LECTURE 4

Nash and Bayes-Nash Equilibria in Extensive Form Games

And

Refinements

MAIN POINTS OF PREVIOUS LECTURE

- **GENERAL WAY OF CALCULATING THE SET OF NASH EQUILIBRIA**
- **THE USE OF BEST REPLY CORRESPONDENCES**

Battle of sexes: the set of Nash equilibria in pure and mixed strategies

probability	1 \ 2	probability: y	$1-y$
		h	t
x	H	3, 1	0, 0
$1-x$	T	0, 0	1, 3

x, y between 0 and 1

That is, $0 \le x \le 1$ and $0 \le y \le 1$

Need to calculate player 1's expected utility from player 2's mixed strategy

1 \ 2 (probability)	h (y)	t ($1-y$)	EU_1:
H	3, 1	0, 0	$3y$
T	0, 0	1, 3	$1 - y$

$EU_1(H) = y \times 3 + (1-y) \times 0 = 3y$

$EU_1(T) = y \times 0 + (1-y) \times 1 = 1 - y$

Player 1 best reply depends on player 2 mixed strategy

$EU_1(H) = y \times 3 + (1- y) \times 0 = 3y$
$EU_1(T) = y \times 0 + (1- y) \times 1 = 1 - y$

1 best reply: H iff $EU_1(H) \geq EU_1(T)$

$\therefore \quad 3y \geq 1 - y$

$\Rightarrow \quad 4y \geq 1$

$\Rightarrow \quad y \geq 1/4$

The best reply correspondence of player 1

$$x = \sigma_1(H) = \begin{cases} 1 & if \quad y = \sigma_2(h) \geq 1/4 \\ \in [0,1] & if \quad y = \sigma_2(h) = 1/4 \\ 0 & if \quad y = \sigma_2(h) \leq 1/4. \end{cases}$$

Need to calculate player 2's expected utility from player 1's mixed strategy

probability	1 \ 2	h	t
x	H	3, 1	0, 0
1- x	T	0, 0	1, 3
	EU_2:	x	3 - 3x

$EU_2(h) = x \times 1 + (1- x) \times 0 = x$

$EU_2(t) = x \times 0 + (1- x) \times 3 = 3 - 3x$

Player 2 best reply depends on player 1 mixed strategy

$EU_2(h) = x \times 1 + (1- x) \times 0 = x$
$EU_2(t) = x \times 0 + (1- x) \times 3 = 3 – 3x$

2 best reply: h iff $EU_2(h) \geq EU_2(t)$

$\therefore \quad x \geq 3 - 3x$

$\Rightarrow \quad 4x \geq 3$

$\Rightarrow \quad x \geq 3/4$

The best reply correspondence of player 2

$$y = \sigma_2(h) = \begin{cases} 1 & if \quad x = \sigma_1(H) \geq 3/4 \\ \in [0,1] & if \quad x = \sigma_1(H) = 3/4 \\ 0 & if \quad x = \sigma_1(H) \leq 3/4. \end{cases}$$

The set of Nash equilibria using best reply correspondences

Thus

$$x = \begin{cases} 1 & if \quad y \geq 1/4 \\ \in [0,1] & if \quad y = 1/4 \\ 0 & if \quad y \leq 1/4. \end{cases}$$

