Game Theory
Ph.D. in Economics - DEFAP
2022

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LECTURE 1

Introduction

&

Models of strategic interaction

Introduction

About Me: Mario Gilli

- B.A. in Discipline Economiche e Sociali, Bocconi University 1987
- Ph.D. in Economics, Cambridge University 1994
- Assistant Professor in Economics, Bocconi University 1992-1998
- Associate Professor, Bari University, 1998-2001
- Full Professor, University of Milan - Bicocca
- **AREAS OF INTERESTS**:
  - Learning and Evolutionary Models in Game Theory and in Social Sciences
  - Economics of Organizations and Institutions
  - Intersections among Economics, Politics and Sociology
  - Conflict theory
  - Political economics

COURSE DESCRIPTION

- *The course consists of*
  1. *Five lectures*
  2. *Three* **group** *homework*
  3. *Three classes*
  4. *Your presentation of a research paper*
- *TOPICS OF THE LECTURES*

1. ***Game forms***
2. ***Rational behavior in strategic setting***
3. ***Properties of solutions and of Nash equilibria***.
4. ***Nash equilibria in extensive form games*** *and* ***refinements***
5. ***Dynamic games with incomplete information***.

Lectures

- PROBLEM: few lectures, many topics.
- IMPLICATION: very teeming lectures
- SOLUTIONS:
- Files available in advance on the web site of the course https://elearning.unimib.it/course/view.php?id=39379

  to allow you to study BEFORE LECTURES and to take notes easily
- **It is crucial that you register yourself at the above website**
- The lectures will illustrate the main concepts through formal definitions and examples, with a particular attention to the calculus of solutions
- Good class participation can improve your evaluation.
- I expect you to come to class prepared to respond intelligently to questions about the readings and assignments

Classes and Homework

- There will be 3 problem sets as homework.
- **You are encouraged to form small group to solve problem sets.**
- The problems will be quite difficult:
  - **you are not expected to be able to answer all the questions correctly, they are useful as training**
  - **Are not a good exam forecast**
- The homework will be correct in the classes
- The solutions of the homework will be on the website downloadable after the class
  - **The homework will be evaluated and will be part of the final grade (20%)**

PRESENTATION

- The presentation will consist in a review of one or more research paper on the topics listed in the syllabus or on a topic suggested by the students.
- The work should be done in a group of 2/4 students.
- **The presentaton will be evaluated and will be part of the final grade (20%)**

EXAM

- The examination consists of three parts:
  1. **The three homework**
  2. **A presentation**
  3. ***<u>a written individual examination.</u>***
- HOMEWORK: 20% of the final mark in Game Theory.
- PRESENTATION: 20% of the final mark in Game Theory
- FINAL EXAM: the final exam will consist in one exercise and it will count for 60% of the final mark in Game Theory.
  - **The marks are relative**
- **Homework and presentation will not count for the resit**

Textbooks

1. Jurgen Eichberger, Game Theory for Economists, Academic Press,1993 = E.
2. Martin Osborne and Ariel Rubinstein, A Course in Game Theory, MIT Press, 1994 = OR.
3. Lecture notes.

Few comments on the books

1. E is a basic, simple and clear book. Personally, I like it very much, unfortunately it is very expensive (more than 100€ on amazon.it), however there are four copies in the Bicocca library
2. OR is a complete, nice and clear book and it freely downloadable from Rubinstein homepage http://arielrubinstein.tau.ac.il/, unfortunately the notation used for extensive form games is very effective but not standard

Introduction to game theory

What is Game Theory? - 1

- *Game theory* has a sexy name but it is actually no more than a collection of concepts and models about **rational behavior in strategic situation**
- that is, in **situations in which the considerations of a rational player depend on how she assumes other players will behave**
- ***Classic*** *Game theory* is the study of mathematical models of conflict and cooperation between ***rational*** *and* ***intelligent*** decision makers.
- *Rational:* **each individual maximizes her expected utility**
- *Intelligent:* **individual understands the situation, including the fact that others are intelligent rational decision makers** (common knowledge)
- The rational player must step into the shoes of the other players, who face a similar task
- This circularity is the source
  - **of the complication and**
  - **of the interest of game theory**

• GAME THEORY TRIES TO INJECT CONTENT INTO THE CONCEPT OF RATIONALITY IN A CONTEXT IN WHICH THE MEANING OF RATIONALITY IS UNCLEAR

- Recently Game Theory has started to consider non rational players (e.g. Evolutionary and Behavioral Game Theory)

COMPARING ECONOMICS AND GAME THEORY

- In Economics there was no overall methodology, but different methods for different problems
- Game Theory defines concepts very broadly so that they can be applied to any interacting situation
- *Game Theory* may be viewed as a sort of umbrella or **"unified field" theory for social sciences based on individual decision making**, where individual are interpreted broadly to include human individuals as well as other collective players such as
  - Corporations
  - Nations
  - Animals
  - Genes
  - Computers
  - Social norms, ecc

Approaches to Game Theory

- There are two main approach
- **Cooperative**:
  - commitments, agreements, promises, threats are fully binding and enforceable
  - deals with the options available to the group, what coalitions form, how the available payoff is divided
- **Non-cooperative**:
  - commitments, agreements, promises, threats are not enforceable.
  - concentrates on the strategic choices of the individual: how each player plays the game, what strategies it chooses to achieve its goals

What is a Game?

Game Theory: Why Games?

- Varieties of games
  - board, card, video, field
  - economic games: bargaining, auctions
- Features of a game
  - rule-governed
  - strategy matters for outcomes
  - strategic interdependence

Example - 1

THE HIDE & SEEK GAME:

- A cruel ruler can hide in one of four palaces (1,2,3,4), where 2 is painted gold, the others white
- The seeker can attack only one palace
- If the ruler is in the attacked palace, the seeker wins, otherwise the ruler wins

Example 2: Both Pay Auction

- $10 is auctioned to highest of two bidders
- Players alternate bidding, with player 1 first
- At each stage, the bidding player must decide either to raise bid by $1 or to quit
- Game ends when one of the two bidders quits in which case the other bidder gets the $10, and *both* bidders pay the auctioneer their bids

Example 3: Trade

- Player 1 wish to sell a unit of an indivisible good which for her has value \$1
- Player 2 wish to buy a unit of this good which for him has value \$2
- Player 1 decides either a high price (\$3) or a low price (\$1)
- player 2 observes the price proposed by the seller and decides whether to accept or to quit.

Example 4: Trade and asymmetric information

- As game 2, but the seller does not know the buyer's reservation value
- The reservation value has two possible values:
  - – 1 with probability p
  - – 3 with probability 1-p.

Comments on examples 1- 4

QUESTIONS:

1. What is the connection between this “fable” and the reality?
2. How can we construct a “model” from this story?
3. What are our predictions?
4. Are these predictions useful?

What is a Game?

A Game is described by four things:

1. The players
2. The rules:
   1. **order of moves,**
   2. **possible actions,**
   3. **information**
3. Outcomes (for each possible actions' profile)
4. Payoffs or (expected) utilities

- **A game provides a "model" for situation of strategic interaction, as previous examples show**

Formal Representations of a Game

- The four forms of a game
  - **extensive form**
  - normal form (**strategic form**)
  - Sequence form
  - [coalition function]
- Sequence form is usually used for computation
- Coalition function form is used to study the gains of cooperation

