LECTURE 2

Dominance and Bayesian Rationality in Strategic Situations

&

Incomplete Information

MAIN POINTS OF PREVIOUS LECTURE

Formal Representations of a Game

- The two forms of a game
  - **extensive form**
  - normal form (**strategic form**)
  - [Sequence form]
  - [Characteristic function]

Extensive Form Games

FORMAL DEFINITION OF EXTENSIVE FORM GAME (1)

- An extensive form game is the following collection:

$$E = \{N; T, \prec; A, \alpha; \iota; H_i; \rho; v_i\}$$

- where

1. *N* is the finite **set of players** (*n* is the number of players)
2. *T* is a **set of nodes**, that together with the binary relation $\prec$ on T represents **precedence** and form an arborescence, i.e. <u>it totally orders the predecessors of each member of T</u>:
   - **it means that each node can be reached by one and only one path**

FORMAL DEFINITION OF EXTENSIVE FORM GAME (2)

- Using the **tree** $T, \prec$ we can define:
  - Predecessors of $x \in T$ $P(x) := \{t \in T | t \prec x\}$
  - Immediate predecessor of x $p_1(x) := \max\{t \in T | t \prec x\}$
  - n-th predecessor of x $p_n(x) := p_1 p_{n-1}((x))$ with $p_0(x) = x$
  - Immediate successors of x $S(x) := p_1^{-1}(x)$
  - Outcomes $Z := \{t \in T \mid S(t) = \varnothing\}$
  - Decision nodes $X := T \setminus Z$
  - Initial nodes $W := \{t \in T \mid P(t) = \varnothing\}$
  - Terminal successors of x $Z(x) := \{z \in Z | x \prec z\}$

FORMAL DEFINITION OF EXTENSIVE FORM GAME (2)

- A is the set of **actions** and $\alpha : T \backslash W \to A$ is a **function** that labels each non-initial node with the **last action taken** to reach it.
  - This function is assumed to be injective
  - $\alpha(S(x))$ is the set of feasible actions at x.
- $\iota : X \to N$ represents the rules for determining **whose move it is** at a decision nodes x

FORMAL DEFINITION OF EXTENSIVE FORM GAME (3)

- **Information** is represented by a partition *H* of X that divides the decision nodes into **information sets.**
  - A cell $H(x) \in H$ contains the nodes that the player cannot distinguish from x: $H(x) \subseteq X$
  - We require $x \in H(x') \Rightarrow \iota(x) = \iota(x') \,\&\, \alpha(S(x)) = \alpha(S(x'))$
- $\rho \in \Delta(W)$ is a **probability distribution** on initial nodes
- $v_i : Z \to \Re$ is the **utility** function of player i.

VERY IMPORTANT NOTION

- ***Information Set*:** for player *i* is a collection of decision nodes satisfying two conditions:
  1. player *i* has the move at every node in the collection, and
  2. *i* doesn't know which node in the collection has been reached.
- ***Meaning*:**
  - **the histories/sequence of actions that lead to the nodes in an information set are not distinguishable for player *i***

Strategies

Further Definitions - 1:

- ***Strategy***: a ***complete*** plan of action (what to do in *every* contingency):

$$s^i: H^i \to A \text{ such that}$$

$$s^i(h^i) \in A(h^i) \text{ for any } h^i \in H^i$$

where:

- H^i is the collection of i's information sets
- A is the set of possible actions
- $A(h^i)$ is the set of actions feasible in h^i

Further Definitions - 2:

- ***Set of Strategy***: under our assumption of finiteness the **set of pure strategy** for player *i* is

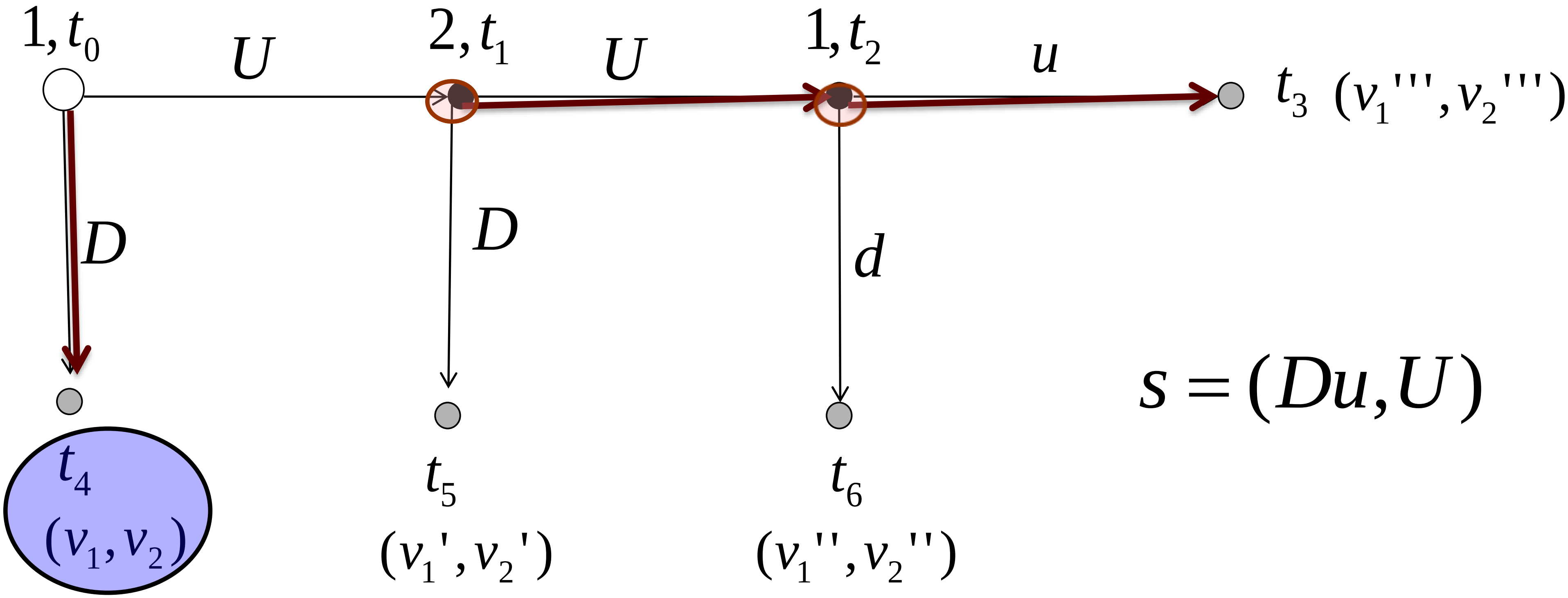
$$S_i = \prod_{h_i \in H_i} A(h_i)$$

- Similarly, the set of strategy profiles is

$$s \in S = \prod_{i \in N} S_i$$

where *s* is a vector

Example of a strategy set and of a strategy profile



$$s = (Du, U)$$

$$S^1 = \{U, D\} \times \{u, d\} = \{(U,u), (U,d), (D,u), (D,d)\}$$

$$S^2 = \{U, D\}$$

Mixed Strategies

- ***Mixed strategy***: a randomization over pure strategies:

  $\sigma^i: S^i \rightarrow [0,1]$

  where $\sigma^i(s^i) = \Pr(\text{i plays pure strategy } s^i)$.
- The set of mixed strategy of player i is $\Delta(S^i)$
- Mixed Strategy Profile

  $\sigma = \{\sigma^1,\ldots,\sigma^n\} \in \Delta(S^1)\times\ldots\times\Delta(S^n)$

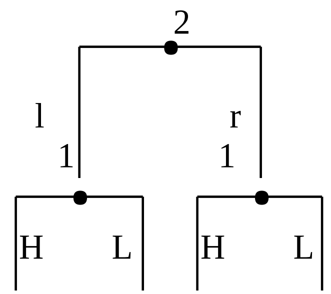
Behavioral strategies

- A ***behavioral strategy*** specifies a probability distribution over feasible actions at each information set.

$$b^i: H^i \rightarrow \Delta(A) \text{ such that}$$

$$b^i(h^i) \in \Delta(A(h^i)) \text{ for any } h^i \in H^i$$

Example



$$S_1 = \{HH, HL, LH, LL\}$$

$$\Sigma_1 = \left\{p_1, p_2, p_3, p_4 : p_i \geq 0 \,\&\, \sum_{i=1}^{4} p_i = 1\right\}$$

$$B_1 = \left\{p, q : p = \Pr\{H \mid l\}, q = \Pr\{H \mid r\}\right\}.$$

PROBLEM

- Mixed and behavioral strategies are different objects
- Set of mixed strategies

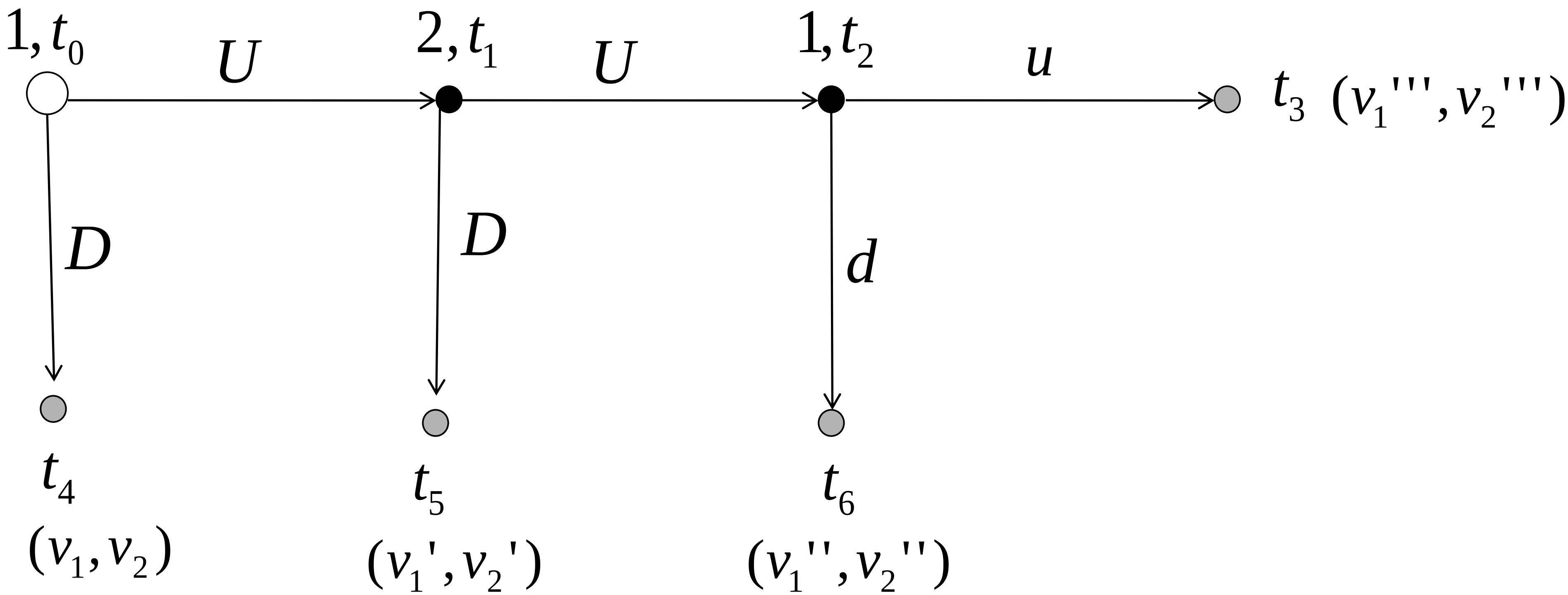
$$\Sigma_i := \Delta(S_i) = \Delta(\prod_{h \in H_i} A(h))$$

