|\{x,x'\}) = \frac{\mu(t_1)\times\pi_1(A|t_1)}{\mu(t_1)\times\pi_1(A|t_1)+\mu(t_2)\times\pi_1(A'|t_2)} = \frac{1/2\times 1}{1/2\times 1+1/2\times 1} = \frac{1}{2}$$

$$\mu(y|\{y,y'\}) = \frac{\mu(t_1)\times\pi_1(B|t_1)}{\mu(t_1)\times\pi_1(B|t_1)+\mu(t_2)\times\pi_1(B'|t_2)} = \frac{1/2\times 0}{1/2\times 0+1/2\times 0} = \frac{0}{0} \in [0;1].$$

   Therefore we need to restrict these beliefs in $\{y,y'\}$ otherwise. Note that type t_1 of player 1 will never deviate from action A since it is getting a higher payoff with respect to everything possible playing B; on the other hand type t_2 of player 1 will not deviate from action A' if and only if player 2 will choose $\pi_2(R|\{y,y'\}) = 1$, which is sequentially rational for player 2 if and only if

$$Eu_2(L|\mu) = -1\mu(y|\{y,y'\}) + 0(1-\mu(y|\{y,y'\})) = -\mu(y|\{y,y'\}) \leq$$
$$\leq Eu_2(R|\mu) = 0\mu(y|\{y,y'\}) - 1(1-\mu(y|\{y,y'\})) = 0$$

   i.e. if and only if

$$\mu(y|\{y,y'\}) \geq \frac{1}{2}.$$

   Therefore we get a SE

$$(AA', RL') \quad \textit{with} \quad \mu(x|\{x,x'\}) = \frac{1}{2} \ \textit{and} \ \mu(y|\{y,y'\}) \geq \frac{1}{2}.$$

3. The first SE is separating, so by construction it satisfies the intuitive criterion. The second SE is pooling, and it relies on the out-of-equilibrium beliefs that it is more likely that type t_1 deviates: this violates the intuitive criterion since type t_1 in equilibrium get the highest possible payoff, while type t_2 deviating and being able to convince player 2 that she is actually type t_2 could actually get the highest possible payoff.

3 Exercise 2

Consider the signalling game of figure 2.

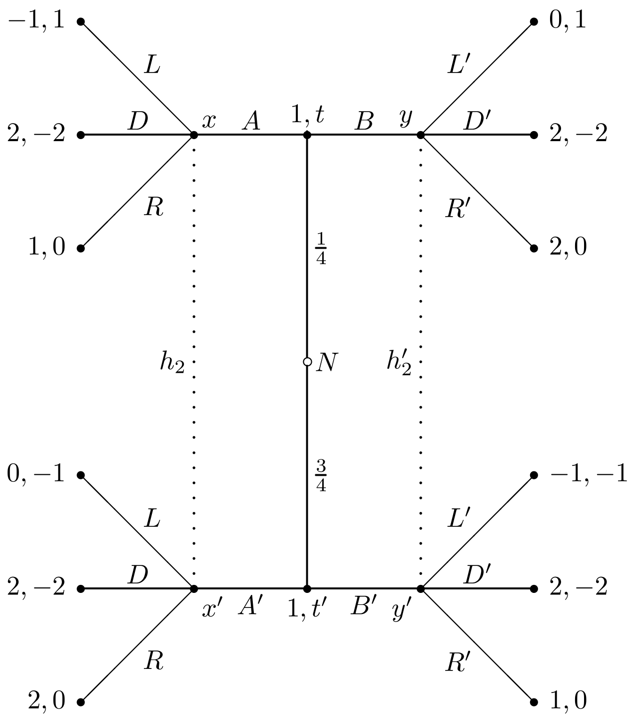

Figure 2

1. Construct the strategic form and calculate the set of Bayes-Nash equilibria in pure strategies;
2. Calculate the set of Subgame Perfect Equilibria in pure strategies;
3. Calculate the set of Weak Perfect Bayesian Equilibria in pure strategies;
4. Calculate the set of Sequential Equilibria in pure strategies;
5. Refine the set of Sequential Equilibria in pure strategies using the intuitive criterion.