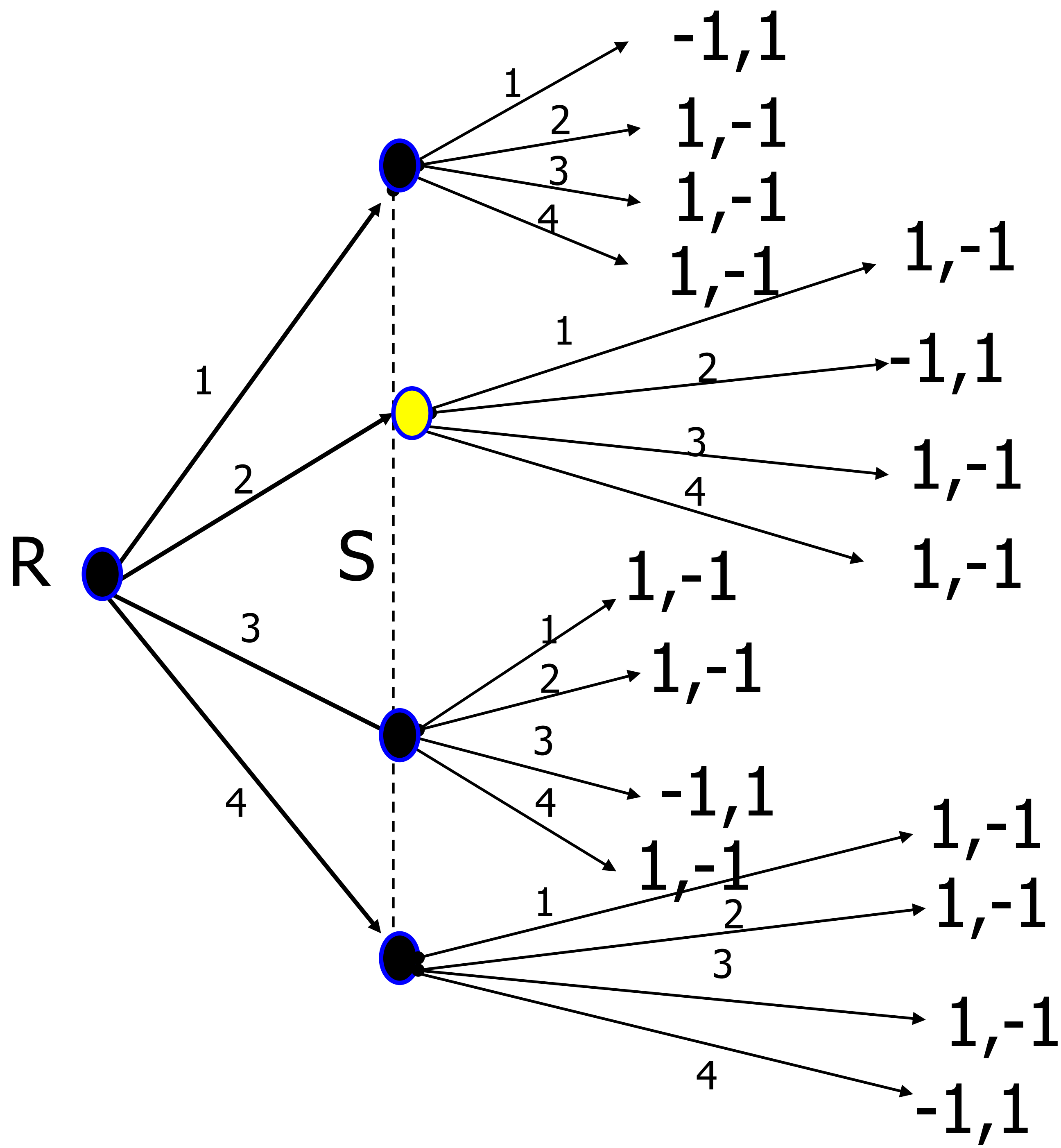
Extensive Form Games

Game in Extensive Form

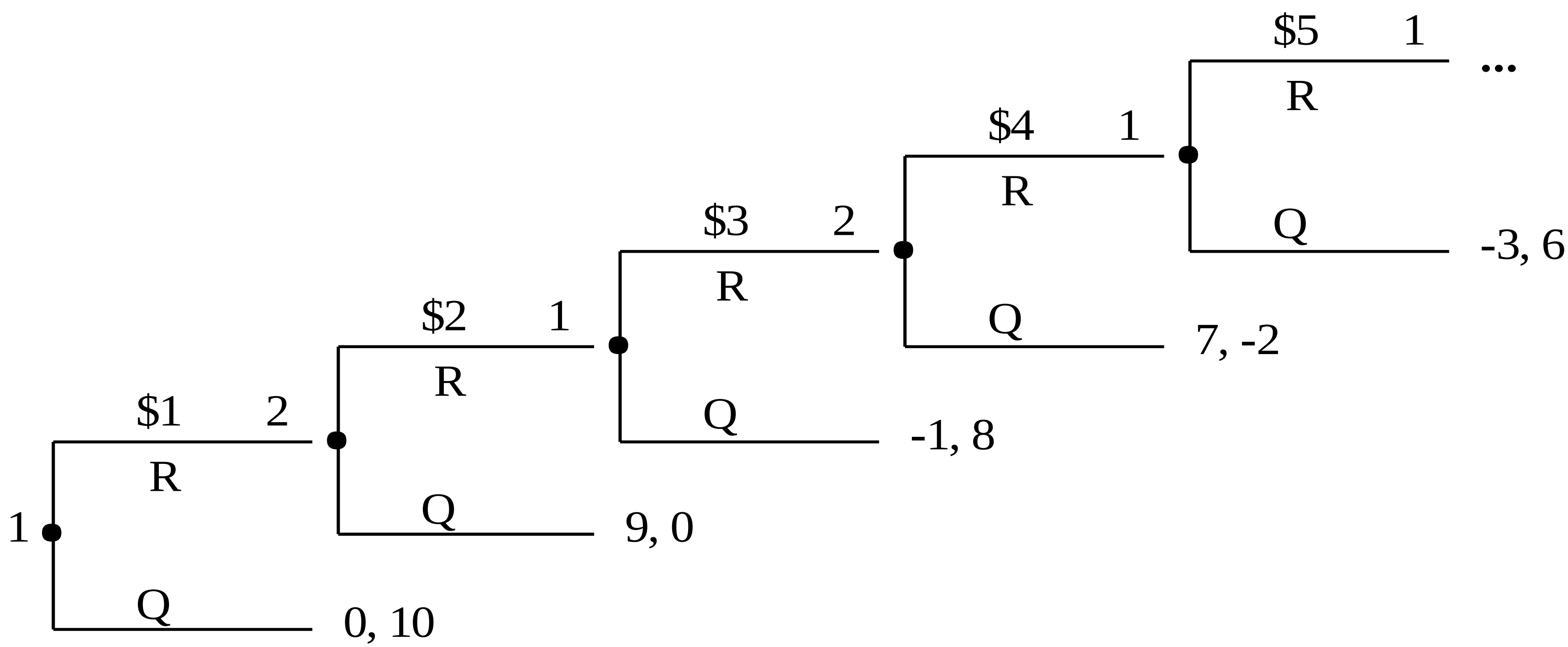
- Who plays when?
- What can they do?
- What do they know?
- What are the payoffs?

- ***Let we consider the geometrical representation***

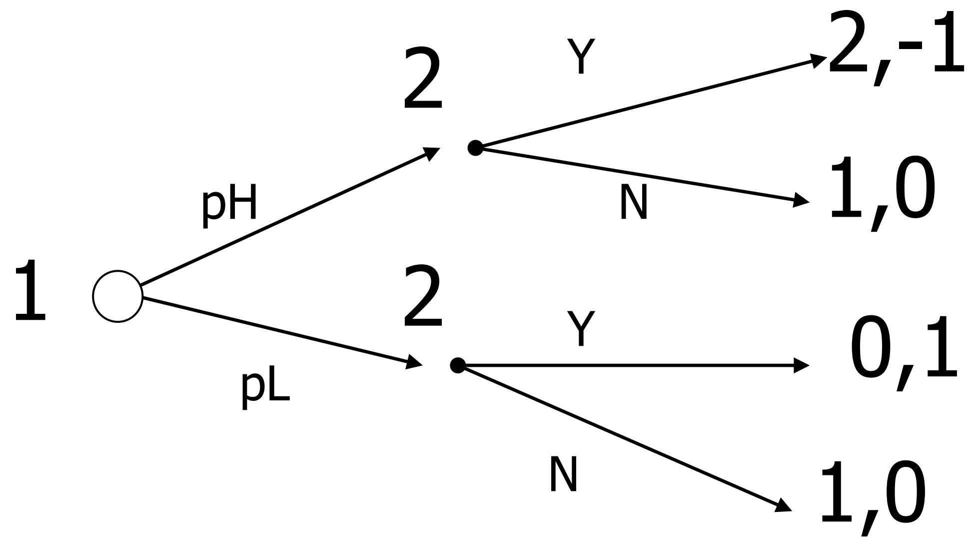
Hide and seek game



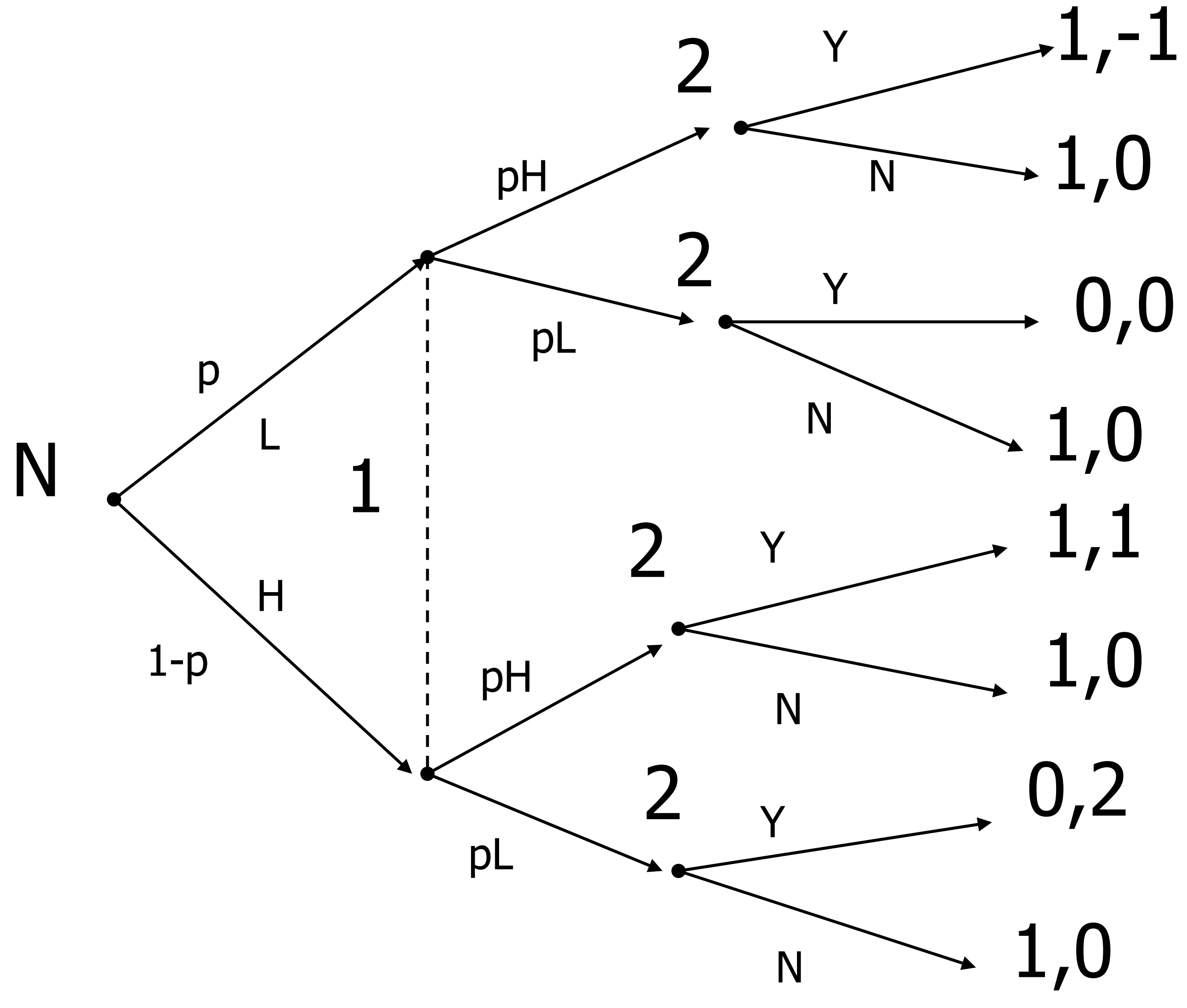
Geometric representation of both pay auction in Extensive Form (Game Tree)



Example 3: trade



Example 4: trade with asymmetric information



DEFINITION OF EXTENSIVE FORM GAME

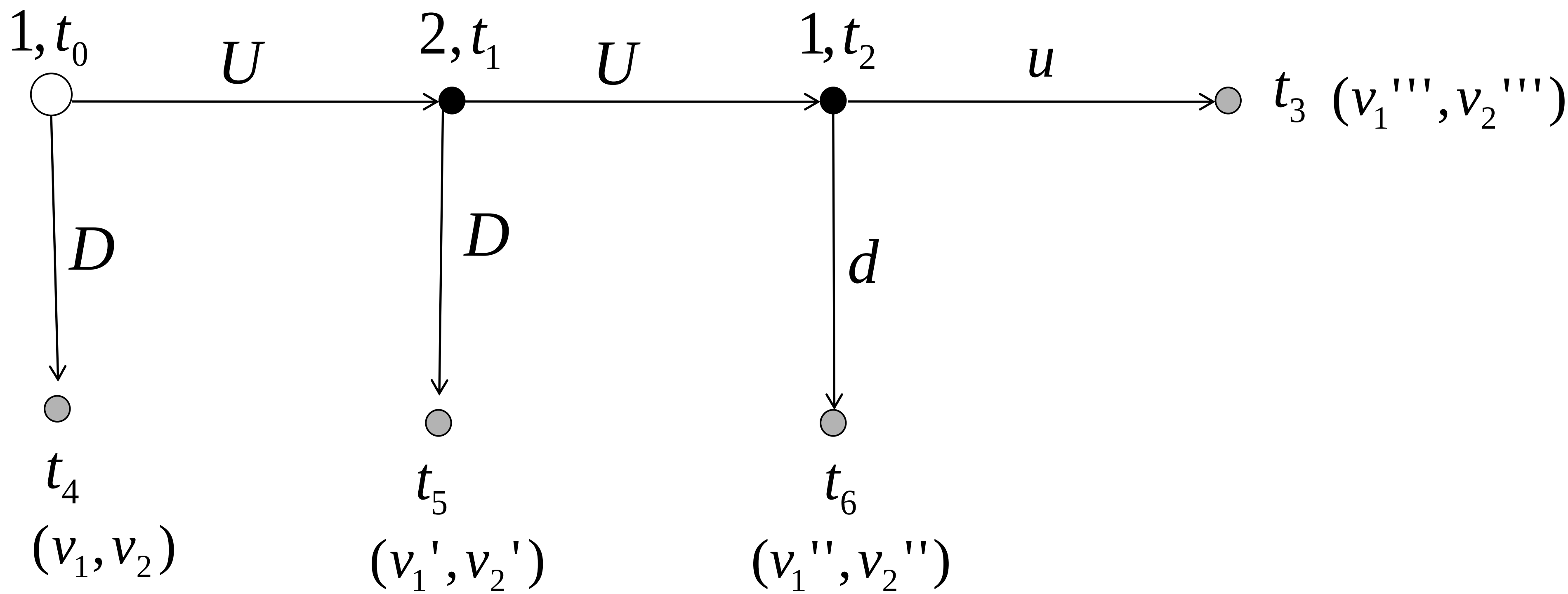
- The extensive form contains the following information:

1. The set of players
2. The order of moves
3. The players' payoffs as a function of the play
4. What the players' choices are when they move
5. What each player knows when it makes its choice
6. The probability distribution over any exogenous event.