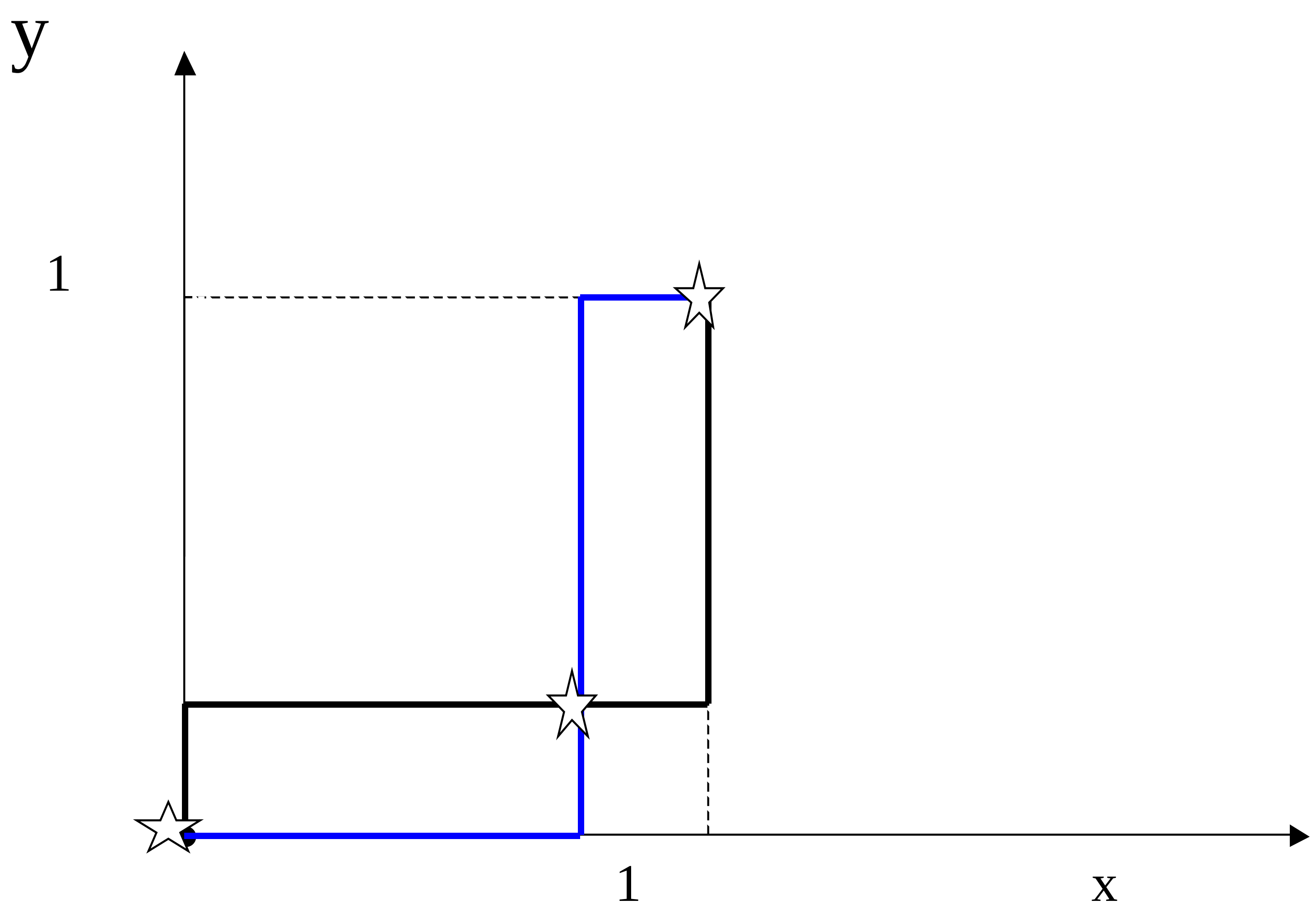
and

$$y = \begin{cases} 1 & if \quad x \geq 3/4 \\ \in [0,1] & if \quad x = 3/4 \\ 0 & if \quad x \leq 3/4. \end{cases}$$

The set of Nash equilibria using best reply correspondences

Pl. 1= black

Pl.2=blue



The set of Nash Equilibria in the battle of sexes

$$NE = \{\sigma_1(H) = 1, \sigma_2(h) = 1\} \cup$$
$$\cup \{\sigma_1(H) = 3/4, \sigma_2(h) = 1/4\} \cup$$
$$\cup \{\sigma_1(H) = 0, \sigma_2(h) = 0\}.$$