4 Solution to exercise 2

1. In figure 2 there is the strategic form of the game of figure 1.

	AA'	AB'	BA'	BB'
LL'	$-1/4,\ -1/2$	$-1,\ -1/2$	$0,\ -1/2$	$-3/4,\ -1/2$
LD'	$-1/4,\ -1/2$	$5/4,\ -5/4$	$1/2,\ -5/4$	$2,\ -2$
LR'	$-1/4,\ -1/2$	$1/2,\ 1/4$	$1/2,\ -3/4$	$5/4,\ 0$
DL'	$2,\ -2$	$-1/4,\ -5/4$	$3/2,\ -5/4$	$-3/4,\ -1/2$
DD'	$2,\ -2$	$2,\ -2$	$2,\ -2$	$2,\ -2$
DR'	$2,\ -2$	$5/4,\ -1/2$	$2,\ -3/2$	$5/4,\ 0$
RL'	$7/4,\ 0$	$-1/2,\ -3/4$	$3/2,\ 1/4$	$-3/4,\ -1/2$
RD'	$7/4,\ 0$	$7/4,\ -3/2$	$2,\ -1/2$	$2,\ -2$
RR'	$7/4,\ 0$	$1,\ 0$	$2,\ 0$	$5/4,\ 0$

Figure 6

Simply from the direct inspection of the matrix and remembering that the first number in each cell is player column payoff and the second number is player raw payoff, it is easy to find the following two pure strategy Bayes Nash equilibria:

$$(AA', RL') \qquad (BB', LR').$$

2. The game has no proper subgame, hence the set of Subgame Perfect equilibria coincides with the set of Nash equilibria;

3. First note that D is always a strictly dominated action for players 2, so it will never be played in a Sequential equilibrium.

   So the "actual" game we are considering is the following:

   (a) Suppose that A, B' is part of a WPBE: then $\mu(x) = \mu(y') = 1$ and thus $L(h_2)$ and $R'(h_2')$ are sequentially rational. But then type t' has an incentive to deviate: it is not a WPBE.

   (b) Suppose that B, A' is part of a WPBE: then $\mu(x') = \mu(y) = 1$ and thus $R(h_2)$ and $L'(h_2')$ are sequentially rational. But then type t' has an incentive to deviate: it is not a WPBE.

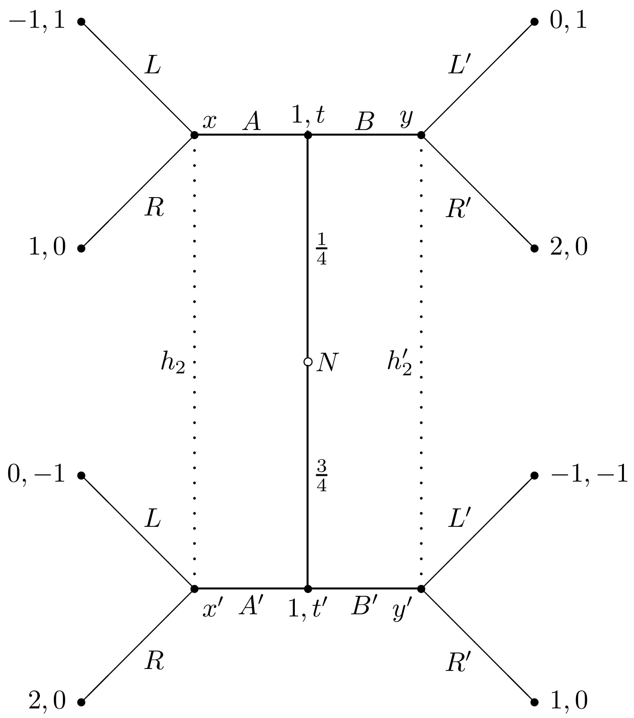

Figure 2: []The “actual" Signalling game

(c) Suppose that A, A' is part of a WPBE: then $\mu(x) = 1/4$ and $\mu(y) \in [0, 1]$. Thus $R(h_2)$ is sequentially rational. Then to avoid deviation of type t' L' should be played in h_2', and this is sequentially rational iff $\mu(y) \geq 1/2$. Therefore the following is a WPBE

$$A, A';\ \ R(h_2), L'(h_2'),\ \ \mu(x) = 1/4,\ \mu(y) \geq 1/2.$$

Note that this WPBE satisfies the intuitive criterion.

(d) Suppose that B, B' is part of a WPBE: then $\mu(y) = 1/4$ and $\mu(x) \in [0, 1]$. Thus $R'(h_2')$ is sequentially rational. Then to avoid deviation of type t' L should be played in h_2, and this is sequentially rational iff $\mu(x) \geq 1/2$. Therefore the following is a WPBE

$$B, B';\ \ L(h_2), R(h_2'),\ \ \mu(y) = 1/4,\ \mu(x) \geq 1/2.$$

4. Sequential equilibria coincide with Weak Perfect Bayesian equilibria in Signalling games with two players.

5. The second WPBE/SE found at point 3

$$B, B';\ \ L(h_2), R'(h_2'),\ \ \mu(y) = 1/4,\ \mu(x) \geq 1/2$$

does not satisfy the intuitive criterion, because type t' has an incentive to signal its type in order to induce player 2 to play $R(h_2)$ and thus getting an higher payoff.