SIMPLIFYING ASSUMPTION

- From now on, when not otherwise stated, we will assume **finite** games:
  - Finite number of players
  - Finite number of actions
  - Finite periods

Example of an extensive form game



Elements of an EFG:

1. Players (1, 2, …) and relation with
2. Nodes ($T = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$)
3. Ordering of nodes $t_0 \prec t_1 \prec t_2 \prec t_3, \;\; t_0 \prec t_1 \prec t_2 \prec t_6, \;\; t_0 \prec t_1 \prec t_5, \;\; t_0 \prec t_4$
4. Actions (*U*, *D*, *U*, *D*, *u*, *d*)
5. Connections between actions and nodes
6. Payoff and **information**

FORMAL DEFINITION OF EXTENSIVE FORM GAME (1)

- An extensive form game is the following collection:

$$E = \{N; T, \prec; A, \alpha; \iota; H_i; \rho; v_i\}$$

- where

1. *N* is the finite **set of players** (*n* is the number of players)
2. *T* is a **set of nodes**, that together with the binary relation $\prec$ on T represents **precedence** and form an arborescence, i.e. <u>it totally orders the predecessors of each member of T</u>:

   - ***<u>it means that each node can be reached by one and only one path</u>***

FORMAL DEFINITION OF EXTENSIVE FORM GAME (2)

- Using the **tree** $T, \prec$ we can define:
  - Predecessors of x $\in T$ $P(x) := \{t \in T \mid t \prec x\}$
  - Immediate predecessor of x $p_1(x) := \max\{t \in T \mid t \prec x\}$
  - n-th predecessor of x $p_n(x) := p_1 p_{n-1}((x))$ with $p_0(x) = x$
  - Immediate successors of x $S(x) := p_1^{-1}(x)$
  - Outcomes $Z := \{t \in T \mid S(t) = \emptyset\}$
  - Decision nodes $X := T \setminus Z$
  - Initial nodes $W := \{t \in T \mid P(t) = \emptyset\}$
  - Terminal successors of x $Z(x) := \{z \in Z \mid x \prec z\}$

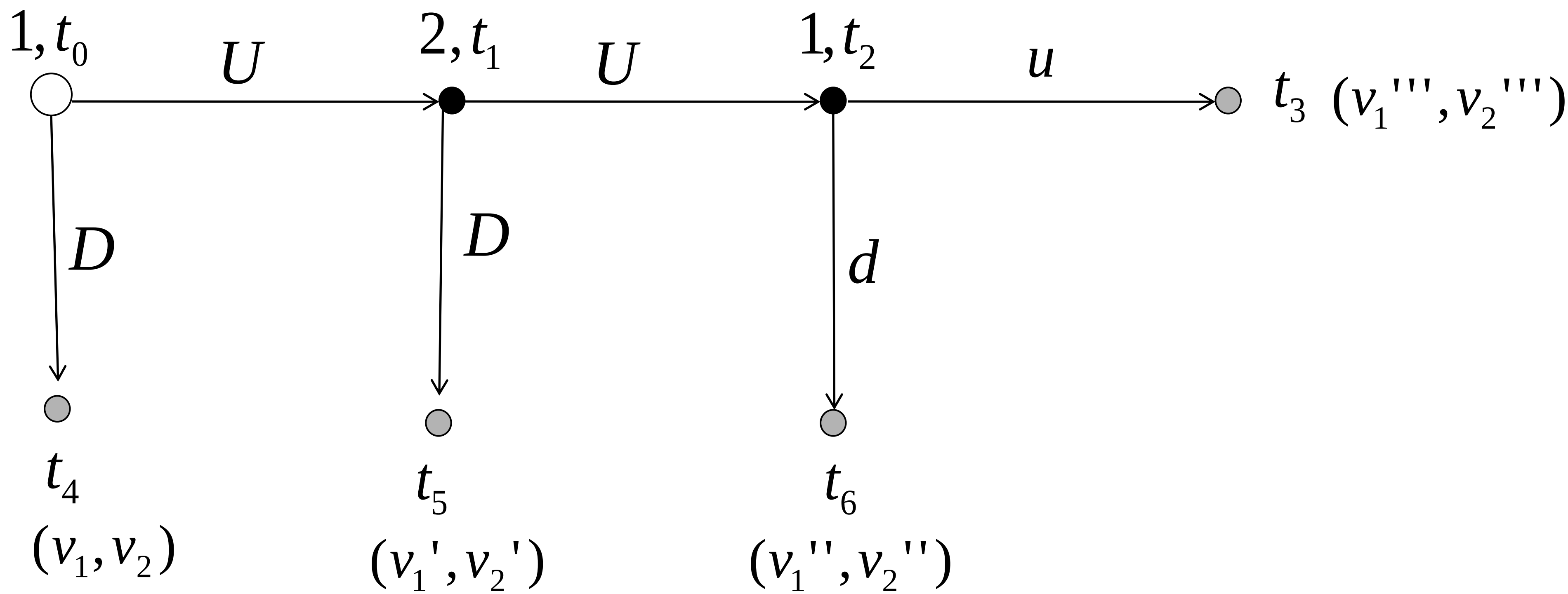
FORMAL DEFINITION OF EXTENSIVE FORM GAME (3)

- A is the set of **actions** and $\alpha : T \setminus W \to A$ is a **function** that labels each non-initial node with the **last action taken** to reach it.
  - This function is assumed to be injective
  - $\alpha(S(x))$ is the set of feasible actions at x.
- $\iota : X \to N$ represents the rules for determining **whose move it is** at a decision nodes x

FORMAL DEFINITION OF EXTENSIVE FORM GAME (4)

- **Information** is represented by a partition H of X that divides the decision nodes into **information sets.**
  - A cell $H(x) \in H$ contains the nodes that the player cannot distinguish from x: $H(x) \subseteq X$
  - We require $x \in H(x') \Rightarrow \iota(x) = \iota(x') \,\&\, \alpha(S(x)) = \alpha(S(x'))$
- $\rho \in \Delta(W)$ is a **probability distribution** on initial nodes
- $v_i : Z \to \Re$ is the **utility** function of player i.

Example of an extensive form game -1

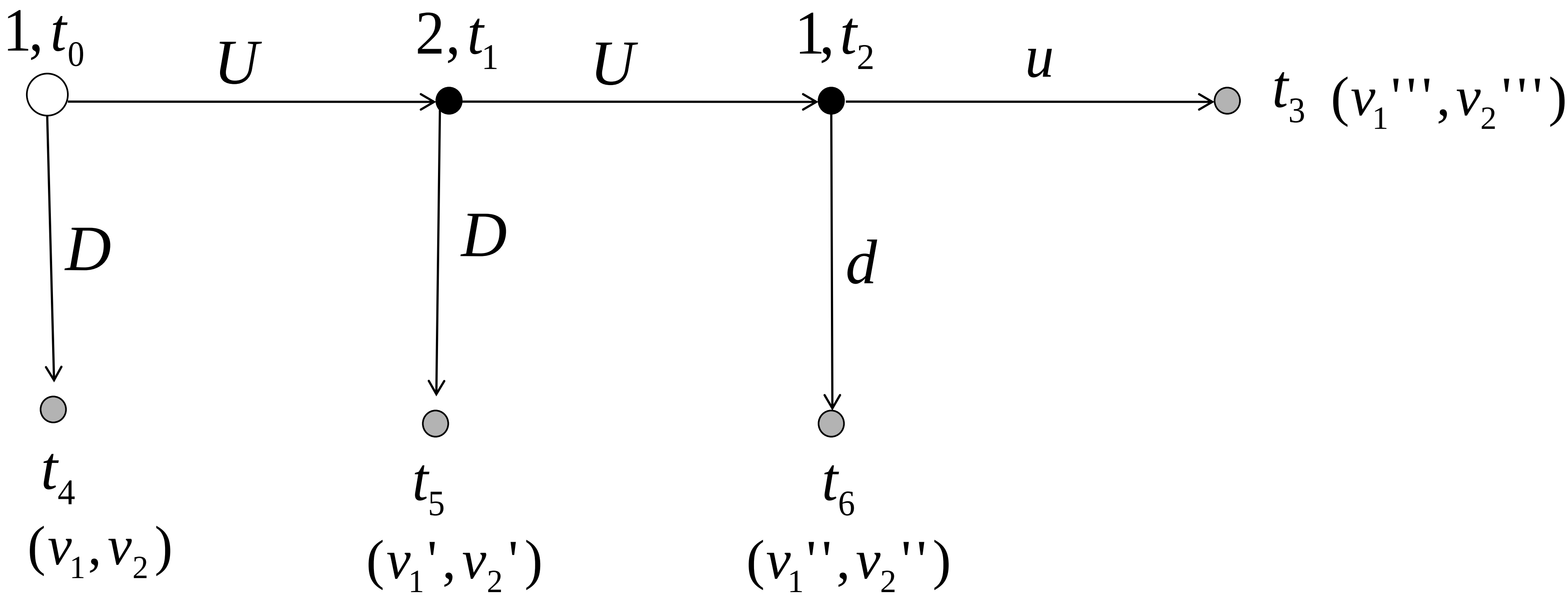


1. N = {1, 2} , $n = 2$
2. $T = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$

$$t_0 \prec t_1 \prec t_2 \prec t_3, \quad t_0 \prec t_1 \prec t_2 \prec t_6, \quad t_0 \prec t_1 \prec t_5, \quad t_0 \prec t_4$$

3. A = {U, D, U, D, u, d }

Example of an extensive form game -2

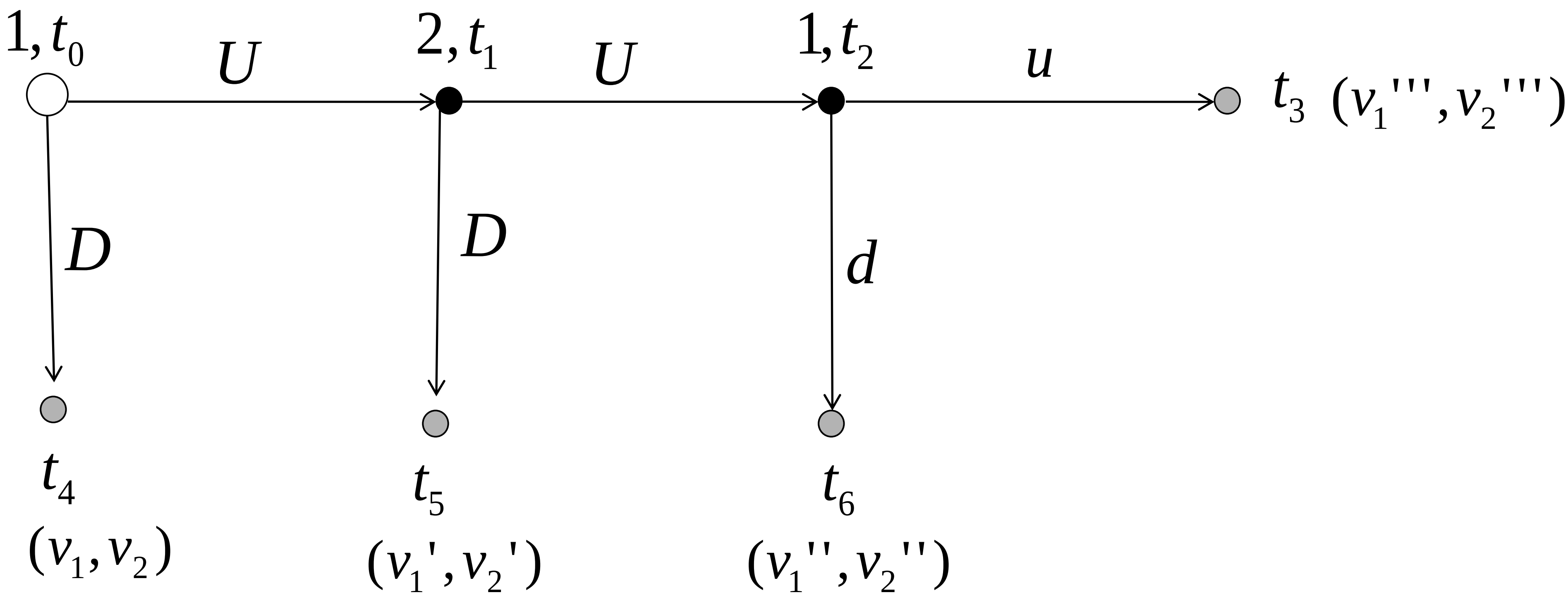


4. $$\alpha(t) = \begin{cases} U & t = t_1 \\ U & t = t_2 \\ u & t = t_3 \\ D & t = t_4 \\ D & t = t_5 \\ d & t = t_6 \end{cases}$$

5. $$\iota(t) = \begin{cases} 1 & t = t_0 \\ 2 & t = t_1 \\ 1 & t = t_2 \end{cases}$$

6. $$H_1 = \{\{t_0\}, \{t_2\}\}$$
$$H_2 = \{\{t_1\}\}$$

Example of an extensive form game -3



7. $\rho(\{t_0\}) = 1$

8. $$v(t) = \begin{cases} (v_1, v_2) & t = t_4 \\ (v_1', v_2') & t = t_5 \\ (v_1'', v_2'') & t = t_6 \\ (v_1''', v_2''') & t = t_3 \end{cases}$$