- Set of behavioral strategies

$$B_i := \prod_{h \in H_i} \Delta(A(h))$$

- Different sets, mixed seems more general
- What is the relationship between mixed and behavioral strategies? To answer we need two further notions

Mixed and behavioral strategies



	U	**D**	
u	$\sigma^1(\{Uu\})$	$\sigma^1(\{Du\})$	$\pi^1(\{u\}\mid t_1)$
d	$\sigma^1(\{Ud\})$	$\sigma^1(\{Dd\})$	$\pi^1(\{d\}\mid t_1)$
	$\pi^1(\{U\}\mid t_0)$	$\pi^1(\{D\}\mid t_0)$	

Expected Payoffs

- Expected payoff with mixed strategy:

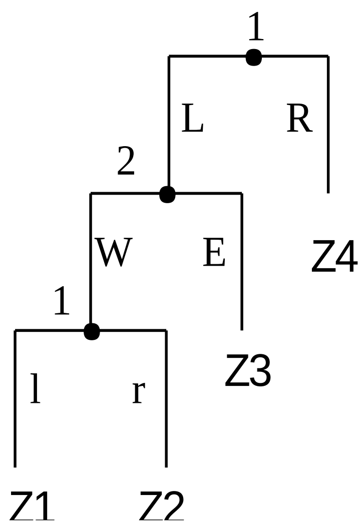
$$v_i(\sigma) := \sum_{z \in Z} v_i(z) P(z \mid \sigma)$$

- Expected payoff with behavioral strategy:

$$v_i(b) := \sum_{z \in Z} v_i(z) P(z \mid b)$$

- ***Two strategy profiles are outcome equivalent if and only if they induce the same probability on the set of final nodes.***

EXAMPLE 1: outcome function



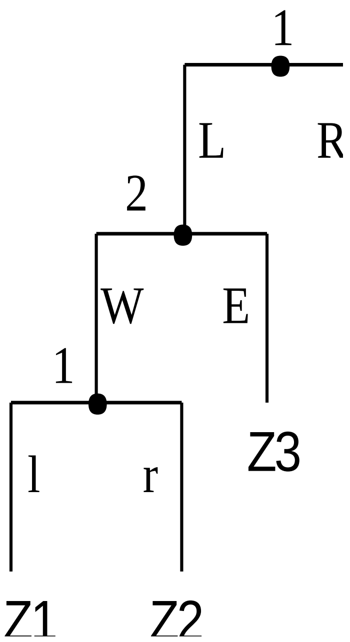
$$\varsigma(s) = \begin{cases} z_1 & se \quad s = (Ll, W) \\ z_2 & se \quad s = (Lr, W) \\ z_3 & se \quad s = (Ll, E), (Lr, E) \\ z_4 & se \quad s = (Rl, W), (Rr, W), (Rl, E), (Rr, E) \end{cases}$$

$$S_1 = \{Ll, Lr, Rl, Rr\} \quad S_2 = \{W, E\}$$

$$S = S_1 \times S_2 = \{Ll, Lr, Rl, Rr\} \times \{W, E\} =$$

$$= \{(Ll, W), (Ll, R), (Lr, W), (Lr, E), (Rl, W), (Rl, E), (Rr, W), (Rr, E)\}$$

Probabilities of outcomes using mixed strategy profiles



$$\sigma_1 = \Pr(Rl) = \Pr(Ll) = 1/2$$

$$\sigma_2 = \Pr(W) = 1$$

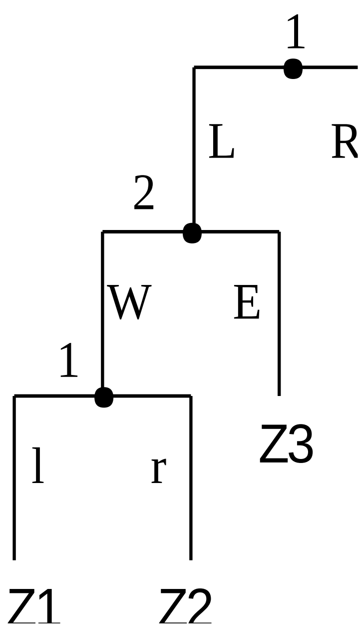
$$\Pr(z_1 \mid \sigma) = \Pr(Ll) \times \Pr(W) = 1/2 \times 1 = 1/2$$

$$\Pr(z_2 \mid \sigma) = \Pr(Lr) \times \Pr(W) = 0 \times 1 = 0$$

$$\Pr(z_3 \mid \sigma) = \Pr(Ll) \times \Pr(E) + \Pr(Lr) \times \Pr(E) = 0$$

$$\Pr(z_4 \mid \sigma) = \Pr(Rl) \times \Pr(W) + \Pr(Rr) \times \Pr(W) +$$
$$+ \Pr(Rl) \times \Pr(E) + \Pr(Rr) \times \Pr(E) = 1/2.$$

Probabilities of outcomes using behavioral strategy profiles



$$\pi^1 = \begin{cases} \Pr(L) = 1/2 \\ \Pr(l) = 1/2 \end{cases} \qquad \pi^2 = \Pr(W) = 1$$

$$\Pr(z_1 \mid b) = \Pr(L) \times \Pr(W) \times \Pr(l) = 1/4$$

$$\Pr(z_2 \mid b) = \Pr(L) \times \Pr(W) \times \Pr(r) = 1/4$$

$$\Pr(z_3 \mid b) = \Pr(L) \times \Pr(E) = 0$$

$$\Pr(z_4 \mid b) = \Pr(R) = 1/2.$$

Relationship between mixed and behavioral strategies

- Clearly mixed strategies are more general than behavioral strategies because allow "correlation" among information sets in the sense of **probabilities of vectors** instead of **vectors of probabilities**:

$$\Sigma_i = \Delta\left(\prod_{h_i \in H_i} A(h_i)\right) \text{ versus} \Pi_i = \prod_{h_i \in H_i} \Delta\big(A(h_i)\big)$$

- Thus it is not surprising that for any behavioral strategy profile there exists an **outcome equivalent** mixed strategy profile:

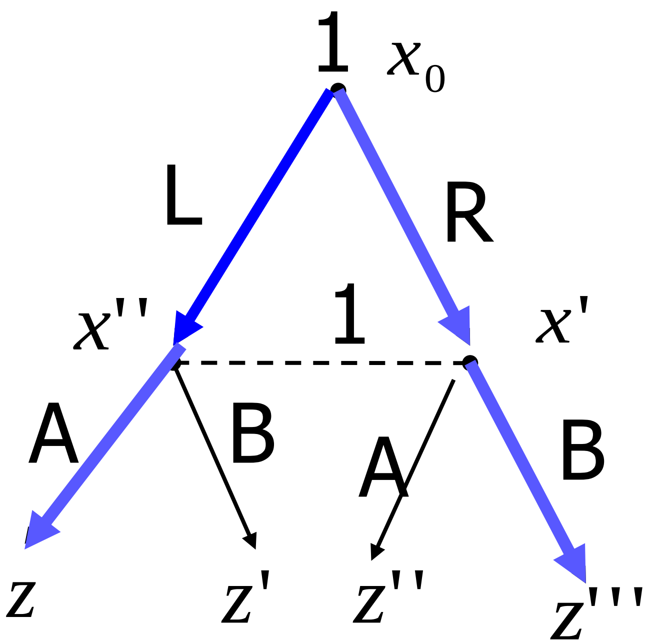
$$\sigma_i(s_i)[b] := \underset{h \in H_i}{\times} b_i[h]\big(s_i(h)\big).$$

- What about the other way ?

KUHN'S THEOREM

For any mixed strategy profile in a finite extensive game with **perfect recall** there is an outcome-equivalent behavioral strategy profile.

Game with imperfect recall where there exists a mixed strategy not equivalent to a behavioral



$$\sigma_1 = \Pr(LA) = \Pr(RB) = 1/2$$

Probabilities of outcomes using σ_1

$$\Pr(z \mid \sigma_1) = \Pr(LA) = 1/2 \qquad \Pr(z' \mid \sigma_1) = \Pr(LB) = 0$$

$$\Pr(z'' \mid \sigma_1) = \Pr(RA) = 0 \qquad \Pr(z''' \mid \sigma_1) = \Pr(RB) = 1/2$$

Probabilities of outcomes using behavioral strategies

$$\Pr(z \mid \pi_1) = \pi_1(L) \times \pi_1(A) = 1/2 \Rightarrow \pi_1(L) > 0 \;\&\; \pi_1(A) > 0$$

$$\Pr(z' \mid \pi_1) = \pi_1(L) \times \pi_1(B) = 0 \Rightarrow \pi_1(L) = 0 \quad \text{or} \quad \pi_1(B) = 0$$

$$\Pr(z'' \mid \pi_1) = \pi_1(R) \times \pi_1(A) = 0 \Rightarrow \pi_1(R) = 0 \quad \text{or} \quad \pi_1(A) = 0$$

$$\Pr(z''' \mid \pi_1) = \pi_1(R) \times \pi_1(B) = 1/2 \Rightarrow \pi_1(R) > 0 \;\&\; \pi_1(B) > 0$$

Contraddiction : $\neg \exists \pi_1$ satisfying such conditions

Strategic Form Games

Alternative representation of a game

- The extensive game is a detailed and thus complex representation of a strategic situation
- A simpler but more concise representation of strategic situations is the STRATEGIC FORM or NORMAL FORM

