SOLUTION MOCK EXAM GAME THEORY
Ph.D. 2023

December 5, 2022

You CAN NOT use books or notes.

Consider the game of figure 1

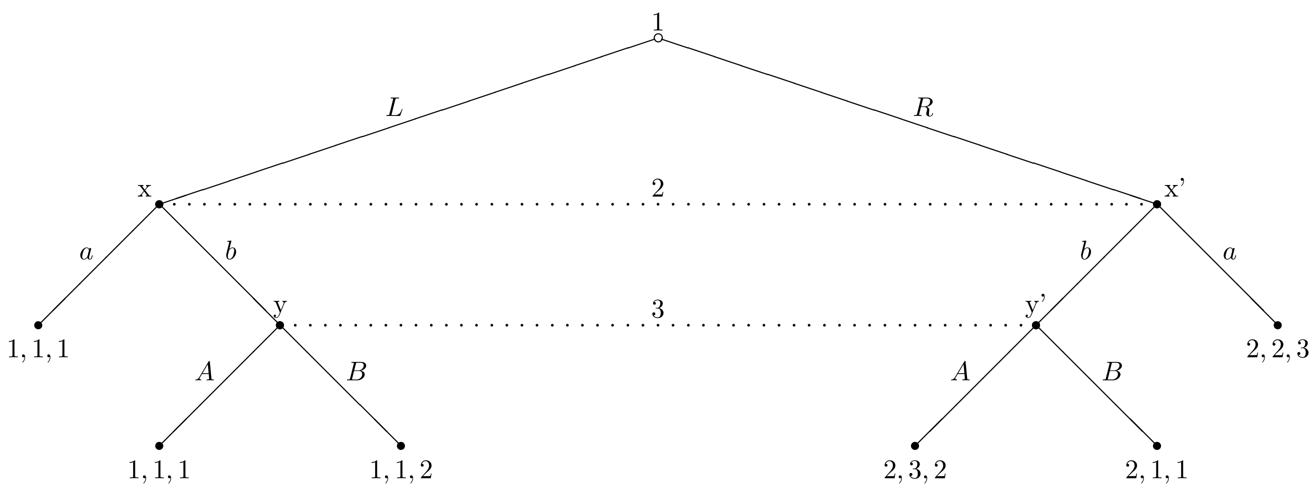

Figure 1

1. Construct the strategic form and calculate the set of Nash equilibria, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria [6 POINTS];

2. Calculate the set of Subgame Perfect Equilibria, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria [6 POINTS];

3. Calculate the set of Weak Perfect Bayesian Equilibria in pure strategies, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria [6 POINTS];

4. Find the set of Sequential Equilibria in pure strategies, the probabilities of outcomes, emphasizing the equilibrium path and the out of equilibrium actions in each of the equilibria [7 POINTS];
5. Discuss the beliefs associated to each WPBE [7 POINTS].

SOLUTION

1. In figure 2 there is the strategic form of the game of figure 1

A	a	b
L	$1,1,1$	$1,1,1$
R	$2,2,3$	$2,3,2$

B	a	b
L	$1,1,1$	$1,1,1$
R	$2,2,3$	$2,1,1$

Figure 2

From figure 1 or from figure 2, it is immediate that the pure strategy L of player 1 is strictly dominated so that player 1 will play R with probability 1 in any Nash equilibrium of this game. Then we can consider the game between player 2 and 3 pictured in figure 3

	A	B
a	$2,3$	$2,3$
b	$3,2$	$1,1$

Figure 3

To find the set of Nash equilibria of the game of figure 3, we need to calculate the players best reply correspondences.

1. For player 2:

   $u_2(a,\sigma_3) = 2\times\sigma_3(A)+2\times(1-\sigma_3(A)) \quad u_2(b,\sigma_3) = 3\times\sigma_3(A)+1\times(1-\sigma_3(A))$

   thus

   - if $\sigma_3(A) \in [0,1/2]$ then 2's best reply is $\sigma_2(a) = 1$,
   - if $\sigma_3(A) \in [1/2,1]$ then 2's best reply is $\sigma_2(a) = 0$,
   - if $\sigma_3(A) = 1/2$ then 2's best reply is $\sigma_2(a) \in [0,1]$.