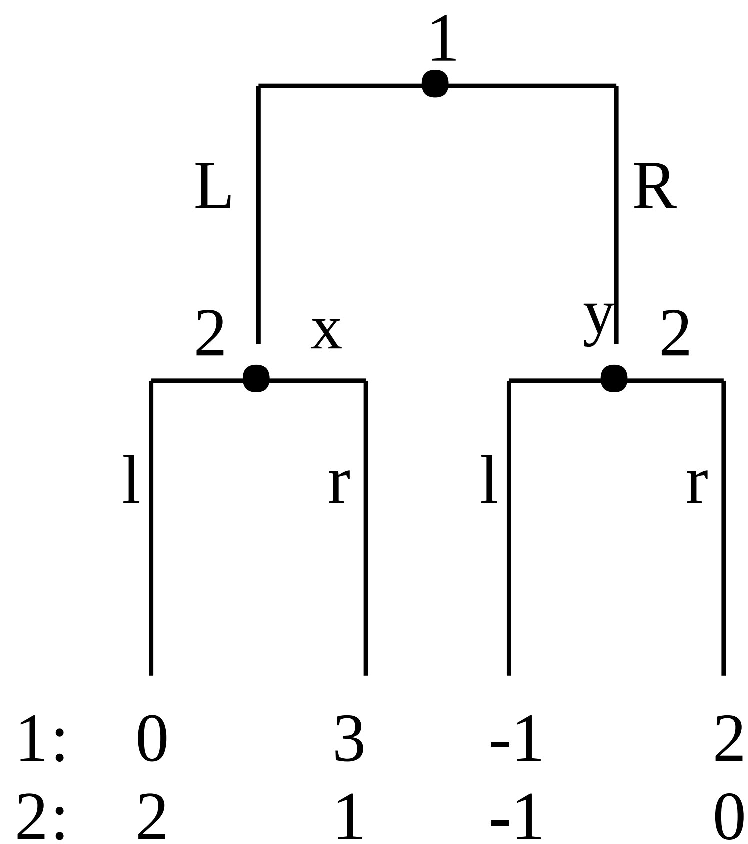
INFORMATION IN A GAME

INFORMATION IN A GAME

- ***Information Set*:** for player *i* is a collection of decision nodes satisfying two conditions:
    1. player i has the move at every node in the collection, and
    2. *i* doesn't know which node in the collection has been reached.

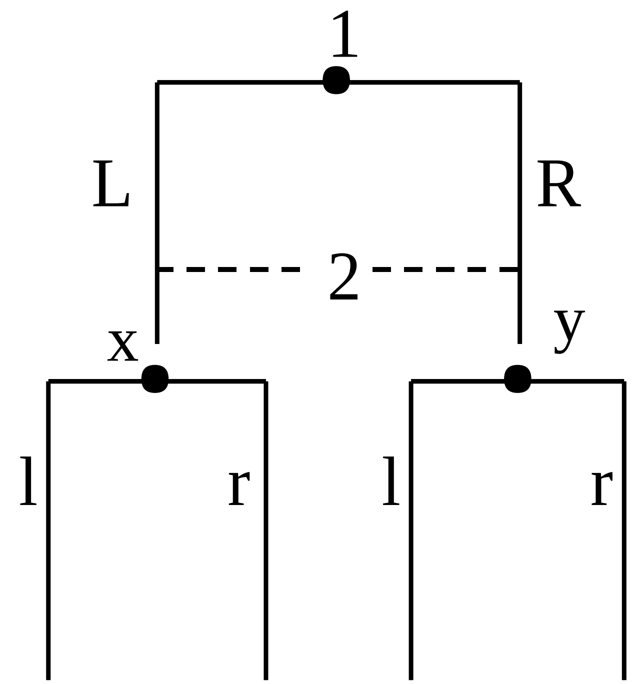
- ***Meaning*:**
    - **the histories/sequence of actions that lead to the nodes in an information set are not distinguishable for player *i***

How many information sets?



$$\{x\} = h'_2 \in H_2, \{y\} = h''_2 \in H_2, H_2 = \{\{x\}, \{y\}\}$$

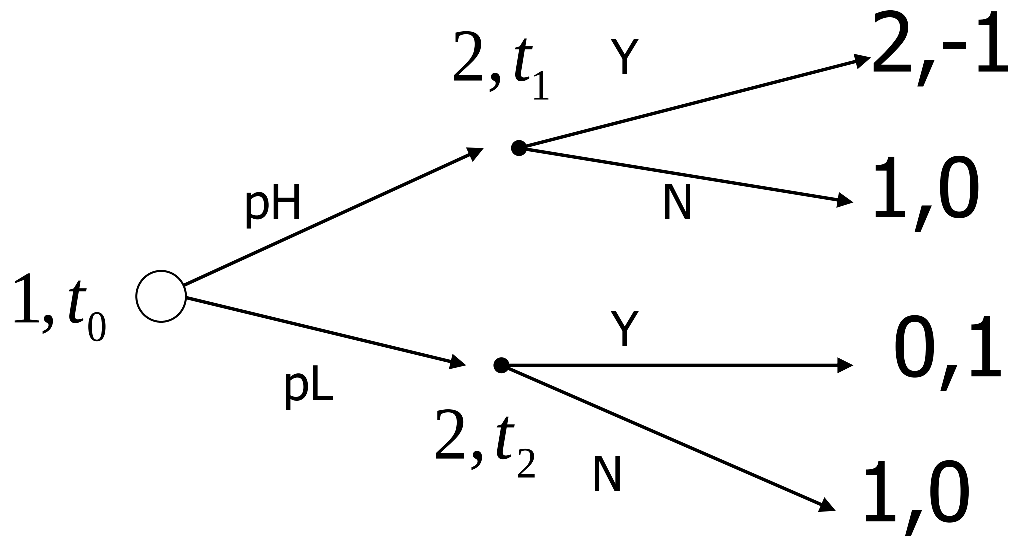
How many information sets?



1:	0	3	-1	2
2:	2	1	-1	0

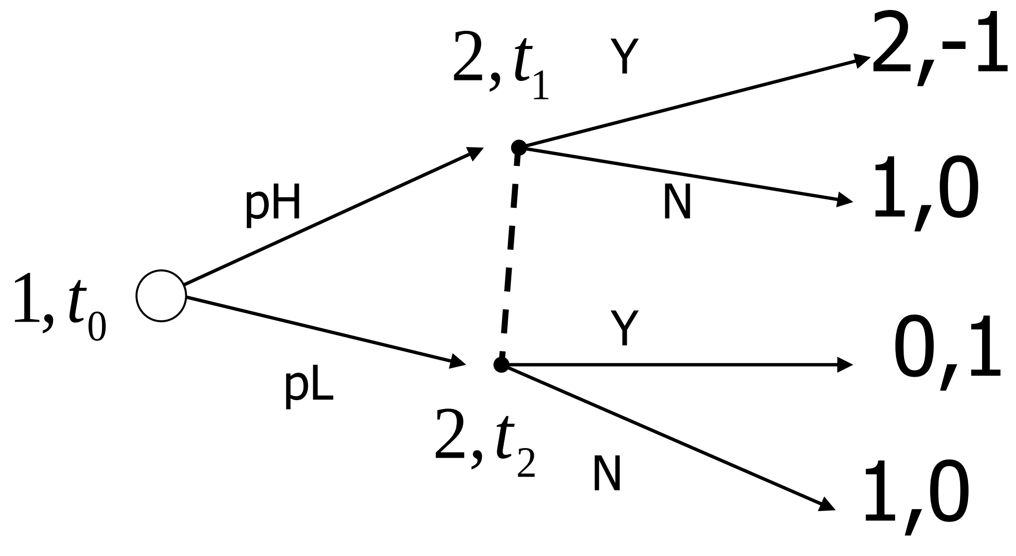
$$\{x, y\} = h_2 \in H_2 \Leftrightarrow x \in h(y), H_2 = \{\{x, y\}\}$$

Example: trade with perfect information



$$H_1 = \{\{t_0\}\}, \quad H_2 = \{\{t_1\},\{t_2\}\}$$

Example: simultaneous trade



$$H_1 = \{\{t_0\}\}, \quad H_2 = \{\{t_1, t_2\}\}$$

More Definitions

- *Perfect Information*: each information set is a single node (Chess, checkers, go, ...)
  - Player *i* has perfect information iff

$$\forall h_i \in H_i \quad |h_i| = 1$$

  - If all players have perfect information, then the game is a game of perfect information

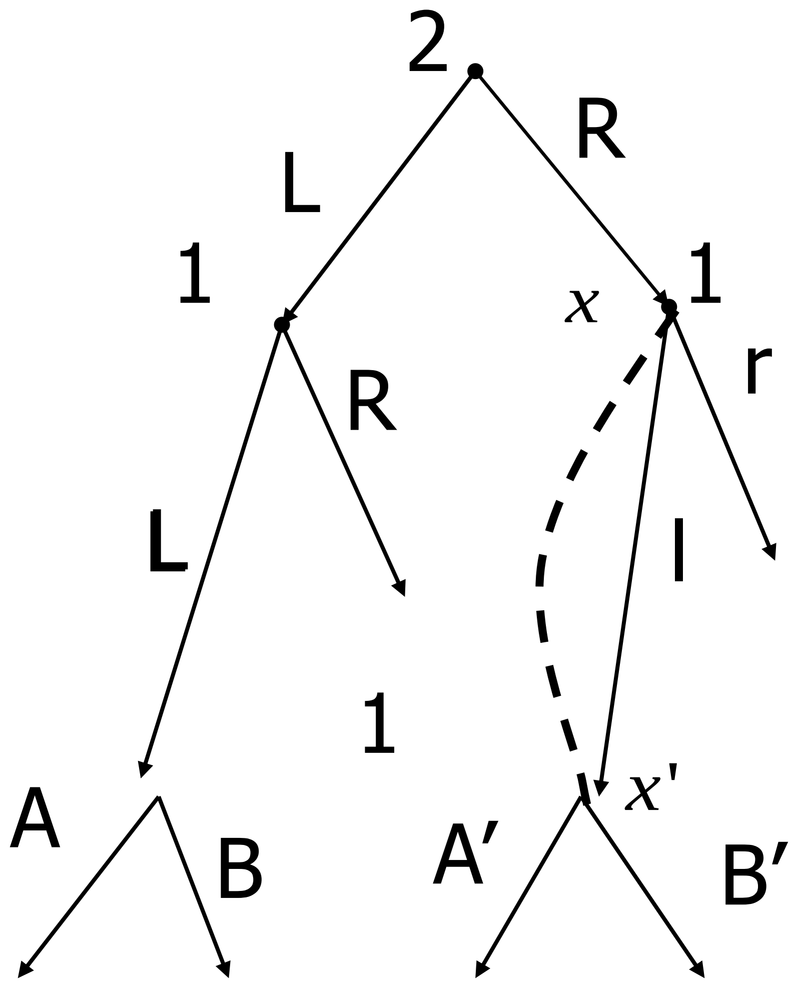
$$\forall i \in N,\ \forall h_i \in H_i \quad |h_i| = 1$$

- *Imperfect Information*: at some point in the tree some player is not sure of the complete history of the game so far

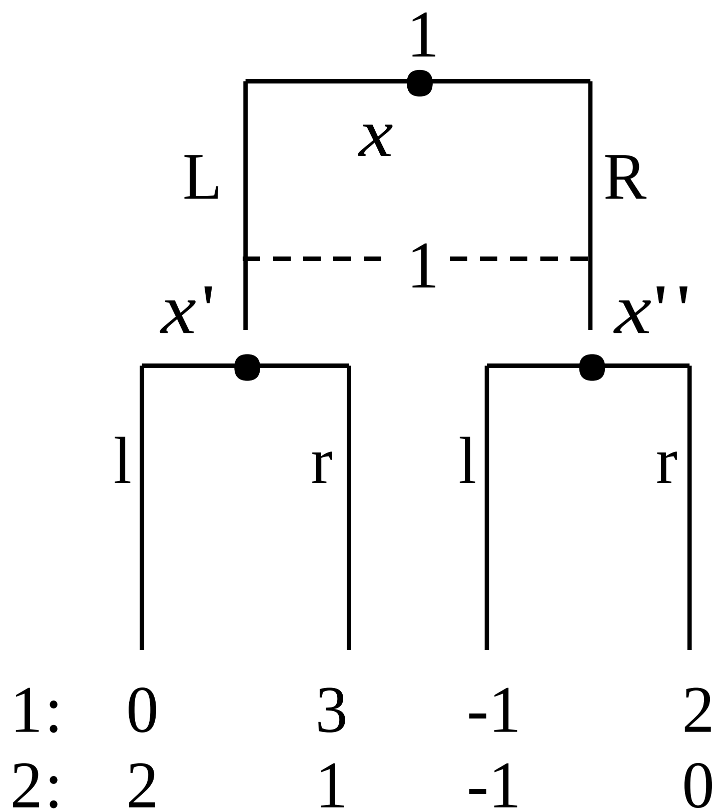
Further Definition: **Perfect Recall**

- A game has *perfect recall* if each player remember
  1. whatever he knew previously,
  2. including his previous actions.