NEW CONCEPTS TO MODEL STRATEGIC INTERACTION

Nash Equilibria
in
Extensive Form Games

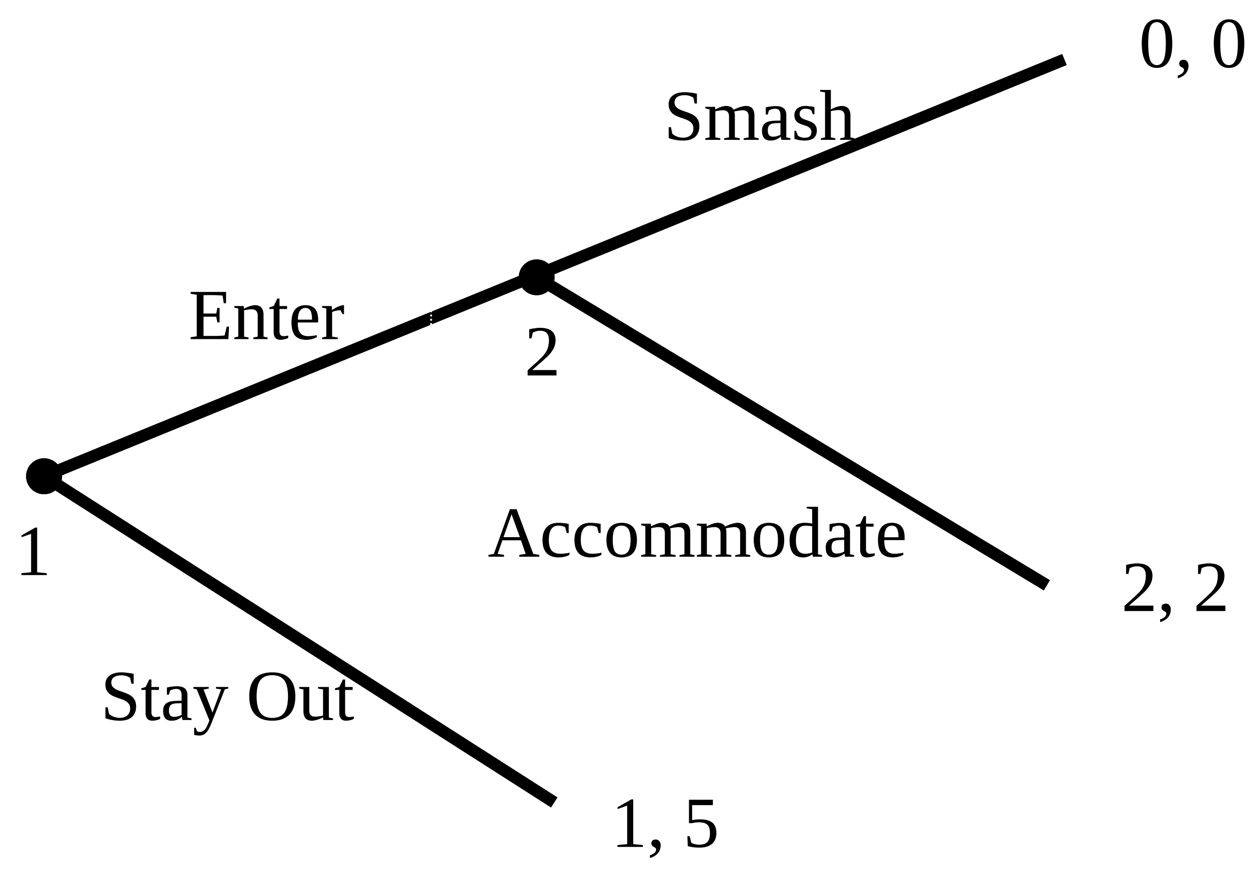
Calculation of Nash Equilibria in EFG

- The definition of Nash equilibrium refers to strategies and payoffs functions

i.e.

- it refers to reduced normal form games
- Therefore to calculate Nash equilibria of an extensive game, first construct the associated reduced normal form.