FORMAL DEFINITION OF STRATEGIC FORM GAME

1. Set of players $N = \{1,...,n\}$
2. Set of strategies S_i
3. payoff function
   - $u_i(s): S \rightarrow \Re,$
- which maps **strategy profiles**

  $s = (s_1,...,s_n) \in S = S_1 \times ... \times S_n$

  into real numbers

- game in normal form

$$\Gamma = \{N, S_1,...,S_{n,} u_1,...,u_n\}$$

A Simple Normal Form Game in Matrix Form

Player 1 \ *Player 2*	Left	Right
Left	A, B	A, B
Right	C, D	E , F

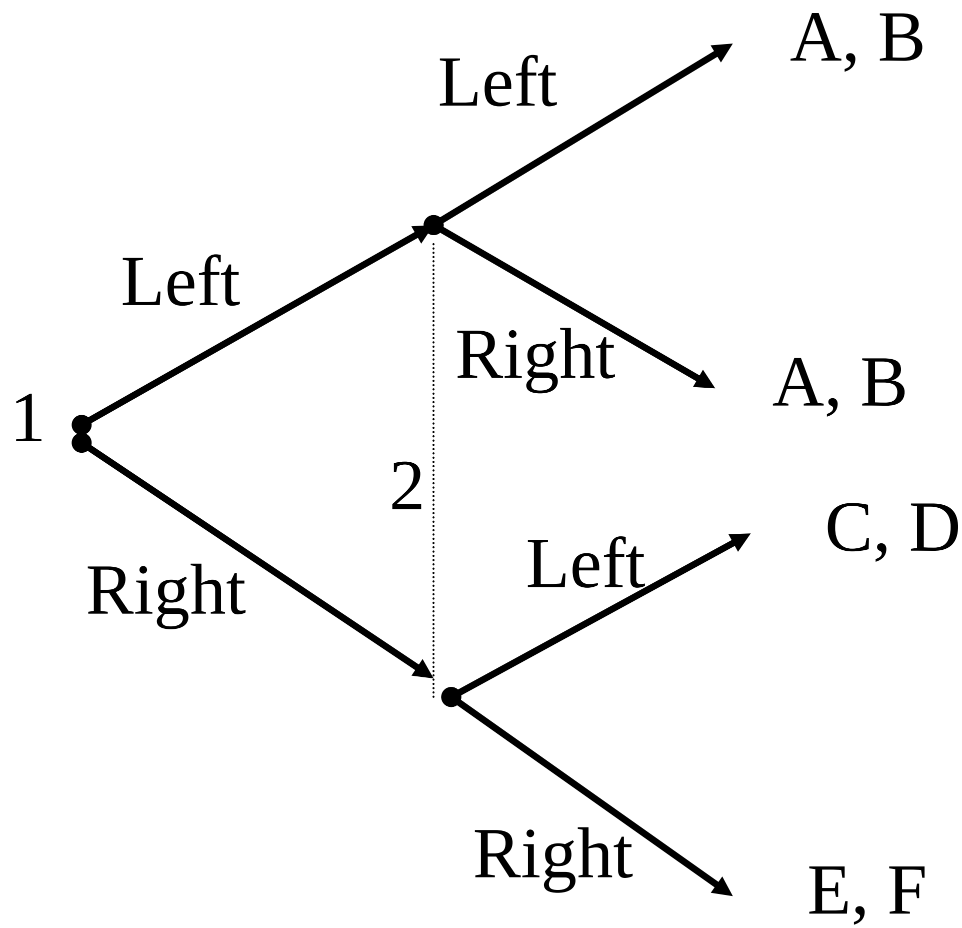
PROBLEM

- WHAT IS THE RELATION BETWEEN EXTENSIVE FORM AND STRATEGIC FORM GAMES ?

From NFG to EFG

Game 1 has the following EFG representation:

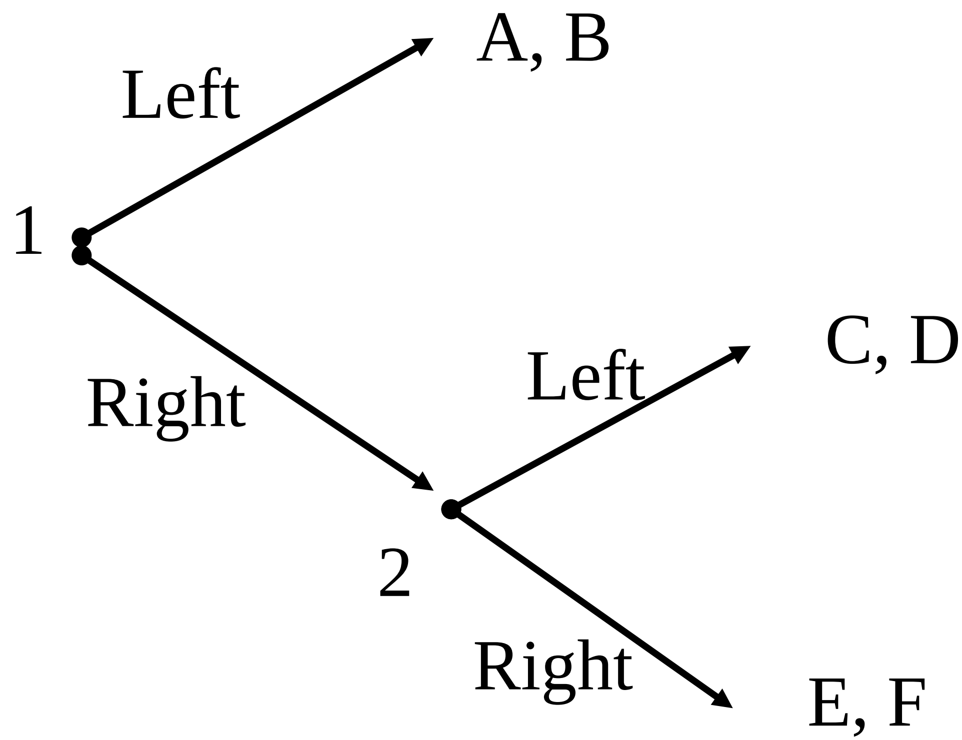
1 \ 2	Left	Right
Left	A, B	A, B
Right	C, D	E , F



From NFG to EFG

Also the following is an EFG representation of game 1:

1 \ 2	Left	Right
Left	A, B	A, B
Right	C, D	E , F



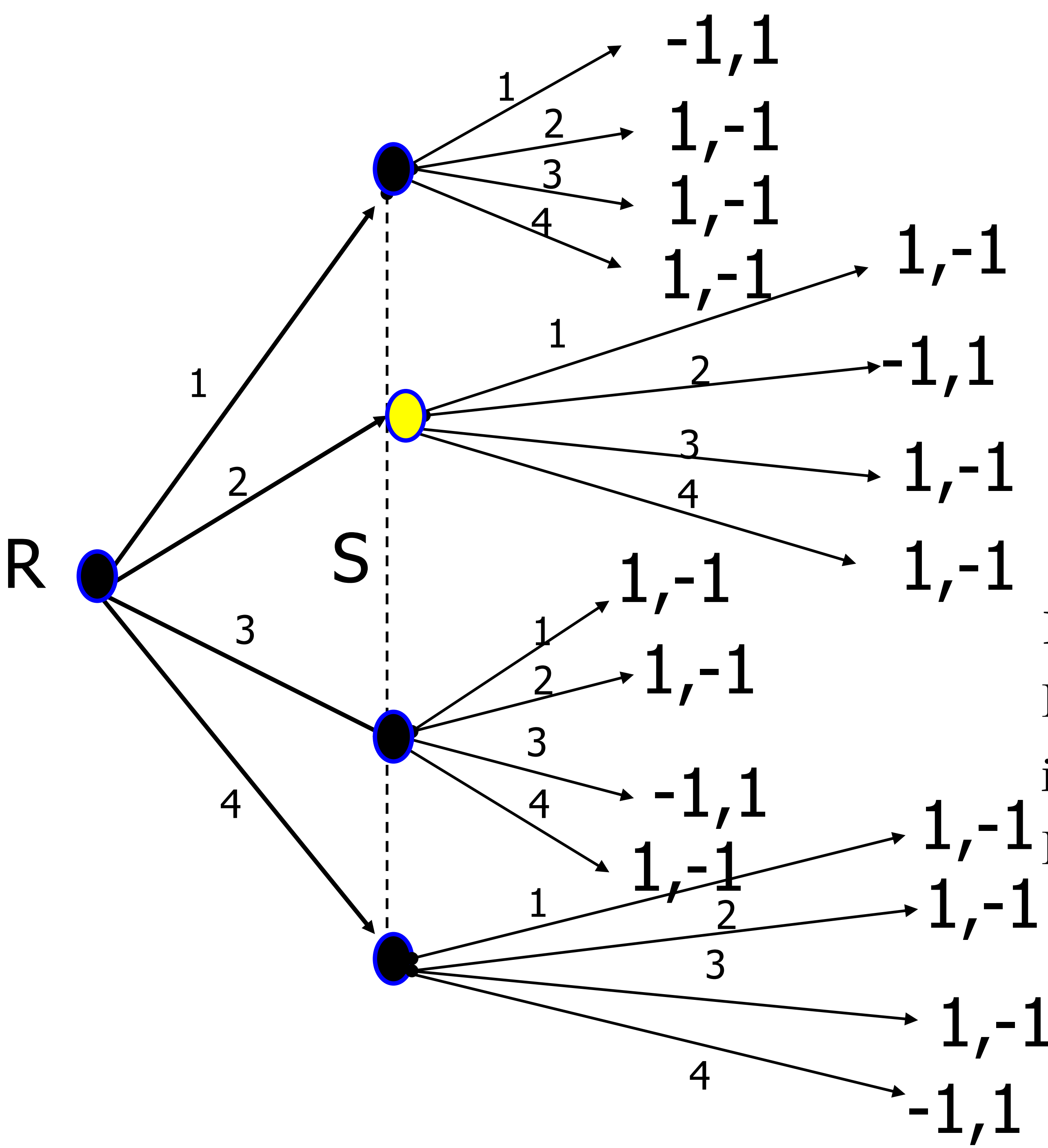
From EFG to NFG

- Use
  1. the previous definition of strategies to construct the set of pure strategies,
  2. the payoff functions are obtained combining the outcome function with v_i

$$u_i = v_i \bullet \varsigma. \qquad \text{In particular}$$

$$u_i(\sigma) = \sum_{s \in S} v_i(\varsigma(s)) \prod_{j=1}^{n} \sigma_j(s_j)$$

Hide and seek game



$S_1 = \{1,2,3,4\}$

$S_2 = \{1,2,3,4\}$

$$\Sigma_1 = \left\{ p_1, p_2, p_3, p_4 \mid p_i \in [0,1] \,\&\, \sum_{i=1}^{4} p_i = 1 \right\}$$

$$\Sigma_2 = \left\{ q_1, q_2, q_3, q_4 \mid q_i \in [0,1] \,\&\, \sum_{i=1}^{4} q_i = 1 \right\}$$