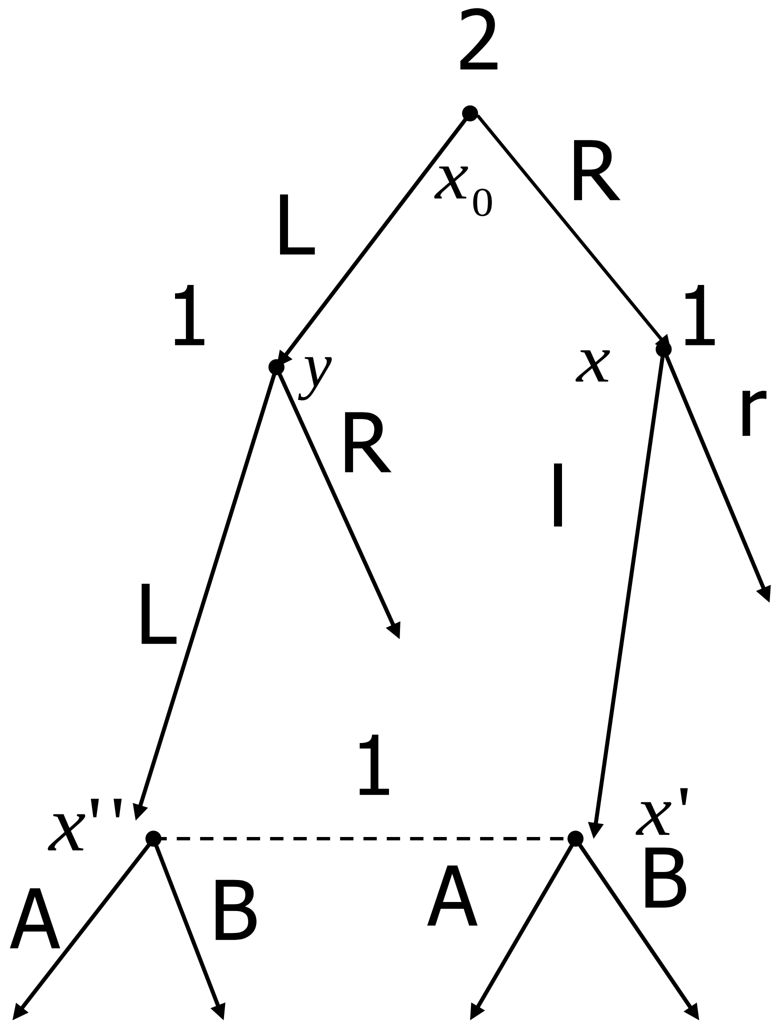
Example 1: Player 1 forget what she chose



Example 2 of imperfect recall: Player 1 forget its own move



Example 3 of imperfect recall:
Player 1 forget what she knew



Strategies

Further Definitions - 1:

- ***Strategy***: a ***complete*** plan of action (what to do in *every* contingency):

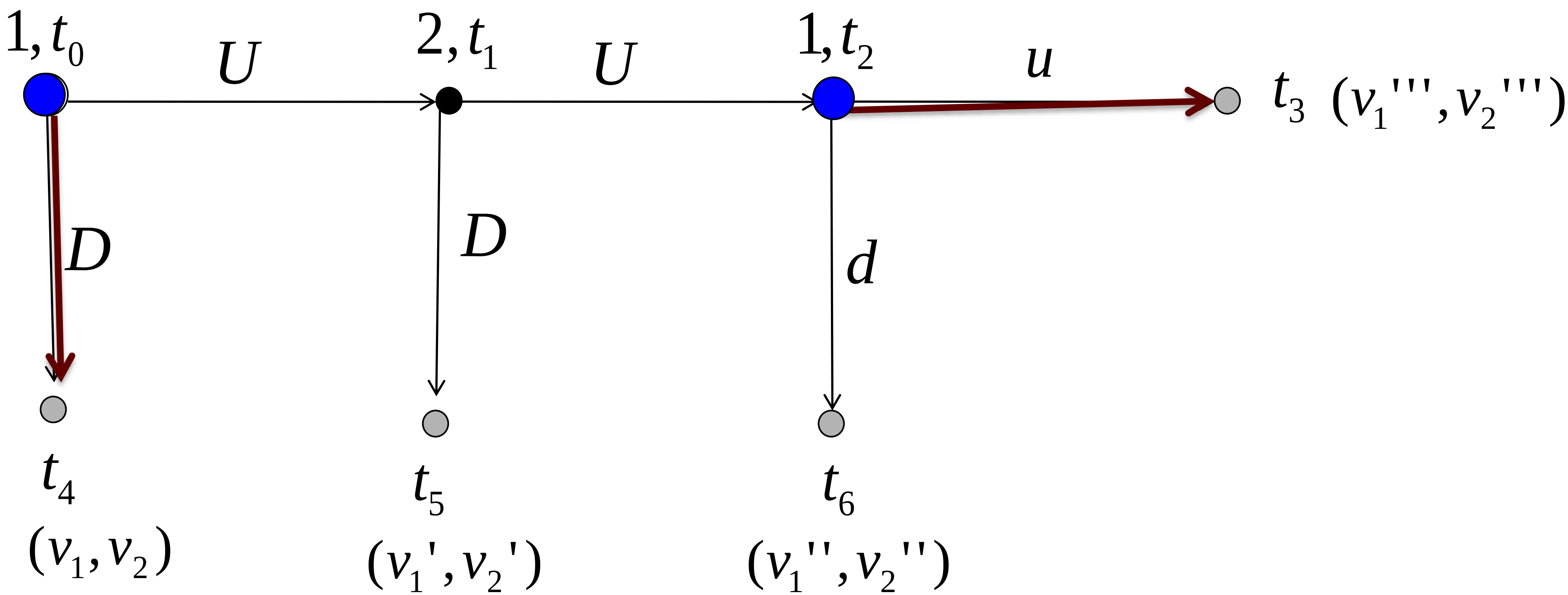
$$s^i: H^i \rightarrow A \text{ such that}$$

$$s^i(h^i) \in A(h^i) \text{ for any } h^i \in H^i$$

where:

- H^i is the collection of i's information sets
- A is the set of possible actions
- $A(h^i)$ is the set of actions feasible in h^i

Example of a strategy for player 1



$$H_1 = \{\{t_0\}, \{t_2\}\}$$

$$s^1(h) = \begin{cases} D & h = \{t_0\} \\ u & h = \{t_2\} \end{cases}$$

Further Definitions - 2:

- ***Set of Strategy***: under our assumption of finiteness the **set of pure strategy** for player *i* is

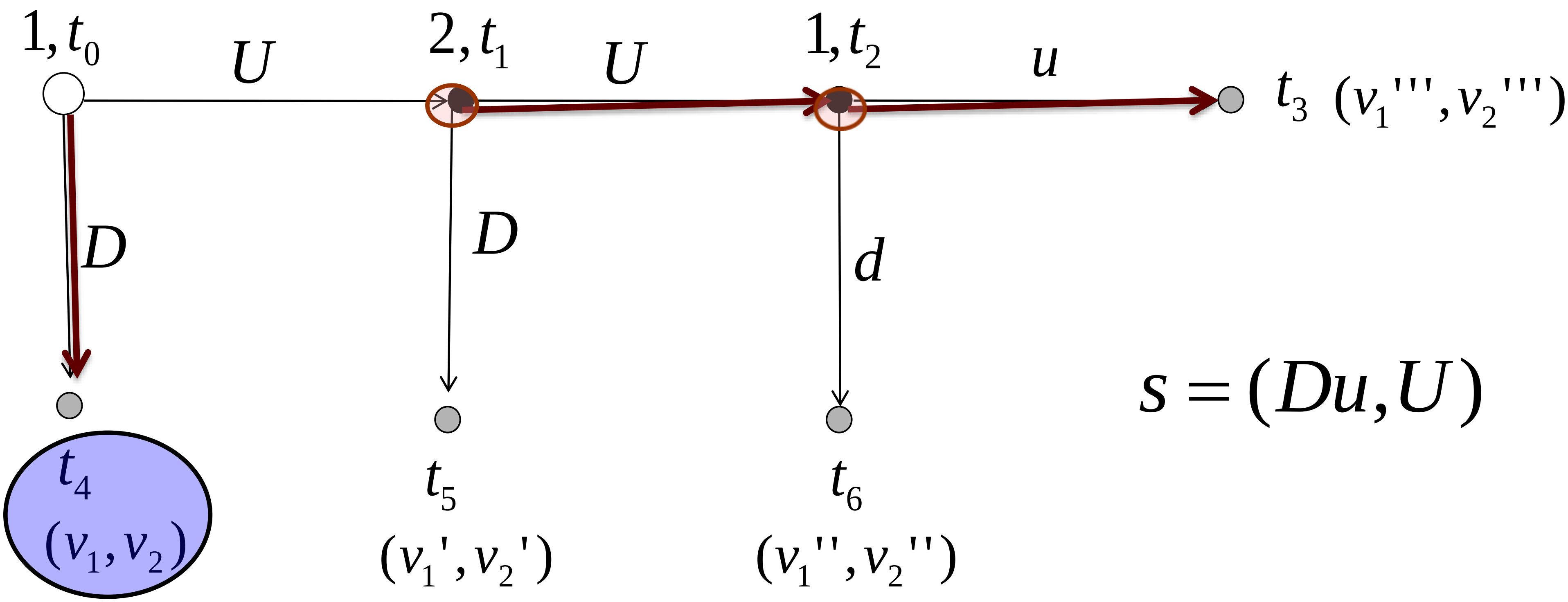
$$S_i = \prod_{h_i \in H_i} A(h_i)$$

- Similarly, the set of strategy profiles is

$$s \in S = \prod_{i \in N} S_i$$

where *s* is a vector

Example of a strategy set and of a strategy profile



$$s = (Du, U)$$

$$S^1 = \{U, D\} \times \{u, d\} = \{(U, u), (U, d), (D, u), (D, d)\}$$

$$S^2 = \{U, D\}$$

Mixed Strategies

- ***Mixed strategy***: a randomization over pure strategies:

  $\sigma^i: S^i \rightarrow [0,1]$

  where $\sigma^i(s^i)$ = Pr(i plays pure strategy s^i).
- The set of mixed strategy of player i is $\Delta(S^i)$
- Mixed Strategy Profile

  $\sigma = \{\sigma^1,\ldots,\sigma^n\} \in \Delta(S^1)\times\ldots\times\Delta(S^n)$