   Similarly, it is possible to calculate player 3' best reply correspondence:

   $u_3(\sigma_2,A) = 3\times\sigma_2(a)+2\times(1-\sigma_2(a)) \quad u_3(\sigma_2,B) = 3\times\sigma_2(a)+1\times(1-\sigma_2(a))$.

   Then

- if $\sigma_2(a) \in [0,1]$ then 3's best reply is $\sigma_3(A) = 1$,
- if $\sigma_2(a) = 1$ then 3's best reply is $\sigma_3(A) \in [0,1]$.

Therefore the two best reply correspondences are:

$$BR_2(\sigma_3) = \begin{cases} \sigma_2(a) = 1 & \text{if } \sigma_3(A) \in [0, 1/2] \\ \sigma_2(a) \in [0,1] & \text{if } \sigma_3(A) = 1/2 \\ \sigma_2(a) = 0 & \text{if } \sigma_3(A) \in [1/2, 1]. \end{cases}$$

$$BR_3(\sigma_2) = \begin{cases} \sigma_3(A) = 1 & \text{if } \sigma_2(a) \in [0,1] \\ \sigma_3(A) \in [0,1] & \text{if } \sigma_2(a) = 1. \end{cases}$$

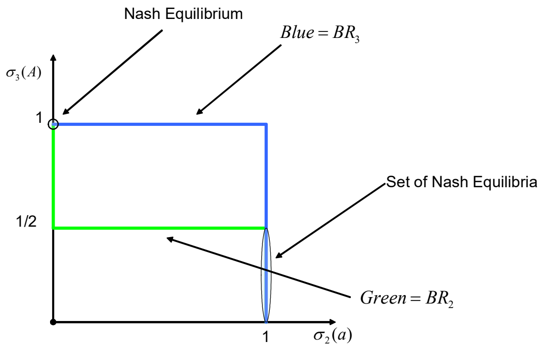

Figure 1:

Graphically:

From the picture or using a system of simultaneous equations it is possible to find the intersection between the best reply correspondences and thus, remembering that player 1 is always choosing R with probability 1, the set of Nash equilibria of the game:

$$NE = \{R, b, A)\} \cup \{(R, a, \sigma_3(A) \in [0, 1/2]\}.$$

2. Since there are no proper subgame, the set of Subgame Perfect equilibria and the set of Nash equilibria coincide

$$SPE = \{R, b, A)\} \cup \{(R, a, \sigma_3(A) \in [0, 1/2]\}.$$

3. The set of Weak Perfect Bayesian equilibria is a subset of Nash equilibria, thus we can start from the Nash equilibria in pure strategies, which are

$$\{R, b, A)\} \cup \{(R, a, B\}.$$

Now we have to compute players' beliefs that sustain such equilibria, if they exists.

(a) Consider (R, b, A). Then by Bayes rule

$$\mu\left(x|\left\{x, x'\right\}\right) = \frac{\pi_1(L)}{\pi_1(L) + \pi_1(R)} = \frac{0}{0+1} = 0$$

Therefore $\mu\left(x'|\left\{x, x'\right\}\right) = 1$

Moreover

$$\mu\left(y\right|\{y,y'\}) = \frac{\pi_1(L)\times\pi_2(b)}{\pi_1(L)\times\pi_2(b)+\pi_1(R)\times\pi_2(b)} = \frac{\pi_1(L)}{\pi_1(L)+\pi_1(R)} = \mu\left(x\right|\{x,x'\}) = 0.$$

Therefore for player 3 is sequentially rational to play A. Moreover also b for player 2 is sequentially rational given previous beliefs and given A by player 3.
Therefore

$$\mu\left(x\right|\{x,x'\}) = 0, \mu\left(y\right|\{y,y'\}) = 0 \quad and \quad R, b, A$$

it is a WPBE in pure strategies.