$B_1 = \Sigma_1, B_2 = \Sigma_2$

Definition of normal form game

$\Gamma = \{N, S_1, \ldots, S_n, u_1, \ldots, u_n\},$

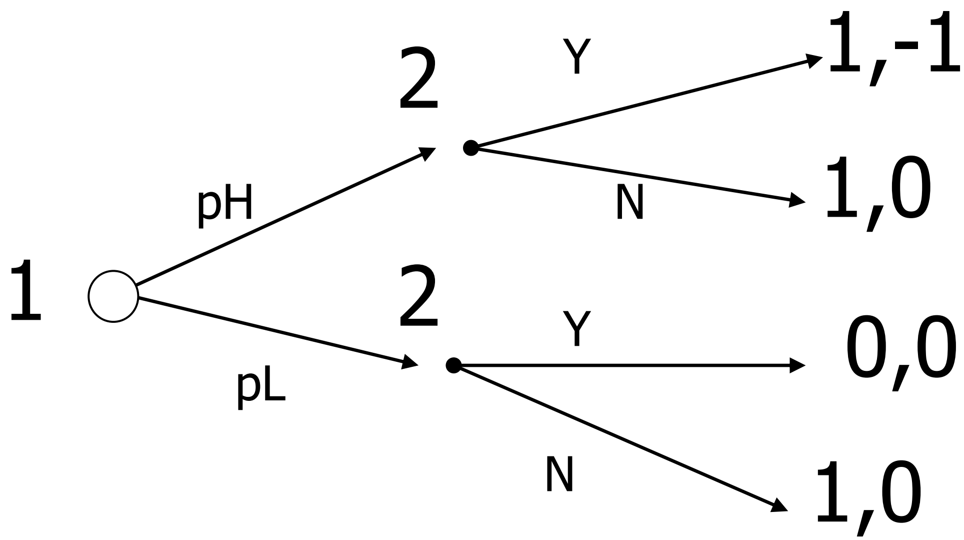
in this example

$\Gamma = \{\{1,2\}, \{1,2,3,4\}, \{1,2,3,4\}, u_1, u_2\}$

Hide & seek game in matrix form

R \ S	1	2	3	4
1	(-1, 1)	(1, -1)	(1, -1)	(1, -1)
2	(1, -1)	(-1, 1)	(1, -1)	(1, -1)
3	(1, -1)	(1, -1)	(-1, 1)	(1, -1)
4	(1, -1)	(1, -1)	(1, -1)	(-1, 1)

Example 2: trade



$$S_1 = \{p_L, p_H\}$$

$$S_2 = \{YY, YN, NY, NN\}$$

$$\Sigma_1 = \left\{ q_1, q_2 \mid q_i \in [0,1] \,\&\, \sum_{i=1}^{2} q_i = 1 \right\}$$

$$\Sigma_2 = \left\{ p_1, p_2, p_3, p_4 \mid p_i \in [0,1] \,\&\, \sum_{i=1}^{4} p_i = 1 \right\}$$

$$B_1 = \Sigma_1$$

$$B_2 = \{b_1(\bullet), b_2(\bullet)\}$$

such that $\quad b_i : H_2 \to \Delta(\{Y, N\})$

Definition of normal form game

$$\Gamma = \{N, S_1, \ldots, S_n, u_1, \ldots, u_n\},$$

in this example $\quad \Gamma = \{\{1,2\}, \{p_L, p_H\}, \{YY, YN, NY, NN\}, u_1, u_2\}$

The normal form game as a bi-matrix

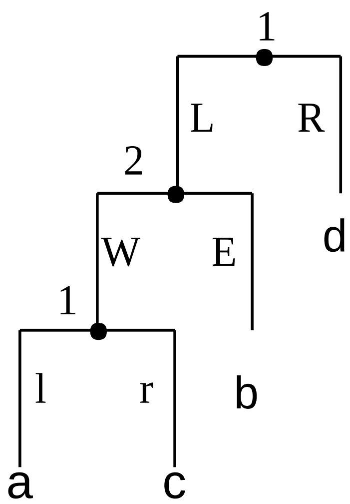
2 \ 1	pL	pH
YY	(0, 0)	(1,-1)
YN	(1, 0)	(1,-1)
NY	(0, 0)	(1, 0)
NN	(1, 0)	(1, 0)

IMPORTANT REMARK

- Usually the strategic game obtained from an extensive game has “equivalent” strategies for some player.
- The strategies are “equivalent” if they give the same payoffs for all possible opponents’ behavior.
- The game obtained reducing to one all equivalent strategies is called

reduced strategic form game.

EXAMPLE 1



$$S_1 = \{Ll, Lr, Rl, Rr\}$$

$$S_2 = \{W, E\}$$

$$\Sigma_1 = \left\{ p_1, p_2, p_3, p_4 \mid p_i \in [0,1] \,\&\, \sum_{i=1}^{4} p_i = 1 \right\}$$

$$\Sigma_2 = \left\{ q_1, q_2 \mid q_i \in [0,1] \,\&\, \sum_{i=1}^{2} q_i = 1 \right\}$$

$$B_1 = \{b_1(\bullet), b_2(\bullet)\}$$

$$B_2 = \Sigma_2$$

Definition of normal form game

$$\Gamma = \{N, S_1, \ldots, S_n, u_1, \ldots, u_n\},$$

in this example $\Gamma = \{\{1,2\}, \{Ll, Lr, Rl, Rr\}, \{W, E\}, u_1, u_2\}$

The normal form game as a matrix

	W	E
Ll	a= (Ll,W)	b= (Ll,E)
Lr	c=(Lr,W)	b=(Lr,E)
Rl	d=(Rl,W)	d=(Rl,E)
Rr	d=(Rr,W)	d=(Rr,E)

The reduced strategic form game

	W	E
Ll	a=(Ll,W)	b=(Ll,E)
Lr	c=(Lr,W)	b=(Lr,E)
R	d=(R,W)	d=(R,E)

RELATIONS BETWEEN EXTENSIVE AND STRATEGIC FORM GAMES

- To each strategic game, we can associate different extensive games, therefore
- Different extensive form may give rise to the same strategic form
- To each extensive game, we can associate a unique strategic game

PROBLEM

WHAT IS THE RIGHT MODEL TO USE?

EXTENSIVE FORM GAMES OR

STRATEGIC FORM GAMES?

TRIVIAL ANSWER

EXTENSIVE FORM GAMES ARE A DETAILLED DESCRIPTION

STRATEGIC FORM GAMES ARE A CONCISE DESCRIPTION

FIRST PROBLEM

ARE EXTENSIVE FORM GAMES TOO DETAILLED?

SECOND PROBLEM

ARE STRATEGIC FORM GAMES TOO CONCISE?

CONCLUSION ON EFGs VERSUS NFGs

- **Normal form games provides enough information**
- **but**
- **they are less intuitive on the sequentiality of behaviour,**
- **so**
- **to discuss dynamic problems EFGs are more useful even if <u>all</u> considerations can be translated in concepts related to NFGs**

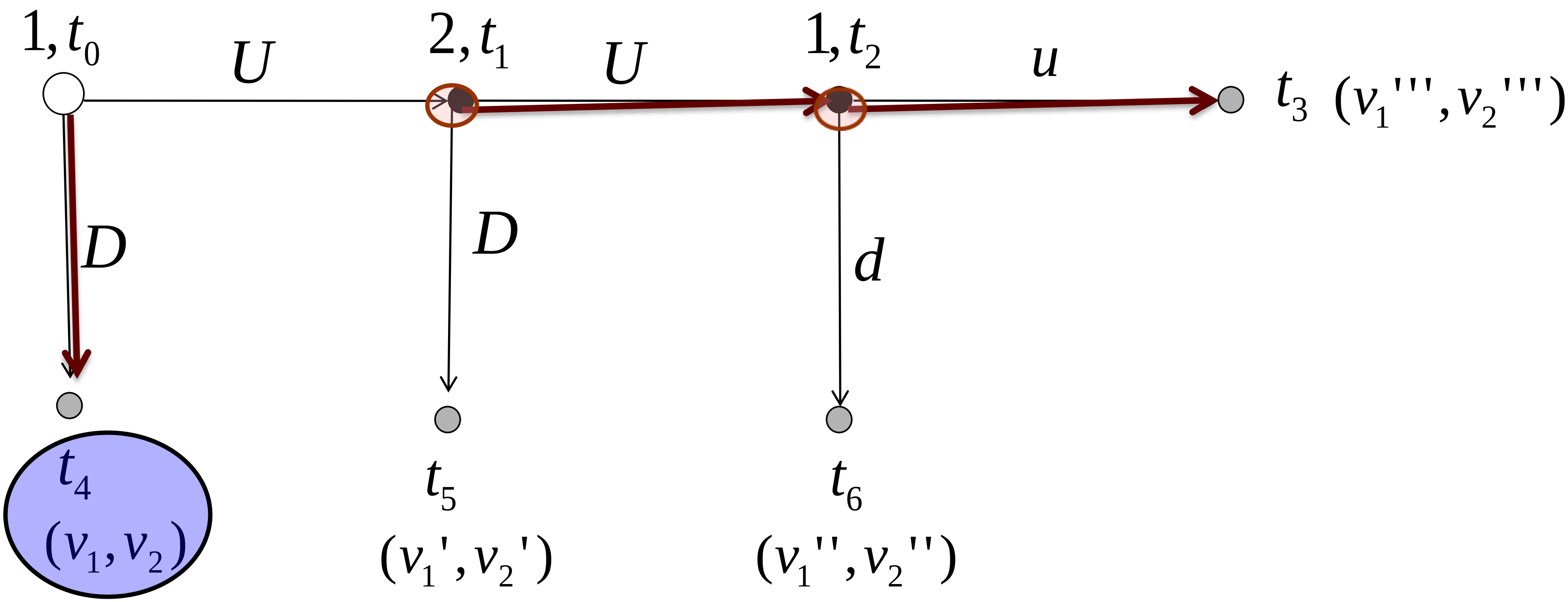
NEW CONCEPTS TO MODEL STRATEGIC INTERACTION

Game’s solution

WHAT IS A GAME'S SOLUTION?