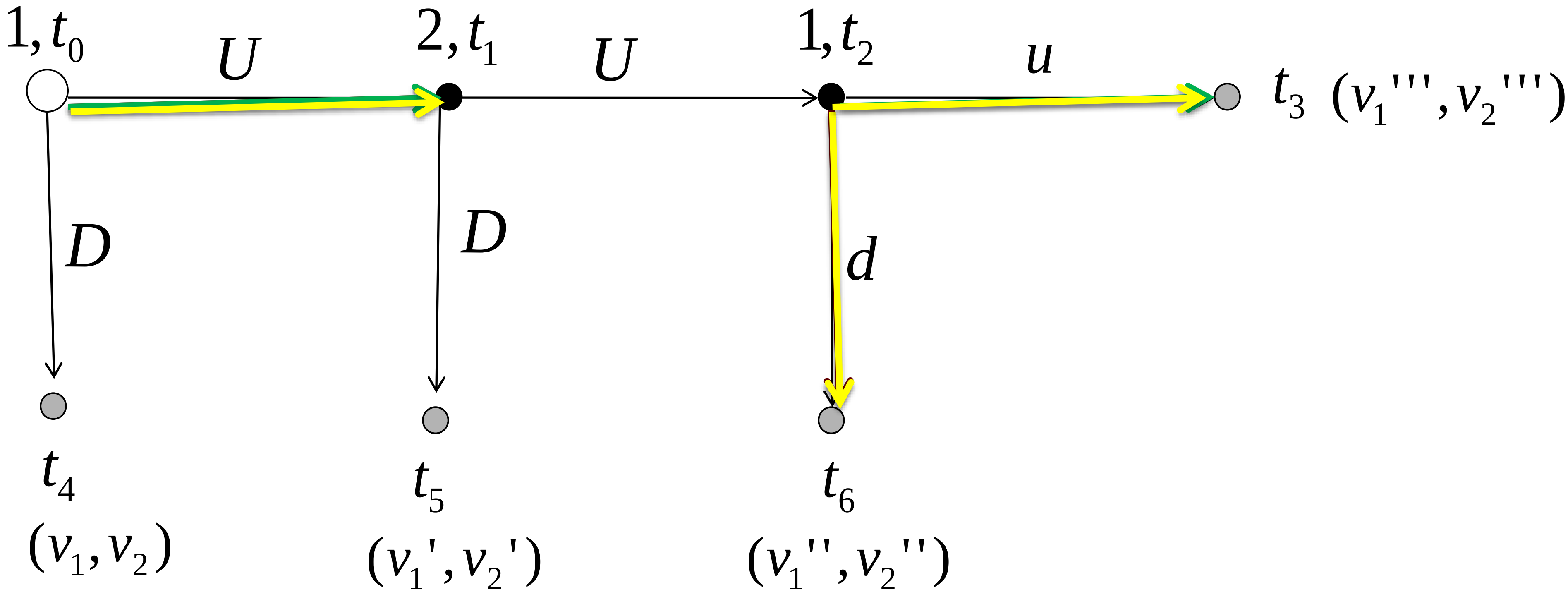
Behavioral strategies

- A ***behavioral strategy*** specifies a probability distribution over feasible actions at each information set.

$$b^i: H^i \rightarrow \Delta(A) \text{ such that}$$

$$b^i(h^i) \in \Delta(A(h^i)) \text{ for any } h^i \in H^i$$

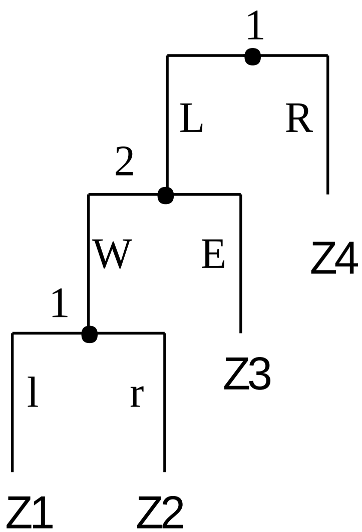
Example of a mixed and of a behavioral strategy



$$\sigma^1(\{Ud\}) = 1/2 \qquad \sigma^1(\{Uu\}) = 1/2$$

$$\pi^1(\{U\} \mid t_0) = 1 \qquad \pi^1(\{u\} \mid t_2) = 1/2$$

EXAMPLE 1



$$S_1 = \{Ll, Lr, Rl, Rr\}$$

$$S_2 = \{W, E\}$$

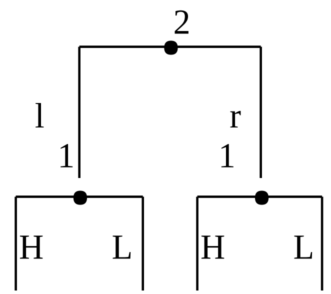
$$\Sigma_1 = \left\{ p_1, p_2, p_3, p_4 \mid p_i \in [0,1] \,\&\, \sum_{i=1}^{4} p_i = 1 \right\}$$

$$\Sigma_2 = \left\{ q_1, q_2 \mid q_i \in [0,1] \,\&\, \sum_{i=1}^{2} q_i = 1 \right\}$$

$$B_1 = \{b_1(\bullet), b_2(\bullet)\}$$

$$B_2 = \Sigma_2$$

Example 2



$$S_1 = \{HH, HL, LH, LL\}$$

$$\Sigma_1 = \left\{p_1, p_2, p_3, p_4 : p_i \geq 0 \,\&\, \sum_{i=1}^{4} p_i = 1\right\}$$

$$B_1 = \left\{p, q : p = \Pr\{H \mid l\}, q = \Pr\{H \mid r\}\right\}.$$

PROBLEM

- Mixed and behavioral strategies are different objects
- Set of mixed strategies

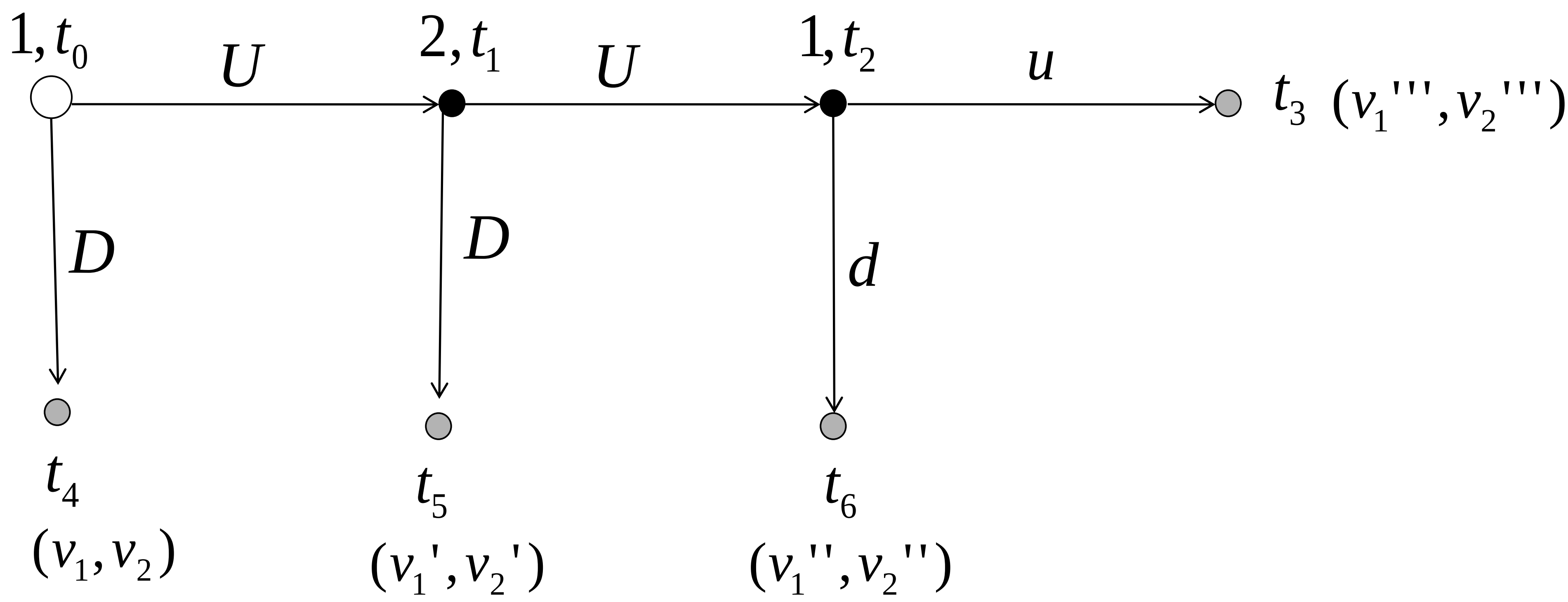
$$\Sigma_i := \Delta(S_i) = \Delta(\prod_{h \in H_i} A(h))$$

- Set of behavioral strategies

$$B_i := \prod_{h \in H_i} \Delta(A(h))$$

- Different sets, mixed seems more general
- **What is the relationship between mixed and behavioral strategies?**
- To answer we need the notion of **outcome equivalence**

Mixed and behavioral strategies



	U	**D**	
u	$\sigma^1(\{Uu\})$	$\sigma^1(\{Du\})$	$\pi^1(\{u\} \mid t_1)$
d	$\sigma^1(\{Ud\})$	$\sigma^1(\{Dd\})$	$\pi^1(\{d\} \mid t_1)$
	$\pi^1(\{U\} \mid t_0)$	$\pi^1(\{D\} \mid t_0)$	

Relationship between mixed and behavioral strategies

- Clearly mixed strategies are more general than behavioral strategies because allow “correlation” among information sets in the sense of **probabilities of vectors** instead of **vectors of probabilities**:

$$\Sigma_i = \Delta\left(\prod_{h_i \in H_i} A(h_i)\right) \text{ versus } \Pi_i = \prod_{h_i \in H_i} \Delta\big(A(h_i)\big)$$

- So, what can we conclude?

OUTCOME EQUIVALENCE

- ***DEFINITION:***

- ***Two strategy profiles are*** ***outcome equivalent***
- ***if and only if they induce the same probability on the set of final nodes.***

Expected Payoffs

- For players' behavior only **outcomes** (probability) matter, since we have

$$v_i(z)$$

- From behavioral and mixed strategy profiles we obtain

$$P(z \mid b) \text{ and } P(z \mid \sigma)$$

- Expected payoff with mixed strategy:

$$v_i(\sigma) := \sum_{z \in Z} v_i(z) P(z \mid \sigma)$$

- Expected payoff with behavioral strategy:

$$v_i(b) := \sum_{z \in Z} v_i(z) P(z \mid b)$$

OUTCOME-EQUIVALENCE

- b and σ are **outcome equivalent** if and only if

$$P(z\,|\,b) = P(z\,|\,\sigma) \quad \forall z \in Z$$

- How to evaluate

$$P(z\,|\,\sigma) \quad \text{and} \quad P(z\,|\,b)?$$

- It's the probability of all the strategies that gives rise to outcome z

Strategies and outcomes: outcome function

- **The rules of a game are such that for each strategy profile, there exists a unique outcome**.
- Therefore for each extensive game, we can define an
- **OUTCOME FUNCTION:**

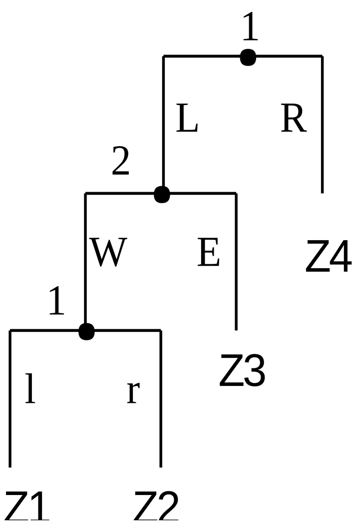
$$\varsigma : S \rightarrow Z$$

- Thus:

$$P(z \mid \sigma) = \sum_{\{s \mid \varsigma(s)=z\}} \sigma(s)$$

- **while *P(z|b)* is evaluated through multiplication of actions' probabilities to reach *z, i.e. along the outcome path*.**