Example of calculation of Nash equilibria of an extensive game



The associated reduced strategic form game

1 \ 2	Smash	Accommodate
Enter	0, 0	2, 2
Stay Out	1, 5	1, 5

Two Nash equilibria in pure strategies

1 \ 2	Smash	Accommodate
Enter	0, 0	**2**, **2** ★
Stay Out	**1**, **5** ★	1, **5**

Nash equilibria in pure and mixed strategies

2 / 1		q	1-q
		Smash	Accommodate
p	Enter	0, 0	**2**, **2**
1-p	Stay Out	**1**, **5**	1, **5**

The set of Nash equilibria using best reply correspondences

$$E[u_1(E,\sigma_2)] = 0q + 2(1-q)$$
$$E[u_1(SO,\sigma_2)] = 1q + 1(1-q)$$
$$\text{Thus}$$
$$p = \begin{cases} 1 & if \quad q \le 1/2 \\ \in [0,1] & if \quad q = 1/2 \\ 0 & if \quad q \ge 1/2. \end{cases}$$

$$E[u_2(S,\sigma_1)] = 0p + 5(1-p)$$
$$E[u_2(A,\sigma_1)] = 2p + 5(1-p)$$
$$\text{Thus}$$
$$q = \begin{cases} 0 & if \quad p \ge 0 \\ \\ \in [0,1] & if \quad p = 0. \end{cases}$$

The set of Nash equilibria using best reply correspondences

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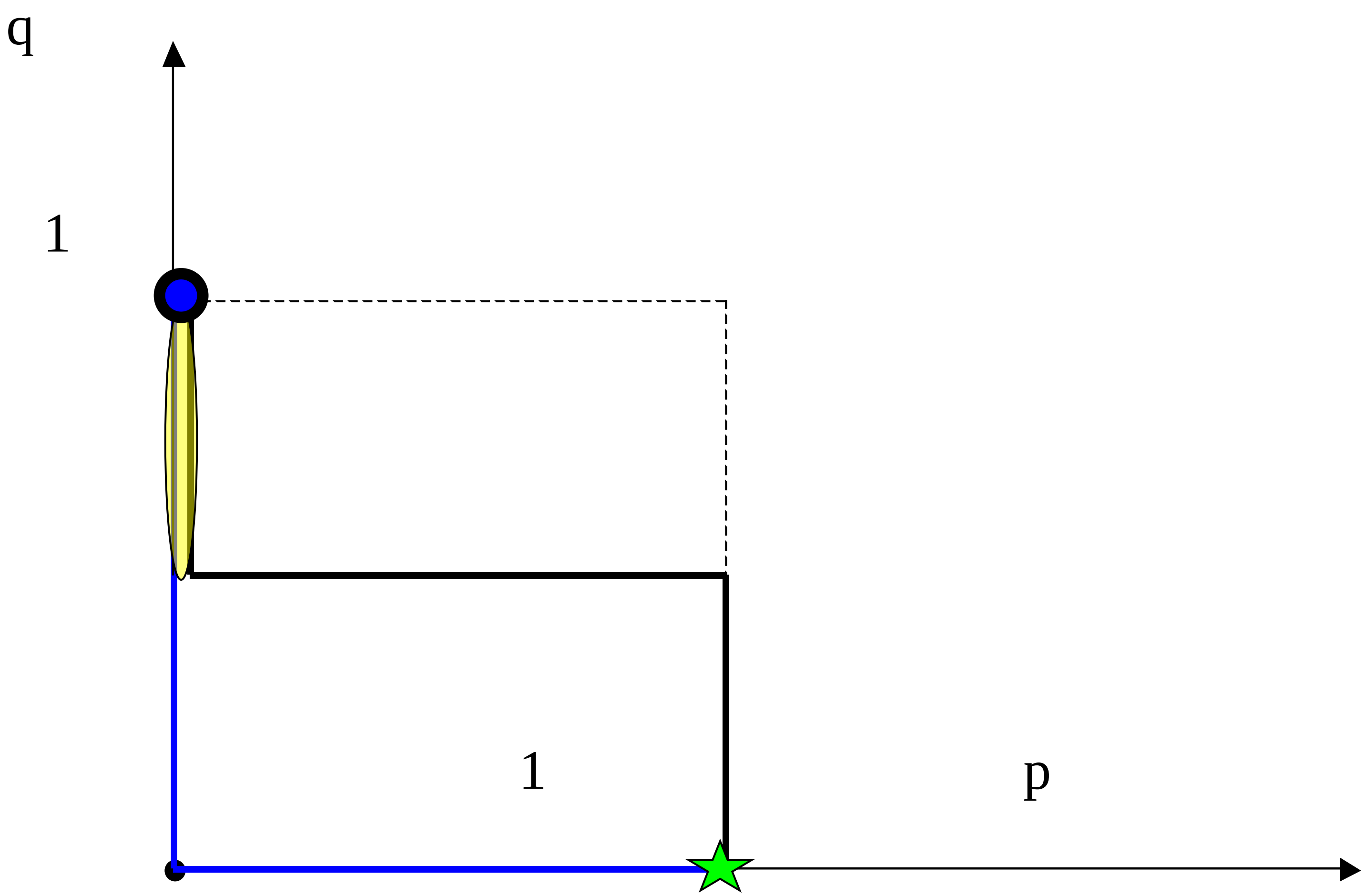