5 Exercise 3

Consider the signalling game of figure 3.

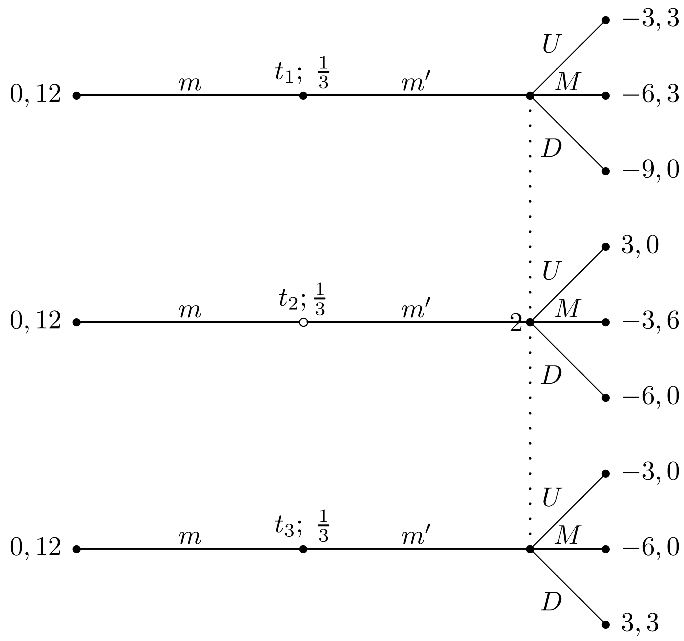

Figure 3

1. Construct the strategic form and calculate the set of Bayes-Nash equilibria in pure strategies;
2. Calculate the set of Subgame Perfect Equilibria in pure strategies;
3. Calculate the set of Weak Perfect Bayesian Equilibria in pure strategies;
4. Calculate the set of Sequential Equilibria in pure strategies;
5. Refine the set of Sequential Equilibria in pure strategies using the intuitive criterion.

6 Solution to exercise 3

1. In figure 7 there is the strategic form of the game of figure 3.

	U	M	D
mmm	$0,\ \underline{12}$	$\underline{\mathbf{0}},\ \underline{\mathbf{12}}$	$0,\ \underline{12}$
mmm'	$-1,\ 8$	$-2,\ 8$	$\underline{\mathbf{1}},\ \underline{\mathbf{9}}$
$mm'm$	$\underline{1},\ 8$	$-1,\ \underline{10}$	$-2,\ 8$
$mm'm'$	$0,\ 4$	$-3,\ 6$	$-1,\ \underline{5}$
$m'mm$	$-1,\ \underline{9}$	$-2,\ \underline{9}$	$-3,\ 8$
$m'mm'$	$-2,\ \underline{5}$	$-4,\ \underline{5}$	$-2,\ \underline{5}$
$m'm'm$	$0,\ 5$	$-3,\ \underline{7}$	$-5,\ 4$
$m'm'm'$	$-1,\ 1$	$-5,\ \underline{3}$	$-4,\ 1$

Figure 7

1. Simply from the direct inspection of the matrix and remembering that the first number in each cell is player column payoff and the second number is player raw payoff, it is easy to find the following two pure strategy Bayes Nash equilibria:

$$(mmm, M) \qquad (mmm', D).$$

2. The game has no proper subgame, hence the set of Subgame Perfect equilibria coincides with the set of Nash equilibria;

3. WPBE are a refinement of Bayes Nash equilibria, therefore we check whether the two Bayes Nash equilibria in pure strategies are Weak perfect too.

   (a) Suppose that

$$s_1^*(t) = \begin{cases} m & \text{if } t = t_1 \\ m & \text{if } t = t_2 \\ m' & \text{if } t = t_3. \end{cases}$$

   $s_2^* = D$ is part of a WPBE: then

$$\mu(t_3|m') = \frac{\mathbf{P}(t_3) \times \mathbf{P}(m'|t_3)}{\mathbf{P}(t_1) \times \mathbf{P}(m'|t_1) + \mathbf{P}(t_2) \times \mathbf{P}(m'|t_2) + \mathbf{P}(t_3) \times \mathbf{P}(m'|t_3)} =$$

$$= \frac{1/3 \times \pi_1^*(m'|t_3)}{1/3 \times \pi_1^*(m'|t_1) + 1/3 \times \pi_1^*(m'|t_2) + 1/3 \times \pi_1^*(m'|t_3)} =$$

$$= \frac{1/3 \times 1}{1/3 \times 0 + 1/3 \times 0 + 1/3 \times 1} = 1.$$

Note that D is a best reply to $\mu(t_3|m') = 1$ and that mmm' is a best reply to D, therefore these strategies are sequentially rational. To conclude the following is a WPBE in pure strategy:

$$s_1^*(t) = \begin{cases} m & \text{if } t = t_1 \\ m & \text{if } t = t_2 \\ m' & \text{if } t = t_3. \end{cases}$$

$s_2^* = D$ and $\mu(t_3|m') = 1$.