- If we want to forecast the likely outcome of a strategic situation, we need to forecast players' behavior, i.e. we need a solution for games.
- A solution is a pattern of players' behavior satisfying some kind of "plausibility" conditions
- **Questions:**
  1. What is a pattern of players' behavior?
  2. What are our plausibility conditions
- **Answers:**
  1. A strategy profile
  2. Rational behavior

Example of a solution



Suppose the solution is $s^* = (Du, U)$,
then the likely outcome is t_4

How to define a game's solution? RATIONALITY

- Problem: how to define players' rationality in strategic situations?

    - Players are **rational and intelligent**

- The problem is to formalize rationality AND intelligence

    - **Let we start assuming rationality**

SOLUTION IN STRATEGIC FORM GAMES

Dominance as solution criterion:
rationality as avoidance of bad choices

Example 1: Prisoner's Dilemma

- Two suspects are arrested and charged with a crime.
- They are held in separate cells.
- The DA separately offers each the chance to turn state's evidence.
- A jail sentence of x years has utility –x.

Example 1 in Normal Form

1 \ 2	Mum	Fink
Mum	-1, -1	-5, 0
Fink	0, -5	**-4, -4**

A small change in Example 1

?,?

?,?

(0, 0)	(-5, 0)
(0, -5)	(-4, -4)

Elimination of dominated strategies

- *Dominated Strategy*:
  - **x *strictly dominates* y** if the player gets a higher payoff from playing x than playing y, *regardless* of what the other players do.
  - **x *weakly dominates* y** if the player's payoff is at least as great by playing x than y, *regardless* of what the other players do.

Example 2: the role of mixed strategies

NB: in the definition of dominance, we can/must use mixed strategies:

	L	R
U	10, 1	0, 4
I	~~4, 2~~	~~4, 3~~
D	0, 5	10, 2

Strict Dominance

- **Def:** s_i is **strictly dominated** for player *i* iff

$$\exists \sigma_i \in \Sigma_i : \forall s_{-i} \in S_{-i}\ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$$

- **Def:** the **set of pure strategies strictly undominated** for player *i* is

$$S_i^1 = \{ s_i \in S_i \mid \neg \exists \sigma_i \in \Sigma_i : \forall s_{-i} \in S_{-i}\ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \}$$

- **Remarks:**
  1. It is the same mixed strategy that should be considered wrt all opponents' strategies
  2. It is not a limitation to consider opponents' pure strategies since expected utility is linear in probabilities and thus can not increase its value
  3. For the same reason a mixed strategy that gives strictly positive probability to a dominated pure strategy is dominated, even if there exists dominated mixed strategy that do not give positive probability to dominated pure strategies

Example 3:

	M	F
M	0, -2	-10, -1
F	-1, -10	-5, -5

Solution by deletion of strictly dominated strategies: {M, F} x {F}, i.e. there are two possible solutions: (M, F) and (F,F).

But:

- If **players are intelligent**, then they must anticipate opponents' rational behavior
- what is the implication of **assuming intelligence** for the elimination of dominated strategies?
- **Iterative solutions:**
  - **iterative deletion of dominated strategies**
- In example 3 if player 1 is intelligent and thus anticipates the opponent's rational behavior, the solution is {F} x {F}.

Example 3:

	M	F
M	0, -2	-10, -1
F	-1, -10	-5, -5

Solution by **iterated** deletion of strictly dominated strategies: {F} x {F}.

Formal definition of the set of strategies iteratively strictly undominated

$$S_i^0 := S_i;$$

$$S_i^t := \left\{ \begin{array}{c} s_i \in S_i^{t-1} \mid \neg\exists \sigma_i \in \Delta(S_i^{t-1}) : \forall s_{-i} \in S_{-i}^{t-1} \\ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \end{array} \right\}$$

$$ISUS_i := \bigcap_{t=0}^{\infty} S_i^t$$

$$ISUS := \prod_{i=1}^{n} ISUS_i$$

Formal definition of ISUS applied to game 3

$$S_1(0) := S_1 = \{M, F\}; S_2(0) := S_2 = \{M, F\}$$

$$S_1(1) := \left\{ \begin{array}{l} s_1 \in S_1(1-1) = \{M, F\} \mid \neg\exists \sigma_1 \in \Delta(S_1(1-1)) = \Delta(\{M, F\}): \\ u_1(\sigma_1, s_2) > u_1(s_1, s_2) \forall s_2 \in S_2(1-1) = \{M, F\} \end{array} \right\} = \{M, F\}$$

since $u_1(M, M) = 0 > u_1(F, M) = -1 \,\&\, u_1(M, F) = -10 < u_1(F, F) = -5$

$$S_2(1) := \left\{ \begin{array}{l} s_2 \in S_2(1-1) = \{M, F\} \mid \neg\exists \sigma_2 \in \Delta(S_2(1-1)) = \Delta(\{M, F\}): \\ u_2(s_1, \sigma_2) > u_2(s_1, s_2) \forall s_1 \in S_1(1-1) = \{M, F\} \end{array} \right\} = \{F\}$$

since $u_2(M, M) = -2 < u_2(M, F) = -1 \,\&\, u_2(F, M) = -10 < u_2(F, F) = -5$

Formal definition of ISUS applied to game 3

$$S_1(2) := \left\{ \begin{array}{l} s_1 \in S_1(2-1) = \{M, F\} \mid \neg\exists \sigma_1 \in \Delta(S_1(2-1)) = \Delta(\{M, F\}): \\ u_1(\sigma_1, s_2) > u_1(s_1, s_2) \forall s_2 \in S_2(2-1) = \{F\} \end{array} \right\} = \{F\}$$

$$\text{since } u_1(M, F) = -10 < u_1(F, F) = -5$$

$$S_2(2) := \left\{ \begin{array}{l} s_2 \in S_2(2-1) = \{F\} \mid \neg\exists \sigma_2 \in \Delta(S_2(2-1)) = \Delta(\{F\}): \\ u_2(s_1, \sigma_2) > u_2(s_1, s_2) \forall s_1 \in S_1(2-1) = \{M, F\} \end{array} \right\} = \{F\}$$

$$\text{since player 2 con choose only } F$$

Formal definition of ISUS applied to game 3

$$S_1(3) := \left\{ \begin{array}{l} s_1 \in S_1(3-1) = \{F\} \mid \neg\exists \sigma_1 \in \Delta(S_1(3-1)) = \Delta(\{F\}): \\ u_1(\sigma_1, s_2) > u_1(s_1, s_2) \forall s_2 \in S_2(3-1) = \{F\} \end{array} \right\} = \{F\}$$

$$\text{since player 1 con choose only } F$$

$$S_2(3) := \left\{ \begin{array}{l} s_2 \in S_2(3-1) = \{F\} \mid \neg\exists \sigma_2 \in \Delta(S_2(3-1)) = \Delta(\{F\}): \\ u_2(s_1, \sigma_2) > u_2(s_1, s_2) \forall s_1 \in S_1(3-1) = \{F\} \end{array} \right\} = \{F\}$$

$$\text{since player 2 con choose only } F$$

$$\ldots$$

$$S_1^\infty := \bigcap_{t=0}^{\infty} S_1(t) = \{M, F\} \cap \{M, F\} \cap \{F\} \cap \{F\} \cap \ldots = \{F\}$$

$$S_2^\infty := \bigcap_{t=0}^{\infty} S_2(t) = \{M, F\} \cap \{F\} \cap \{F\} \cap \{F\} \cap \ldots = \{F\}$$

$$S^\infty := \prod_{i=1}^{n} S_i^\infty = \{F\} \times \{F\} = \{F, F\}.$$

Iterated Strictly Undominated strategies

- **Remarks:**
  1. $S_i^{\infty} \neq \phi$ since it is the infinite intersection of a decreasing sequence of non empty compact sets
  2. In the definition, at each stage we consider the simultaneous deletion of all strictly dominated strategies, but it is possible to prove that the order of deletion does not matter

Bayesian rationality and rationalizability

rationality as search for possible good choices

An alternative notion of solution: Bayesian rationality

- A strategy is **Bayesian rational** iff it maximizes expected utility with respect to some beliefs on opponents' behavior:

$$s_i \in BR_i$$

$$\text{if and only if}$$

$$\exists \mu_i \in \Delta(S_{-i}) : \forall s'_i \in S_i$$

$$u_i(s_i, \mu_i) \geq u_i(s'_i, \mu_i)$$

Example 1 again

1 \ 2	μ_1 Mum	$1-\mu_1$ Fink
μ_2 Mum	-1, -1	-5, 0
$1-\mu_2$ Fink	0, -5	**-4,-4**

$$u_i\left(F,\mu_i\right)\geq u_i\left(M,\mu_i\right)\Leftrightarrow$$

$$0\mu_i-4\left(1-\mu_i\right)\geq -1\mu_i-5\left(1-\mu_i\right)$$

Solution of the Prisoner's Dilemma using Bayesian Rationality

$BR_1 = \{Fink\}$ since

$u_1(F, \mu_1) = 0 \times \Pr(M) - 4 \times \Pr(F)$

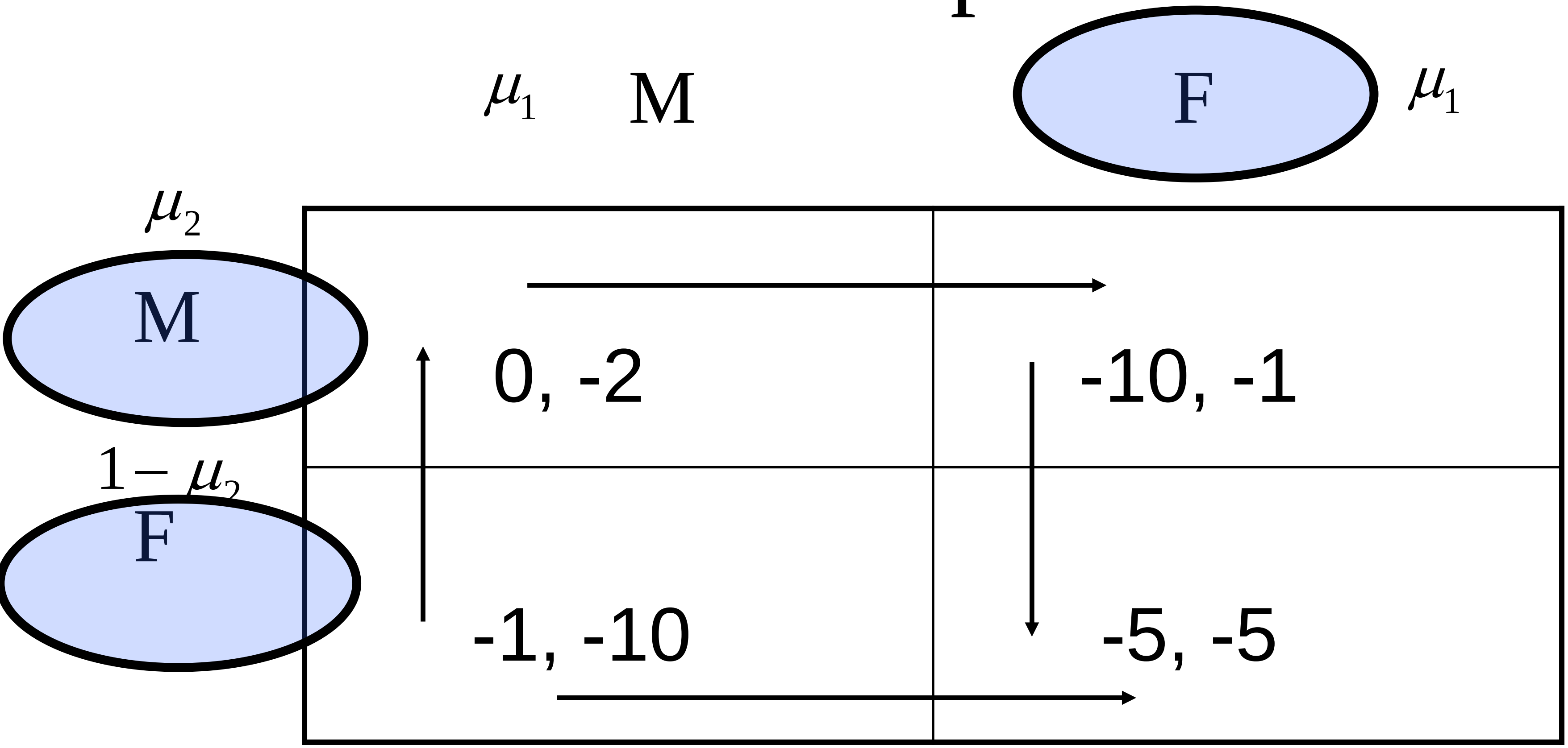
$u_1(M, \mu_1) = -1 \times \Pr(M) - 5 \times \Pr(F)$

Therefore $\forall \mu_1 \in \Delta(\{M, L\})$

$u_1(F, \mu_1) \geq u_1(M, \mu_1)$, i.e. there exists no conjecture μ_1 such that player 1 maximizes her expected utility playing M.

Similarly for player 2 $BR_2 = \{Fink\}$.

Example 3:



$$u_1(F,\mu_1) \ge u_2(M,\mu_1) \Leftrightarrow -1\mu_1 - 5(1-\mu_1) \ge 0\mu_1 - 10(1-\mu_1) \Leftrightarrow \mu_1 \le \frac{5}{6}$$

$$u_2(F,\mu_2) \ge u_2(M,\mu_2) \Leftrightarrow -1\mu_2 - 5(1-\mu_2) \ge -2\mu_2 - 10(1-\mu_2)$$

Solution by Bayesian Rationality: {M, F} x {F}, i.e. there are two possible solutions: (M, F) and (F,F).

But:

- If **players are intelligent**, then they must anticipate opponents' rational behavior
- what is the implication of **assuming intelligence** for Bayesian rationality?
- **Iterative solutions:**
  - **rationalizability**
- In example 3 if player 1 is intelligent and thus anticipates the opponent's rational behavior, the solution is {F} x {F}.

Example 3:

	M	F
M	0, -2	-10, -1
F	-1, -10	-5, -5

Solution by iterated Bayesian Rationality: {F} x {F}.

Formal definition of Rationalizability

$$R_i(1) := S_i;$$

$$R_i(t) := \left\{ \begin{array}{l} s_i \in R_i(t-1) \mid \exists \mu_i \in \Delta(R_{-i}(t-1)): \\ u_i(s_i, \mu_i) \geq u_i(s_i', \mu_i) \forall s_i' \in R_i(t-1) \end{array} \right\}$$

$$R_i := \bigcap_{t=1}^{\infty} R_i(t)$$

$$R := \prod_{i=1}^{n} R_i$$

$$N.B.: \quad R_i(2) \equiv BR_i$$

Formal definition of Rationalizability applied to game 3

$$R_1(1) := S_1 = \{M, F\}; R_2(1) := S_2 = \{M, F\}$$

$$R_1(2) := \left\{ \begin{array}{l} s_1 \in R_1(2-1) = \{M, F\} \mid \exists \mu_1 \in \Delta(R_2(2-1)) = \Delta(\{M, F\}): \\ u_1(s_1, \mu_1) \geq u_1(s_1', \mu_1) \forall s_1' \in R_1(2-1) = \{M, F\} \end{array} \right\} = \{M, F\}$$

since $u_1(M, M) = 0 \geq u_1(F, M) = -1 \,\&\, u_1(F, F) = -5 \geq u_1(M, F) = -10$

$$R_2(2) := \left\{ \begin{array}{l} s_2 \in R_2(2-1) = \{M, F\} \mid \exists \mu_2 \in \Delta(R_1(2-1)) = \Delta(\{M, F\}): \\ u_2(s_2, \mu_2) \geq u_2(s_2', \mu_2) \forall s_2' \in R_2(2-1) = \{M, F\} \end{array} \right\} = \{F\}$$

since $u_2(M, F) = -1 \geq u_2(M, M) = -2 \,\&\, u_2(F, F) = -5 \geq u_2(F, M) = -10$

Formal definition of Rationalizability applied to game 3

$$R_1(3) := \left\{ \begin{matrix} s_1 \in R_1(3-1) = \{M, F\} \mid \exists \mu_1 \in \Delta(R_2(3-1)) = \Delta(\{F\}): \\ u_1(s_1, \mu_1) \geq u_1(s_1', \mu_1) \forall s_1' \in R_1(3-1) = \{M, F\} \end{matrix} \right\} = \{F\}$$

$$\text{since } u_1(F, F) = -5 \geq u_1(M, F) = -10$$

$$R_2(3) := \left\{ \begin{matrix} s_2 \in R_2(3-1) = \{F\} \mid \exists \mu_2 \in \Delta(R_1(3-1)) = \Delta(\{M, F\}): \\ u_2(s_2, \mu_2) \geq u_2(s_2', \mu_2) \forall s_2' \in R_2(3-1) = \{F\} \end{matrix} \right\} = \{F\}$$

$$\text{since player 2 con choose only } F$$

Formal definition of Rationalizability applied to game 3

$$R_1(4) := \left\{ \begin{array}{l} s_1 \in R_1(4-1) = \{F\} \mid \exists \mu_1 \in \Delta(R_2(4-1)) = \Delta(\{F\}): \\ u_1(s_1, \mu_1) \geq u_1(s_1', \mu_1) \forall s_1' \in R_1(4-1) = \{F\} \end{array} \right\} = \{F\}$$

$$\text{since player 1 con choose only } F$$

$$R_2(4) := \left\{ \begin{array}{l} s_2 \in R_2(4-1) = \{F\} \mid \exists \mu_2 \in \Delta(R_1(4-1)) = \Delta(\{F\}): \\ u_2(s_2, \mu_2) \geq u_2(s_2', \mu_2) \forall s_2' \in R_2(4-1) = \{F\} \end{array} \right\} = \{F\}$$

$$\text{since player 2 con choose only } F$$

$$\ldots$$

$$R_1 := \bigcap_{t=1}^{\infty} R_1(t) = \{M, F\} \cap \{M, F\} \cap \{F\} \cap \{F\} \cap \ldots = \{F\}$$

$$R_2 := \bigcap_{t=1}^{\infty} R_2(t) = \{M, F\} \cap \{F\} \cap \{F\} \cap \{F\} \cap \ldots = \{F\}$$

$$R := \prod_{i=1}^{n} R_i = \{F\} \times \{F\} = \{F, F\}.$$

Further example of rationalizability

- Consider a partnership between two people:
  - They share a profit

$$P = 4(x + y + 0.25xy)$$

  - that depends on their effort, x and y
  - The effort is any real number in [0,4] and cost to each player respectively x^2 and y^2
  - The players choose the effort simultaneously and indipendently.
  - The game in strategic form is:

$$N = \{1,2\}, \qquad S_i = [0,4],$$

$$v_1(x, y) = 2(x + y + 0.25xy) - x^2$$

$$v_2(x, y) = 2(x + y + 0.25xy) - y^2$$

First best: Pareto efficient efforts

- Find x, y to maximize the joint profit

$$4(x + y + 0.25xy) - x^2 - y^2$$

$$FOC: \quad 4 + \text{y-2x} = 0 \ \text{ and } \ 4 + x - 2y = 0$$

$$x^{FB} = 4 \quad y^{FB} = 4.$$

Solution by rationalizability

- Find the best reply function:

$$\frac{\partial v_1}{\partial x} = 2 + 0.5y - 2x = 0 \Rightarrow x = BR_1(y) = 0.25y + 1$$

$$\frac{\partial v_2}{\partial y} = 2 + 0.5x - 2y = 0 \Rightarrow y = BR_2(x) = 0.25x + 1$$