The set of Nash Equilibria in the extensive game

$$NE\left(E.G.\right)=\left\{\sigma_1\left(E\right)=0,\sigma_2\left(S\right)\in\left[\frac{1}{2},1\right]\right\}\cup$$

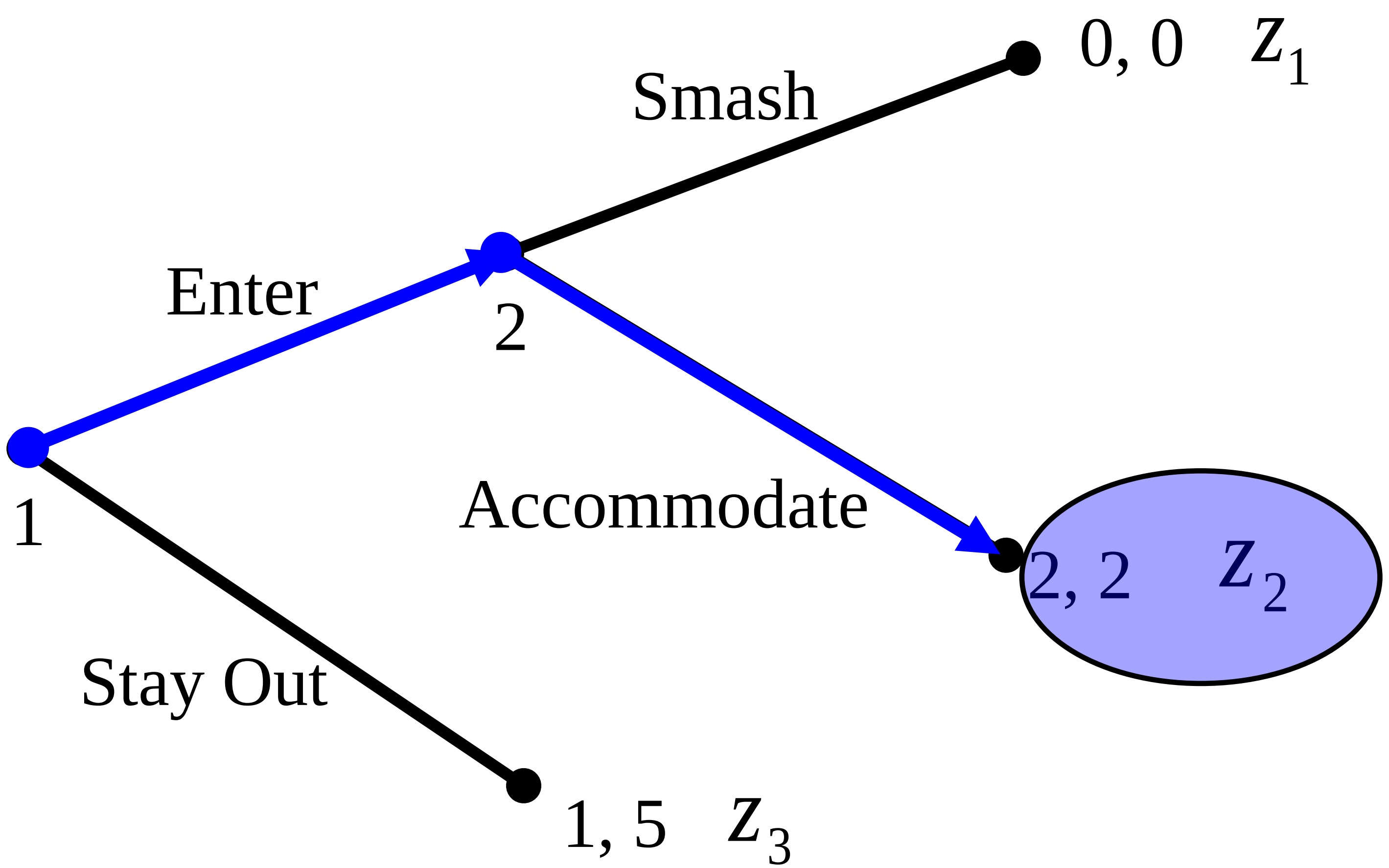
$$\cup\left\{\sigma_1\left(E\right)=1,\sigma_2\left(S\right)=0\right\}$$