(b) Suppose that

$$s_1^*(t) = \begin{cases} m & \text{if } t = t_1 \\ m & \text{if } t = t_2 \\ m & \text{if } t = t_3. \end{cases}$$

$s_2^* = M$ is part of a WPBE: then $\forall t \in \{t_1, t_2, t_3\}$

$$\mu(t|m') = \frac{\mathbf{P}(t) \times \mathbf{P}(m'|t)}{\mathbf{P}(t_1) \times \mathbf{P}(m'|t_1) + \mathbf{P}(t_2) \times \mathbf{P}(m'|t_2) + \mathbf{P}(t_3) \times \mathbf{P}(m'|t_3)} =$$

$$= \frac{1/3 \times \pi_1^*(m'|t)}{1/3 \times \pi_1^*(m'|t_1) + 1/3 \times \pi_1^*(m'|t_2) + 1/3 \times \pi_1^*(m'|t_3)} =$$

$$= \frac{1/3 \times 0}{1/3 \times 0 + 1/3 \times 0 + 1/3 \times 0} = \frac{0}{0} \in [0, 1]$$

since the ratio is indeterminate. Note that we need $2\mu(t_1|m') + 3\mu(t_2|m') \geq 1$ to guarantee that M is sequentially rational and thus that types t_2 and t_3 do not deviate. To conclude the following is a WPBE in pure strategy:

$$s_1^*(t) = \begin{cases} m & \text{if } t = t_1 \\ m & \text{if } t = t_2 \\ m & \text{if } t = t_3. \end{cases}$$

$s_2^* = M$ and $2\mu(t_1|m') + 3\mu(t_2|m') \geq 1$.

4. Sequential equilibria coincide with Weak Perfect Bayesian equilibria in Signalling games with two players.

5. The beliefs of the first sequential equilibrium are derived from Bayes rule therefore can not be discussed.

   Consider then the second (set of) Sequential Equilibrium: in this case the beliefs are not derived though Bayes rule and therefore are partially arbitrary. From an intuitive point of view note the following aspects

   (a) in equilibrium type t_1 get 0, while deviating get for sure something smaller, notwithstanding player 3 behaviour. Therefore according to some form of forward induction, we might expect $\mu(t_1|m') = 0$.

(b) using this restriction on the set of possible beliefs, then we might conclude that U is never a best reply for player 2 and consequently even type t_2 deviating would get something strictly smaller than 0. Therefore according to some form of forward induction, we might expect $\mu(t_2|m') = 0$ too.

Therefore the second (set of) Sequential equilibrium doesn't seem consistent with forward induction.

Now we try to formalize these ideas using the intuitive criterion. We should prove that there exists a out-of-equilibrium message m' and a subset of types J such that

(a) $\forall t \in J \; \forall a \in BR(T, m'^*(t) > U^S(t, m', a)$

(b) $\exists t' \in T \setminus J : \;\; \forall a \in BR(T \setminus J, m'^*(t'^S(t', m', a)$.

Let be

$$T := \{t_1, t_2, t_3\}, \; J = \{t_1\}.$$

Then

$$BR(T, m') := \bigcup_{\{\mu|\mu(T)=1\}} BR(\mu, m') = \{U, M, D\}$$

and thus

$$U^*(t) = U^*(t_1) = 0 > U^S(t_1, m', a) \; \forall a \in \{U, M, D\}.$$

Moreover let it be

$$t' = t_3 \in T \setminus \{t_1\},$$

so that

$$BR(T \setminus \{t_1\}, m') := \bigcup_{\{\mu|\mu(\{t_2,t_3\})=1\}} BR(\mu, m') = \{M, D\}.$$

Thus

$$U^*(t'^*(t_3) = 0 < U^S(t_3, m', D) = 3$$

but

$$U^*(t'^*(t_3) = 0 > U^S(t_3, m', M) = -6.$$

Thus we can not reject this (set of) Sequential equilibrium using the intuitive criterions since it does satisfy the intuitive criterion.