The set of rationalizable strategies

$$R_i(1) := S_i = [0,4];$$

$$R_i(2) := BR_i(S_j) = 0.25([0,4]) + 1 = [1,2]$$

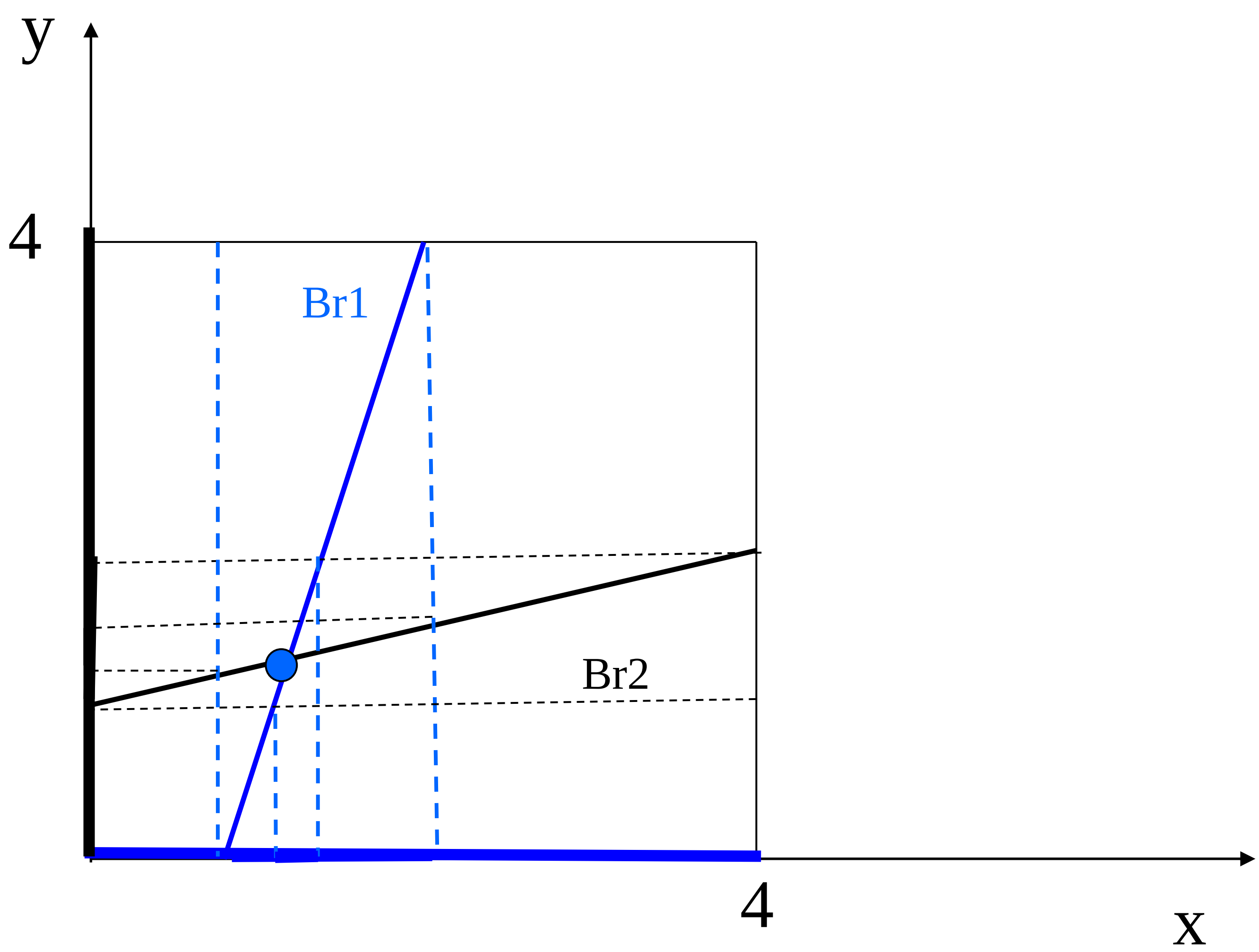
$$R_i(3) := BR_i(R_j(2)) = 0.25([1,2]) + 1 = [1.25, 1.5]$$

$$R_i(4) := BR_i(R_j(3)) = 0.25([1.25, 1.5]) + 1 = [1.31, 1.37]$$

$$\ldots$$

$$R_i := \bigcap_{t=1}^{\infty} R_i(t) = 4/3.$$

Graphically



PROBLEM

What are the connections between rationalizability and iterative deletion of dominated strategies ?

THEY ARE STRATEGICALLY EQUIVALENT

SECOND CRUCIAL PROBLEM

Intelligent players anticipate opponents' rational behavior implying **iterative solutions**

HOWEVER TO MAKE OPERATIVE THIS ANTICIPATION OF OPPONENTS' RATIONAL BEHAVIOR, PLAYERS NEED TO KNOW

1. **OPPONENTS' STRATEGY SETS**
2. **OPPONENTS' PAYOFF FUNCTIONS**

I.E.

THE GAME

However standard models do not specify players' information on the game itself: *information sets regard actions only*

IMPERFECT INFORMATION

VS

INCOMPLETE INFORMATION

Imperfect Information vs. Incomplete Information

- Standard models do not specify players' information on the game itself***: information sets regard actions only***
- Standard **informal assumption**:

The **game** is **common knowledge**, i.e.

1. all the players know the **game**
2. All the players know that all the players know the **game**
3. Etc. ad infinitum

- If a game satisfies this assumption is called **complete information game**

Imperfect Information vs. Incomplete Information

Definitions

- Game of ***imperfect information***: one or more players do not know the full history of the game, i.e. previous moves.
- Game of ***incomplete information***: the players have private information about the **game**, which we will call the **state of nature**.
- We need new formal tools to deal with incomplete information: **information sets are not enough since they regard players' actions**

Example 1: the problem when the true game being played is unknown - 1

State of nature 1

Player 1 \ Player 2	L	R
T	0, 1	1, 0
B	1, 0	0, 1

State of nature 2

Player 1 \ Player 2	L	R
T	1, 0	0, 1
B	0, 1	1, 0

Example 1: players' best response as function of: Prior belief Opponent's strategy

Player 1 rational behavior

	Player 2: L	Player 2: R
$p<0.5$	T	B
$p>0.5$	B	T

$p=Pr^1$ {s=State of nature 1} by player 1

Player 2 rational behavior

Player 1	$q<0.5$	$q>0.5$
T	R	L
B	L	R

$q=Pr^2$ {s=State of nature 1} by player 2

Example 1: the problem when the true game being played is unknown - 3

- As the previous slide shows
  - 1's optimal strategy depends on
    1. Prior belief p **and**
    2. The strategy of 2, which in turn depend on
        1. Prior belief q **and**
        2. The strategy of 1, which in turn depend on
            1. Prior belief p **and**
            2. The strategy of 2, which in turn depend on **…**
- Therefore when we don't know the s.o.n., it is not enough to have beliefs on it (first order beliefs), but we need beliefs on beliefs (second order beliefs), etc. i.e. we need
  - **Infinite hierarchy of beliefs**

Example 1: the problem when the true game being played is unknown - 4

- According to the **Bayesian approach**, each player has a belief on the unknown s.o.n.
- But unlike to decision making problem, in an interactive situation we are naturally lead, as previously shown, to
  - **Infinite hierarchies of beliefs**
- But this object is cumbersome and hardly manageable
- This is the **explicit approach** and its complexity was the main obstacle to the development of the theory of games of incomplete information
- Till a breakthrough by Harsanyi

Bayesian games
and
the Harsanyi approach

Imperfect Information vs. Incomplete Information: Harsanyi idea

- The key to analyze games of incomplete information is to transform them into games of imperfect information by letting nature move first, randomly selecting each possible "state of nature" and "players' information on it", i.e. on each possible **type** **(Harsanyi transformation).**

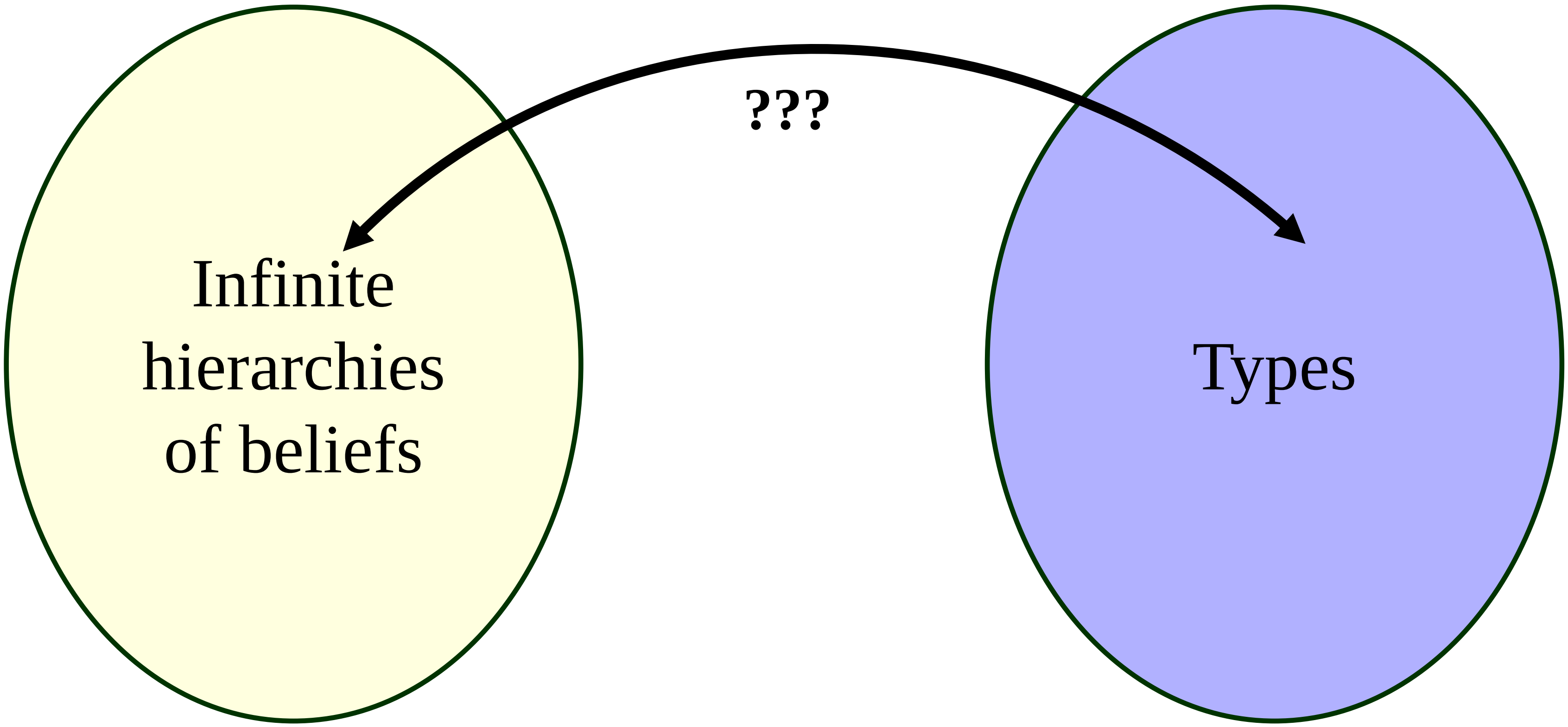
The notion of Bayesian game

- Using the Harsanyi approach, the situation of incomplete information is reinterpreted as a game of imperfect information
- Nature makes the first move, choosing realizations of the random variables that determine
  - each player's **TYPE,**
  - **i.e. each player's PRIVATE INFORMATION ON THE RULE OF THE GAME, INCLUDING OTHER PLAYERS' POSSIBLE PRIVATE INFORMATION**
- Each players observes the realization of only his type
- This sort of game is called **BAYESIAN GAME.**