Credibility and out-of-equilibrium information sets

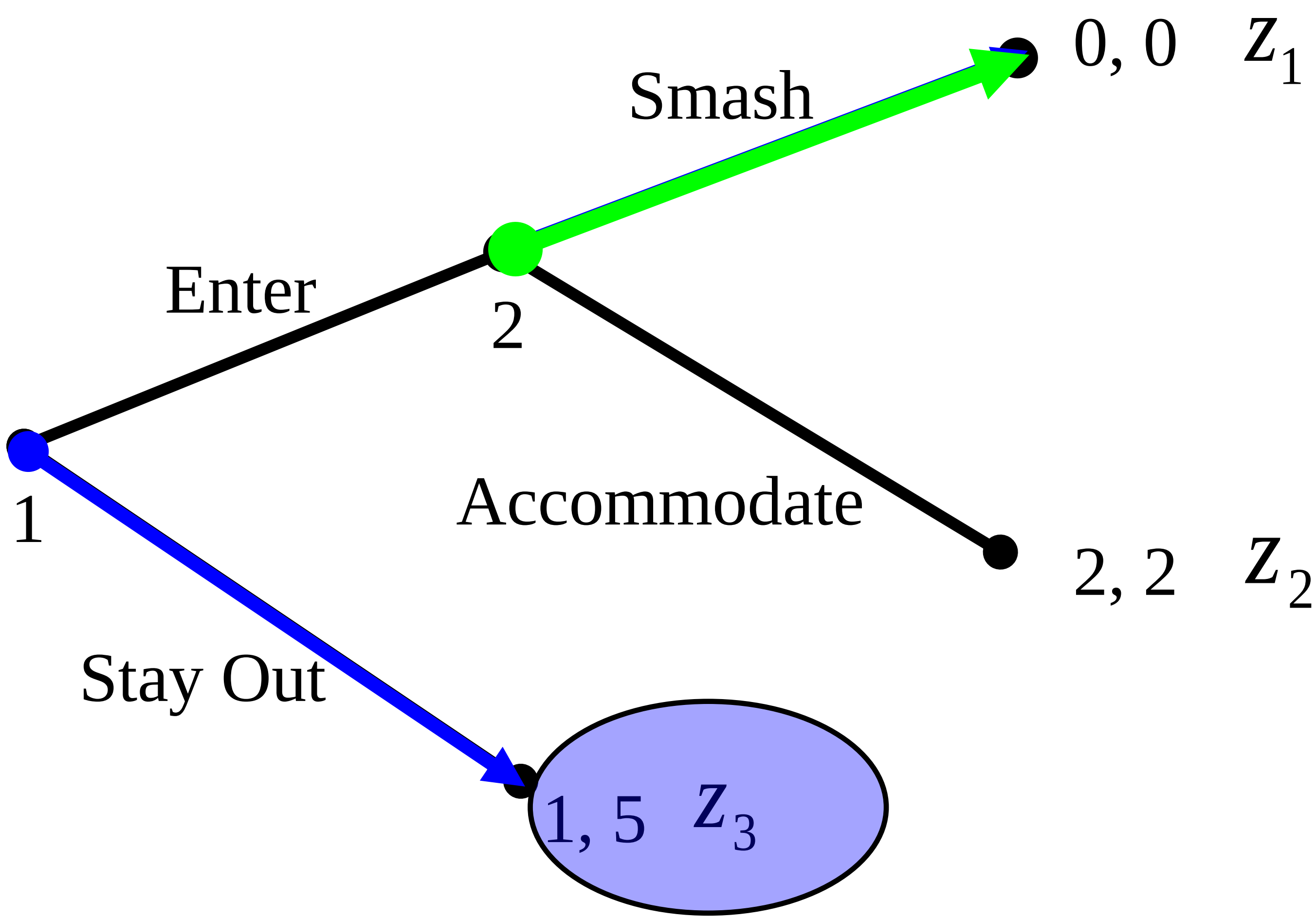
Consider pure strategy Nash equilibria

$$NE^{PS} = \{(E,A)\} \cup \{(S,S)\}.$$