EXAMPLE 1:
outcome function



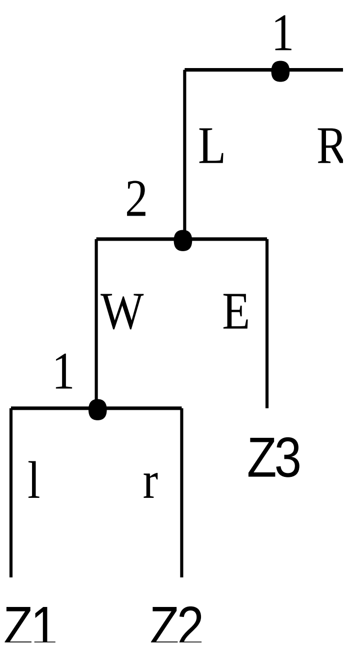
$$\varsigma(s)=\begin{cases} z_1 & se \quad s=(Ll,W) \\ z_2 & se \quad s=(Lr,W) \\ z_3 & se \quad s=(Ll,E),(Lr,E) \\ z_4 & se \quad s=(Rl,W),(Rr,W),(Rl,E),(Rr,E) \end{cases}$$

$$S_1=\{Ll,Lr,Rl,Rr\} \quad S_2=\{W,E\}$$

$$S=S_1\times S_2=\{Ll,Lr,Rl,Rr\}\times\{W,E\}=$$

$$=\{(Ll,W),(Ll,R),(Lr,W),(Lr,E),(Rl,W),(Rl,E),(Rr,W),(Rr,E)\}$$

Probabilities of outcomes using mixed strategy profiles



$$\sigma_1 = \Pr(Rl) = \Pr(Ll) = 1/2$$

$$\sigma_2 = \Pr(W) = 1$$

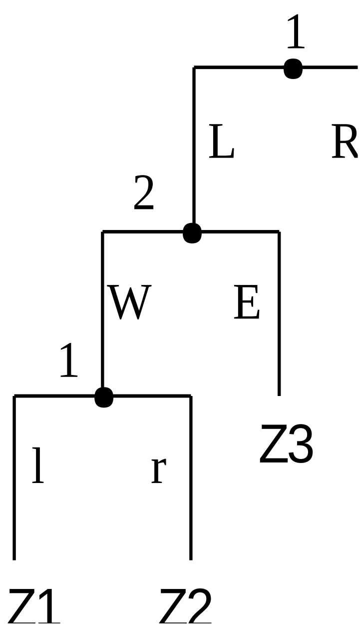
$$\Pr(z_1 \mid \sigma) = \Pr(Ll) \times \Pr(W) = 1/2 \times 1 = 1/2$$

$$\Pr(z_2 \mid \sigma) = \Pr(Lr) \times \Pr(W) = 0 \times 1 = 0$$

$$\Pr(z_3 \mid \sigma) = \Pr(Ll) \times \Pr(E) + \Pr(Lr) \times \Pr(E) = 0$$

$$\Pr(z_4 \mid \sigma) = \Pr(Rl) \times \Pr(W) + \Pr(Rr) \times \Pr(W) + \\ + \Pr(Rl) \times \Pr(E) + \Pr(Rr) \times \Pr(E) = 1/2.$$

Probabilities of outcomes using behavioral strategy profiles



$$\pi^1 = \begin{cases} \Pr(L) = 1/2 \\ \Pr(l) = 1/2 \end{cases} \qquad \pi^2 = \Pr(W) = 1$$

$$\Pr(z_1 \mid b) = \Pr(L) \times \Pr(W) \times \Pr(l) = 1/4$$

$$\Pr(z_2 \mid b) = \Pr(L) \times \Pr(W) \times \Pr(r) = 1/4$$

$$\Pr(z_3 \mid b) = \Pr(L) \times \Pr(E) = 0$$

$$\Pr(z_4 \mid b) = \Pr(R) = 1/2.$$

Relationship between mixed and behavioral strategies

- Clearly mixed strategies are more general than behavioral strategies because allow "correlation" among information sets in the sense of **probabilities of vectors** instead of **vectors of probabilities**:

$$\Sigma_i = \Delta\left(\prod_{h_i \in H_i} A(h_i)\right) \text{ versus } \Pi_i = \prod_{h_i \in H_i} \Delta\big(A(h_i)\big)$$

- Thus it is not surprising that for any behavioral strategy profile there exists an **outcome equivalent** mixed strategy profile:

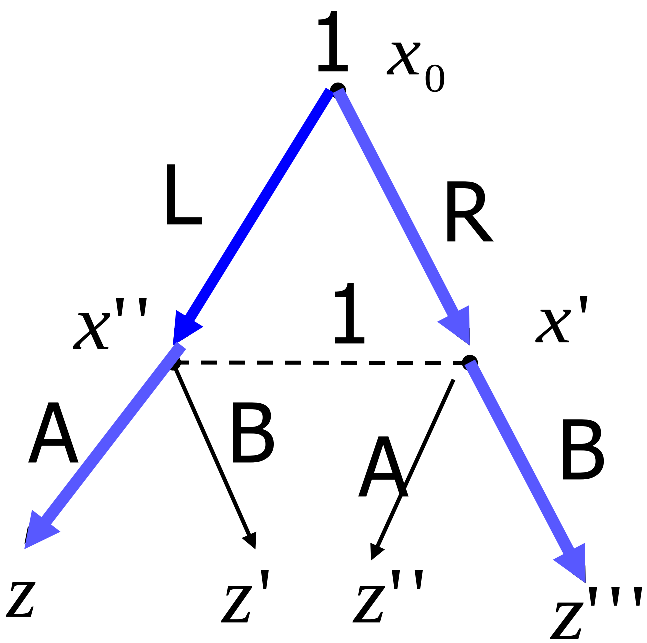
$$\sigma_i(s_i)[b] := \underset{h \in H_i}{\times} b_i\big(s_i(h) \mid h\big).$$

- What about the other way ?

KUHN'S THEOREM

For any mixed strategy profile in a finite extensive game with **perfect recall** there is an outcome-equivalent behavioral strategy profile.

Game with imperfect recall where there exists a mixed strategy not equivalent to a behavioral



$$\sigma_1 = \Pr(LA) = \Pr(RB) = 1/2$$

Probabilities of outcomes using σ_1

$$\Pr(z \mid \sigma_1) = \Pr(LA) = 1/2 \qquad \Pr(z' \mid \sigma_1) = \Pr(LB) = 0$$

$$\Pr(z'' \mid \sigma_1) = \Pr(RA) = 0 \qquad \Pr(z''' \mid \sigma_1) = \Pr(RB) = 1/2$$

Probabilities of outcomes using behavioral strategies

$$\Pr(z \mid \pi_1) = \pi_1(L) \times \pi_1(A) = 1/2 \Rightarrow \pi_1(L) > 0 \,\&\, \pi_1(A) > 0$$

$$\Pr(z' \mid \pi_1) = \pi_1(L) \times \pi_1(B) = 0 \Rightarrow \pi_1(L) = 0 \quad \text{or} \quad \pi_1(B) = 0$$

$$\Pr(z'' \mid \pi_1) = \pi_1(R) \times \pi_1(A) = 0 \Rightarrow \pi_1(R) = 0 \quad \text{or} \quad \pi_1(A) = 0$$

$$\Pr(z''' \mid \pi_1) = \pi_1(R) \times \pi_1(B) = 1/2 \Rightarrow \pi_1(R) > 0 \,\&\, \pi_1(B) > 0$$

Contraddiction : $\neg \exists \pi_1$ satisfying such conditions

Strategic Form Games

Alternative representation of a game

- The extensive game is a detailed and thus complex representation of a strategic situation
- A simpler but more concise representation of strategic situations is the STRATEGIC FORM or NORMAL FORM

FORMAL DEFINITION OF STRATEGIC FORM GAME

1. Set of players $N = \{1,\ldots,n\}$
2. Set of strategies S_i
3. payoff function

- $u_i(s): S \rightarrow \Re,$

- which maps **strategy profiles**

$$s = (s_1,\ldots,s_n) \in S = S_1 \times \ldots \times S_n$$

into real numbers

- game in normal form

$$\Gamma = \{N, S_1,\ldots,S_{n,} u_1,\ldots,u_n\}$$

A Simple Normal Form Game in Matrix Form

Player 1 \ *Player 2*	Left	Right
Left	A, B	A, B
Right	C, D	E , F

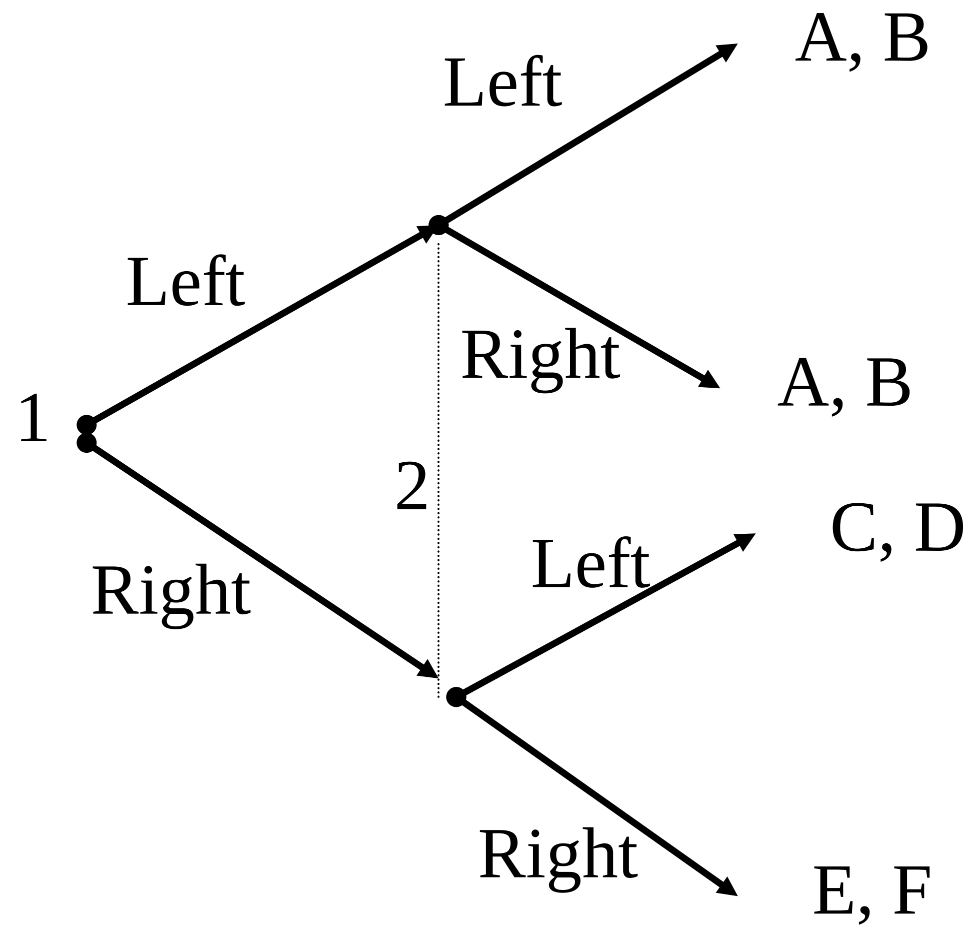
PROBLEM

- WHAT IS THE RELATION BETWEEN EXTENSIVE FORM AND STRATEGIC FORM GAMES ?

From NFG to EFG

Game 1 has the following EFG representation:

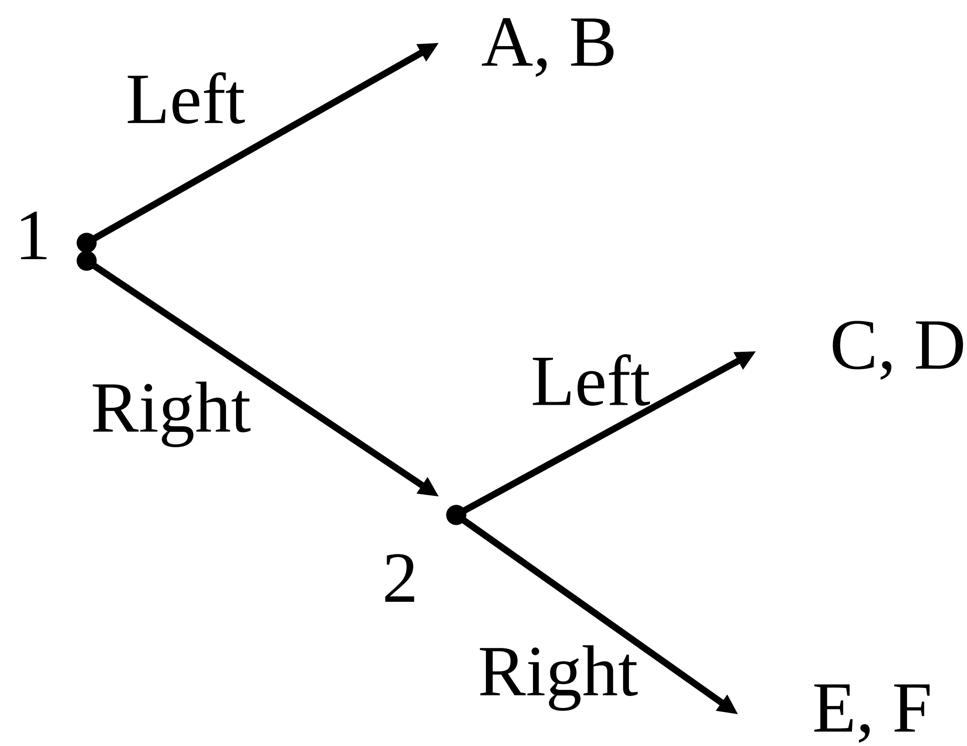
1 \ 2	Left	Right
Left	A, B	A, B
Right	C, D	E , F