(b) Consider (R, a, B). Then by Bayes rule

$$\mu\left(x\right|\{x,x'\}) = \frac{\pi_1(L)}{\pi_1(L)+\pi_1(R)} = \frac{0}{0+1} = 0$$

Moreover

$$\mu\left(y\right|\{y,y'\}) = \frac{\pi_1(L)\times\pi_2(b)}{\pi_1(L)\times\pi_2(b)+\pi_1(R)\times\pi_2(b)} = \frac{0\times 0}{0\times 0+0\times 0} = \frac{0}{0} \in [0,1].$$

Then Bayes rule does not restrict $\mu\left(y\right|\{y,y'\})$. But B is sequentially rational only if

$$Eu_3(B|\mu(y)) \geq Eu_3(A|\mu(y))$$

i.e.

$$1\times\mu(y) + 1\times(1-\mu(y)) \geq 1\times\mu(y) + 2\times(1-\mu(y))$$

which means $\mu\left(y\right|\{y,y'\}) = 1$.
Moreover player 2 sequential rational strategy given R and B is a, which in turn implies that R is sequentially rational for player 1.
Therefore

$$\mu\left(x\right|\{x,x'\}) = 0, \mu\left(y\right|\{y,y'\}) = 1 \quad and \quad R, a, B$$

it is a (continuum of) WPBE in pure strategy.

4, The set of Sequential equilibria is a refinement of the set of WPBE. In the previous point of the homework, we have derived the set of Weak Perfect Bayesian equilibria in pure strategies:

(a) (R, b, A) with beliefs $\mu(x|\{x,x'\}) = \mu(y|\{y,y'\}) = 0$

(b) (R, a, B) with beliefs $\mu(x|\{x,x'\}) = 0$ and $\mu(y|\{y,y'\}) = 1$.

Note that in the first WPBE there are no out-of-equilibrium information set, thus beliefs satisfy consistency and it is a Sequential equilibrium.

Consider the second (set of) WPBE. When there are out-of-equilibrium nodes, the beliefs are derived from Bayes rule assuming strictly mixed behavioral strategies. Then applying Bayes rule, the beliefs are

$$\mu(y|\{y,y'\}) = \frac{\pi_1(L) \times \pi_2(b)}{\pi_1(R) \times \pi_2(b) + \pi_1(L) \times \pi_2(b)} = \frac{\pi_1(L)}{\pi_1(R) + \pi_1(L)} = \mu(x|\{x,x'\})$$

where the fractions are well defined because by assumption $\pi_i(\cdot) > 0$.

Hence in any sequential equilibrium $\mu(x|\{x,x'\}) = \mu(y|\{y,y'\})$ and consequently the second (set of) WPBE is not a sequential equilibrium. Thus the set of Sequential equilibria in pure strategies is a singleton:

$$(R, b, A), \quad \mu(x|\{x,x'\}) = \mu(y|\{y,y'\}) = 0.$$

5. In the first WPBE there are no out-of-equilibrium information set, thus there is no room for discussion of players' beliefs.

Let consider the second (set of) WPBE, which are not SE and where $\mu(x|\{x,x'\}) = 0$ and $\mu(y|\{y,y'\}) \geq 1/2$: in this case the beliefs $\mu(y|\{y,y'\}) \geq 1/2$ are not restricted by Bayes rules, so in this sense they are arbitrary.

From the game tree, node y is reached with probability $\pi_1(L) \times \pi_2(b)$, while y' is reached with probability $\pi_1(R) \times \pi_2(b)$. Hence:

$$\mu(y|\{y,y'\}) = 1 \Leftrightarrow \pi_1(L) > \pi_1(R)$$

and using dominance is possible to conclude that L is a more costly "mistake" than R. Thus, it is possible to exclude $\mu(y|\{y,y'\}) = 1$ using FI arguments, however it is also immediate using independent vanishing mistakes because node y is reached with two independent infinitesimal mistakes, i.e. with probability $\pi_1(L) \times \pi_2(b) = \varepsilon \times \delta$, while y' is reached with probability $\pi_1(R) \times \pi_2(b) = (1 - \varepsilon) \times \delta$.