The first equilibrium: Enter, Accomodate



The second equilibrium: Stay Out-Smash



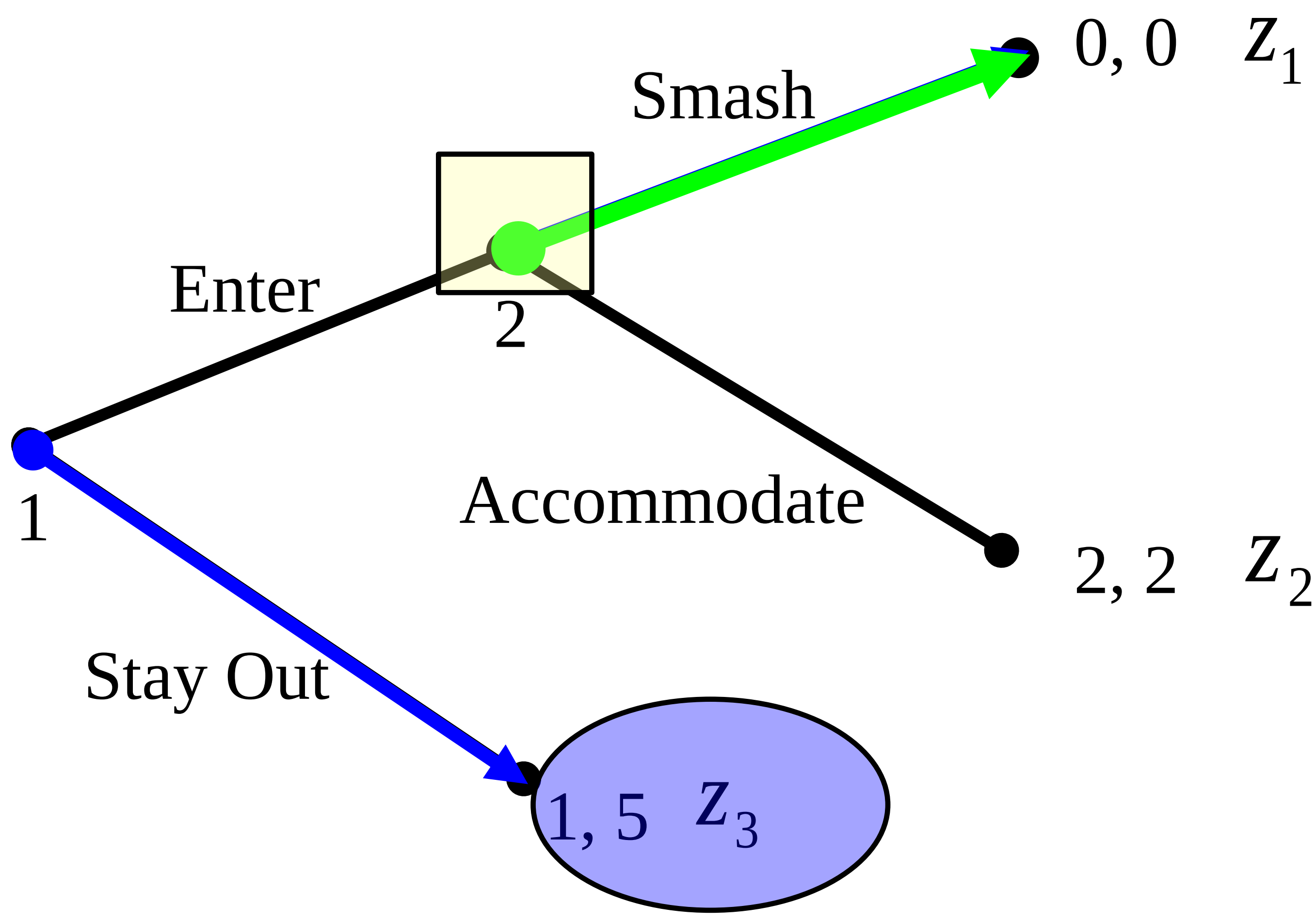
Meaning of the second equilibrium: Stay Out, Smash