From NFG to EFG

Also the following is an EFG representation of game 1:

1 \ 2	Left	Right
Left	A, B	A, B
Right	C, D	E , F



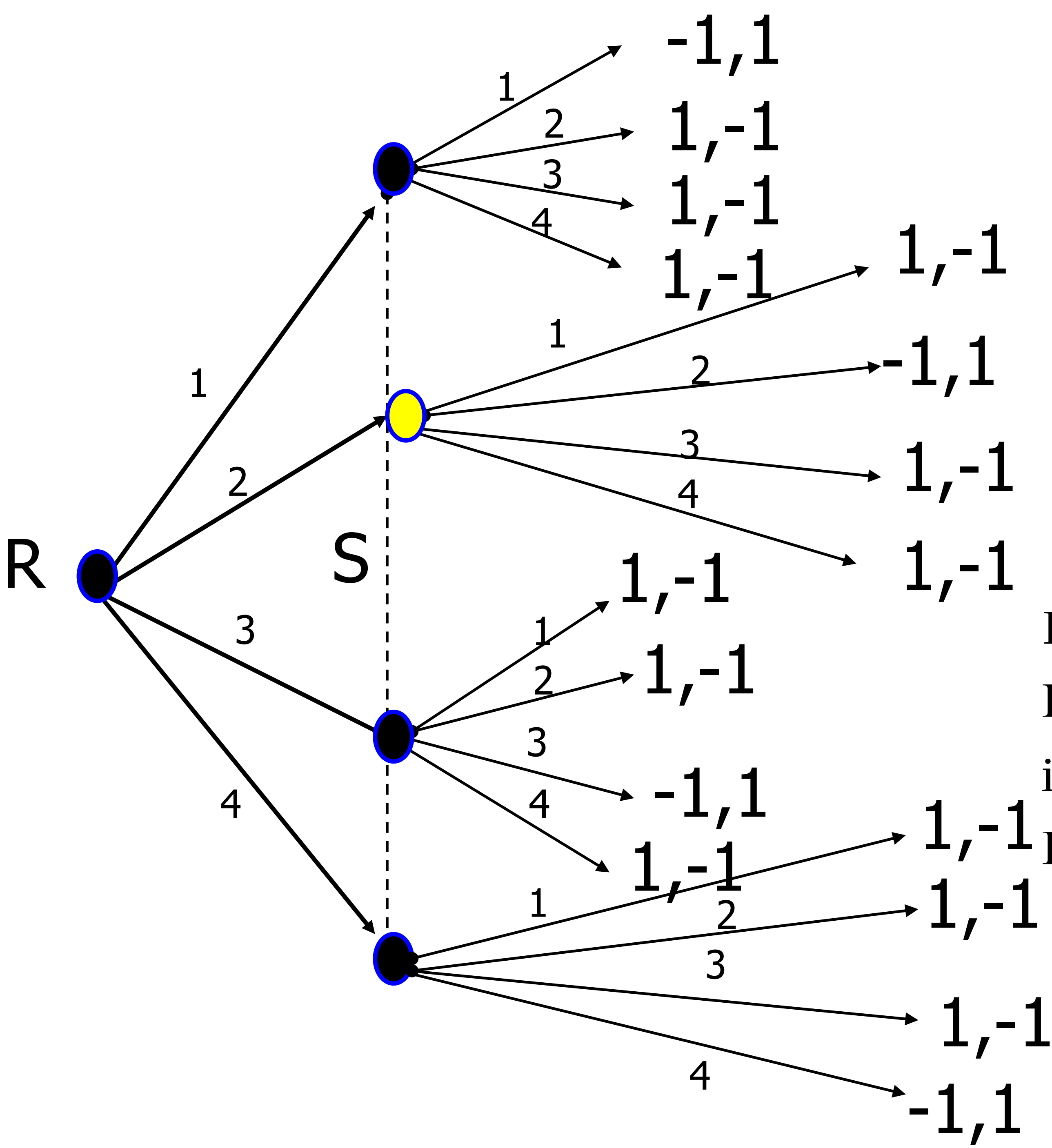
From EFG to NFG

- Use
  1. the previous definition of strategies to construct the set of pure strategies,
  2. the payoff functions are obtained combining the outcome function with v_i

$$u_i = v_i \bullet \varsigma. \qquad \text{In particular}$$

$$u_i(\sigma) = \sum_{s \in S} v_i(\varsigma(s)) \prod_{j=1}^{n} \sigma_j(s_j)$$

Hide and seek game



$S_1 = \{1,2,3,4\}$

$S_2 = \{1,2,3,4\}$

$$\Sigma_1 = \left\{ p_1, p_2, p_3, p_4 \mid p_i \in [0,1] \,\&\, \sum_{i=1}^{4} p_i = 1 \right\}$$

$$\Sigma_2 = \left\{ q_1, q_2, q_3, q_4 \mid q_i \in [0,1] \,\&\, \sum_{i=1}^{4} q_i = 1 \right\}$$

$B_1 = \Sigma_1, B_2 = \Sigma_2$

Definition of normal form game

$\Gamma = \{N, S_1, \ldots, S_n, u_1, \ldots, u_n\},$

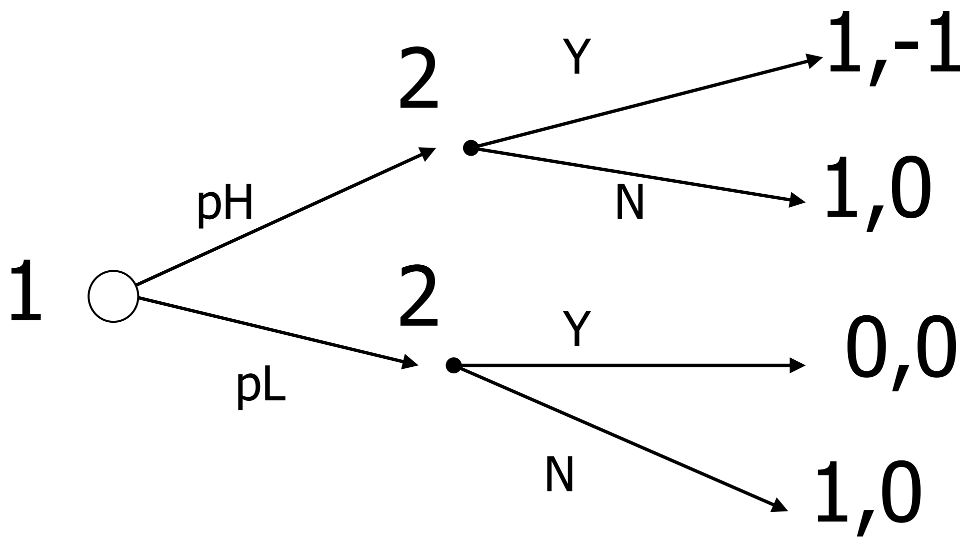
in this example

$\Gamma = \{\{1,2\}, \{1,2,3,4\}, \{1,2,3,4\}, u_1, u_2\}$

Hide & seek game in matrix form

S \ R	1	2	3	4
1	(-1, 1)	(1, -1)	(1, -1)	(1, -1)
2	(1, -1)	(-1, 1)	(1, -1)	(1, -1)
3	(1, -1)	(1, -1)	(-1, 1)	(1, -1)
4	(1, -1)	(1, -1)	(1, -1)	(-1, 1)

Example 2: trade



$$S_1 = \{p_L, p_H\}$$

$$S_2 = \{YY, YN, NY, NN\}$$

$$\Sigma_1 = \left\{q_1, q_2 \mid q_i \in [0,1] \,\&\, \sum_{i=1}^{2} q_i = 1\right\}$$

$$\Sigma_2 = \left\{p_1, p_2, p_3, p_4 \mid p_i \in [0,1] \,\&\, \sum_{i=1}^{4} p_i = 1\right\}$$

$$B_1 = \Sigma_1$$

$$B_2 = \{b_1(\bullet), b_2(\bullet)\}$$

such that $\quad b_i : H_2 \to \Delta(\{Y, N\})$

Definition of normal form game

$$\Gamma = \{N, S_1, \ldots, S_n, u_1, \ldots, u_n\},$$

in this example $\quad \Gamma = \{\{1,2\}, \{p_L, p_H\}, \{YY, YN, NY, NN\}, u_1, u_2\}$

The normal form game as a bi-matrix

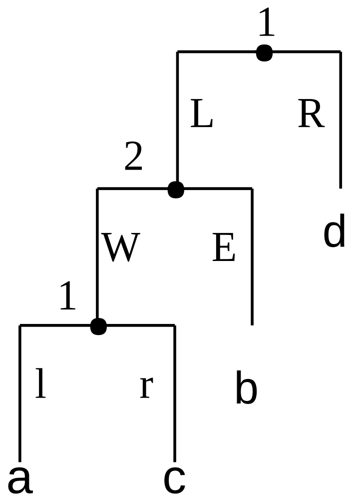
2 \ 1	pL	pH
YY	(0, 0)	(1,-1)
YN	(1, 0)	(1,-1)
NY	(0, 0)	(1, 0)
NN	(1, 0)	(1, 0)

IMPORTANT REMARK

- Usually the strategic game obtained from an extensive game has "equivalent" strategies for some player.
- The strategies are "equivalent" if they give the same payoffs for all possible opponents' behavior.
- The game obtained reducing to one all equivalent strategies is called

reduced strategic form game.

EXAMPLE 1



$$S_1 = \{Ll, Lr, Rl, Rr\}$$

$$S_2 = \{W, E\}$$

$$\Sigma_1 = \left\{ p_1, p_2, p_3, p_4 \mid p_i \in [0,1] \,\&\, \sum_{i=1}^{4} p_i = 1 \right\}$$

$$\Sigma_2 = \left\{ q_1, q_2 \mid q_i \in [0,1] \,\&\, \sum_{i=1}^{2} q_i = 1 \right\}$$

$$B_1 = \{b_1(\bullet), b_2(\bullet)\}$$

$$B_2 = \Sigma_2$$

Definition of normal form game

$$\Gamma = \{N, S_1, \ldots, S_n, u_1, \ldots, u_n\},$$

in this example $\Gamma = \{\{1,2\}, \{Ll, Lr, Rl, Rr\}, \{W, E\}, u_1, u_2\}$

The normal form game as a matrix

	W	E
Ll	a= (Ll,W)	b= (Ll,E)
Lr	c=(Lr,W)	b=(Lr,E)
Rl	d=(Rl,W)	d=(Rl,E)
Rr	d=(Rr,W)	d=(Rr,E)

The reduced strategic form game

	W	E
Ll	a=(Ll,W)	b=(Ll,E)
Lr	c=(Lr,W)	b=(Lr,E)
R	d=(R,W)	d=(R,E)

RELATIONS BETWEEN EXTENSIVE AND STRATEGIC FORM GAMES

- To each strategic game, we can associate different extensive games, therefore
- Different extensive form may give rise to the same strategic form
- To each extensive game, we can associate a unique strategic game

PROBLEM

WHAT IS THE RIGHT MODEL TO USE?

EXTENSIVE FORM GAMES OR

STRATEGIC FORM GAMES?

TRIVIAL ANSWER

EXTENSIVE FORM GAMES ARE A DETAILLED DESCRIPTION

STRATEGIC FORM GAMES ARE A CONCISE DESCRIPTION

FIRST PROBLEM

ARE EXTENSIVE FORM GAMES TOO DETAILLED?

SECOND PROBLEM

ARE STRATEGIC FORM GAMES TOO CONCISE?

CONCLUSION ON EFGs VERSUS NFGs

- **Normal form games provides enough information**
- **but**
- **they are less intuitive on the sequentiality of behaviour,**
- **so**
- **to discuss dynamic problems EFGs are more useful even if <u>all</u> considerations can be translated in concepts related to NFGs**