The notion of TYPE

- A **PLAYER'S SET OF TYPES** is a random variable, its realization is a **PLAYER'S TYPE** representing the **player's private information.**
- In other words **a type is a full description of**
  - **Player's beliefs on the rule of the game i.e. on state of nature**
  - **Beliefs on other players' beliefs on s.o.n. and its own beliefs**
  - **Etc.**
- **NB: there is a circular element in the definition of type, which is unavoidable in interactive situations**
- **i.e. the Harsanyi approach solves the problem of modelling incomplete information in a simple ingenious way at the cost of making the set of possible types potentially extremely complex**

Types and infinite hierarchies of beliefs



Bayesian Games

(Harsanyi, *Management Science* 1967-8)

- u_i = utility function for i, $u_i(a,t)$ depends on both actions a and types t.
- normal form game $G = \{N;A_1,\ldots,A_n;\ u_1,\ldots,u_n\}$
- Bayesian game $\Gamma = \{N;A_1,\ldots,A_n;\ T_1,\ldots,T_n;\ p_1,\ldots,p_n;\ u_1,\ldots,u_n\}$
- A_i = strategy set for i, actions in the Bayesian Game:

  $a = (a_1,\ldots,a_n) \in A = A_1\times\ldots\times A_n$.
- T_i = type set for i, types: $t = (t_1,\ldots,t_n) \in T = T_1\times\ldots\times T_n$
- p_i = beliefs for i, $p_i(t_{-i} \mid t_i)$ = i's belief about types t_{-i} given type t_i.

Bayesian Games
(Harsanyi, *Management Science* 1967-8)

- Beliefs $\{p_1,\ldots,p_n\}$ are ***consistent*** if they can be derived using Bayes' rule from a common joint distribution p(t) on T; i.e., there exists p(t) such that

$$p_i(t_{-i}|t_i) = \frac{p(t)}{p(t_i)} \quad \text{where} \qquad p(t_i) = \sum_{t_{-i} \in T_{-i}} p(t_{-i}, t_i)$$

for all i and t_i.

- Beliefs are **consistent** if nature moves first and types are determined according to the **common prior** p(t) and each i is informed only of t_i.
- Plausible (?) if types are interpreted as full description of a player’s private information

Beliefs derived from common prior - 1

- **EXAMPLE: joint & marginal probability**

	A low costs	A high costs	Marginal Pr of B costs
B low costs	0.45	0.05	0.5
B high costs	0.15	0.35	0.5
Marginal Pr of A costs	0.6	0.4	1

Beliefs derived from common prior - 2

- **EXAMPLE: conditional probability**

Pr{B cost | A cost}

		A INFORMATION	
		A low costs	A high costs
UNCERETAIN EVENT	B low costs	0.45/0.6= =0.75	0.05/0.4= =0.125
	B high costs	0.15/0.6= =0.25	0.35/0.4= =0.875

Beliefs derived from common prior - 3

- **EXAMPLE: conditional probability**

Pr{A cost | B cost}

		B INFORMATION	
UNCERETAIN EVENT		**B low costs**	**B high costs**
	A low costs	**0.45/0.5= =0.9**	**0.15/0.5= =0.3**
	A high costs	**0.05/0.5= =0.1**	**0.35/0.5= =0.7**

Definition

- A ***strategy in a Bayesian game*** for i is a plan of action for each of i's possible types

$$d_i\text{: } T_i \rightarrow A_i$$

- As usual it says what to do in every possible contingency (each of the possible types).

Example 1: a modified prisoner's dilemma with different possible payoffs

- Prisoner 2 has two possible different payoffs:
  - With probability m the players' payoffs are that of figure 1
  - With probability 1-m the players' payoffs are that of figure 2
  - Player 2 payoffs are 2's **private information**
- Thus the players are possibly playing two different games, with player 2 informed of the true game and player 1 not informed (asymmetric information).

The possible payoffs of player 2

Figure 1

Player 1 \ Player 2	DC	C
DC	0, -2	-10, -1
C	-1, -10	-5, -5

Figure 2

Player 1 \ Player 2	DC	C
DC	0, -2	-10, **-7**
C	-1, -10	-5, **-11**

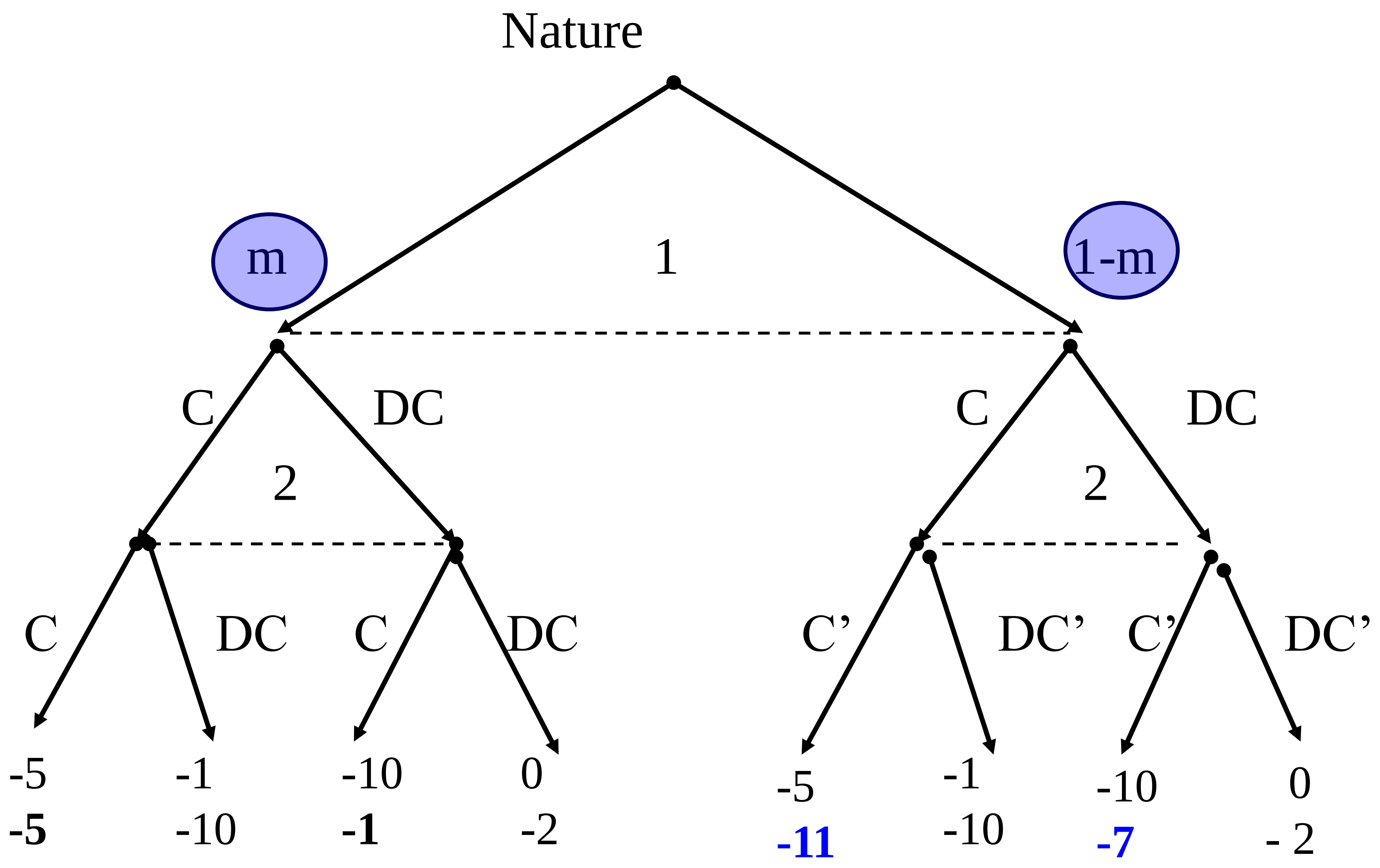
The Harsanyi approach applied to example 1

- According to this approach each player's preferences are determined by the realization of a random variable;
- The random variable's actual realization is observed only by the player
- **Its ex ante probability distribution is assumed to be common knowledge among all the players**
- **Players' types:**
  - player 1 set of types is the null set since player 1 has no private information: $T_1 = \{\varnothing\}$
  - player 2 set of types has two element, the payoffs of figure 1 and figure 2: $T_2 = \{t', t''\}$
- **Players' Beliefs:**
  - $p_1\{t' \mid \varnothing\} = m$
  - $p_2\{\varnothing \mid t'\} = p_2\{\varnothing \mid t''\} = 1.$

The Extensive Form of example 1

$T = \{(Fig\ 1, \emptyset), (Fig\ 2, \emptyset)\}$ $p(t') = Pr\{(Fig\ 1, \emptyset)\} = m \Rightarrow$

$p_1(t'|t_1) = Pr\{Fig\ 1|\emptyset\} = m$ & $p_2(t'|t_2) = Pr\{\emptyset|Fig\ 2\} = Pr\{\emptyset|Fig\ 1\}$=1



The Bayesian strategic form of example 1

1 \ 2	C-C'	C-DC'	DC-C'	DC-DC'
C	-5, -5m-11(1-m)	-5m-1(1-m), -5m-10(1-m)	-5, -5m-11(1-m)	-1, -10
DC	-10, -1m-7(1-m)	-10m+0(1-m), -1m-2(1-m)	0m-10(1-m), -2m-7(1-m)	0, -2

SUMMING UP - 1

- **BAYESIAN GAME:** a game in which players are uncertain on payoff relevant parameters
- **STATE OF NATURE:** payoff relevant data. It is convenient to think of a s.o.n. as a full description of a game form
- **TYPE**: full description of player's relevant characteristics, therefore it fully describes

1. **Player's beliefs (i.e. information) on s.o.n.**
2. **Player's beliefs on others' beliefs**
3. **Player's beliefs on others' beliefs on its beliefs**
4. **Etc. ad infinitum**

SUMMING UP - 2

- **STATE OF THE WORLD:** a specification of s.o.n. ***and*** players' types. i.e. of
  1. Payoff relevant parameters
  2. Beliefs of all levels
- **COMMON PRIOR AND CONSISTENT BELIEFS:** players' beliefs are said to be **consistent** if they are derived from the same probability distribution (the **common prior**) by conditioning on each player's private information. Therefore if beliefs are consistent, the only source of differences in beliefs is difference in information