- **Threat by 2**: if you will enter, I will smash you
- But **once 2 is called to play**, will 2 have the incentive to carry out the threat?
  - If YES, the threat is credible
  - If NO, the threat is noncredible

The second equilibrium: Stay Out-Smash

In this equilibrium, if 2 will be asked to play, then 2 will prefer to accomodate: the threat is non credible

How is it possible in a Nash equilibrium?



Problems with Nash equilibria

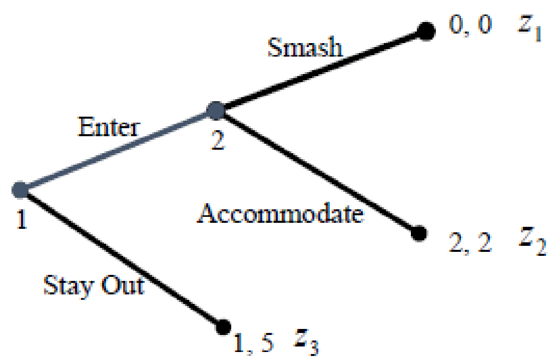
- ***Nash Equilibrium***: each player must act optimally given the other players' strategies, i.e., **play a best response to the others' equilibrium strategies**.

- ***Problem***: **Optimality condition on strategies, i.e. only at the beginning of the game**.
  - Hence, **some Nash equilibria of sequential games involve actions which will not be played in equilibrium**
  - This allows noncredible threats in equilibrium.

Out of equilibrium information sets

- In sequential games there are **equilibrium paths that do not reach some information sets**: these are the **out-of-equilibrium information sets**
- <u>The optimality conditions of **Nash equilibria does not constrain behavior at these nodes**,</u>
- <u>but</u>
- ***<u>these information sets are out-of-equilibrium because of the actions the players are supposed to play at these nodes</u>***
- In other words,
  - **reaching these nodes in equilibrium is a zero probability event, hence it does not matter for expected payoff**
  - **but this probability is *<u>endogeneous</u>*, because it is derived from the players' equilibrium behavior**
  - **And players' equilibrium behavior depends on this zero probability events**

Out of equilibrium information sets in the entry game



- Formally, for any strategy profile:

$$v_2(\pi_1, \pi_2) =$$

$$= v_2(z_1) \times P(z_1 \mid \pi_1, \pi_2) + v_2(z_2) \times P(z_2 \mid \pi_1, \pi_2) +$$

$$+ v_2(z_3) \times P(z_3 \mid \pi_1, \pi_2) =$$

$$= 0 \times \cancel{\pi_1(E)} \times \pi_2(S) + 2 \times \cancel{\pi_1(E)} \times \pi_2(A) + 5 \times \pi_1(SO).$$

- Suppose **1 plays Stay out, i.e.** $\pi_1(SO) = 1 \,\&\, \pi_1(E) = 0$
- Then **player 2's payoff does not depend on his strategy**

$$v_2(\pi_1, \pi_2) = 5$$

- **Therefore any 2's strategy is a best reply to 1's SO**

WHAT IS A POSSIBLE SOLUTION TO THIS PROBLEM?

Sequential rationality

Sequential Rationality

- An optimal strategy for a player should maximize his or her payoff, **conditional on every information set at which this player has the move**
- In other words, player i's strategy should specify an "optimal" action from each of player i's information sets, **even those that have zero endogenous probability to be reached**

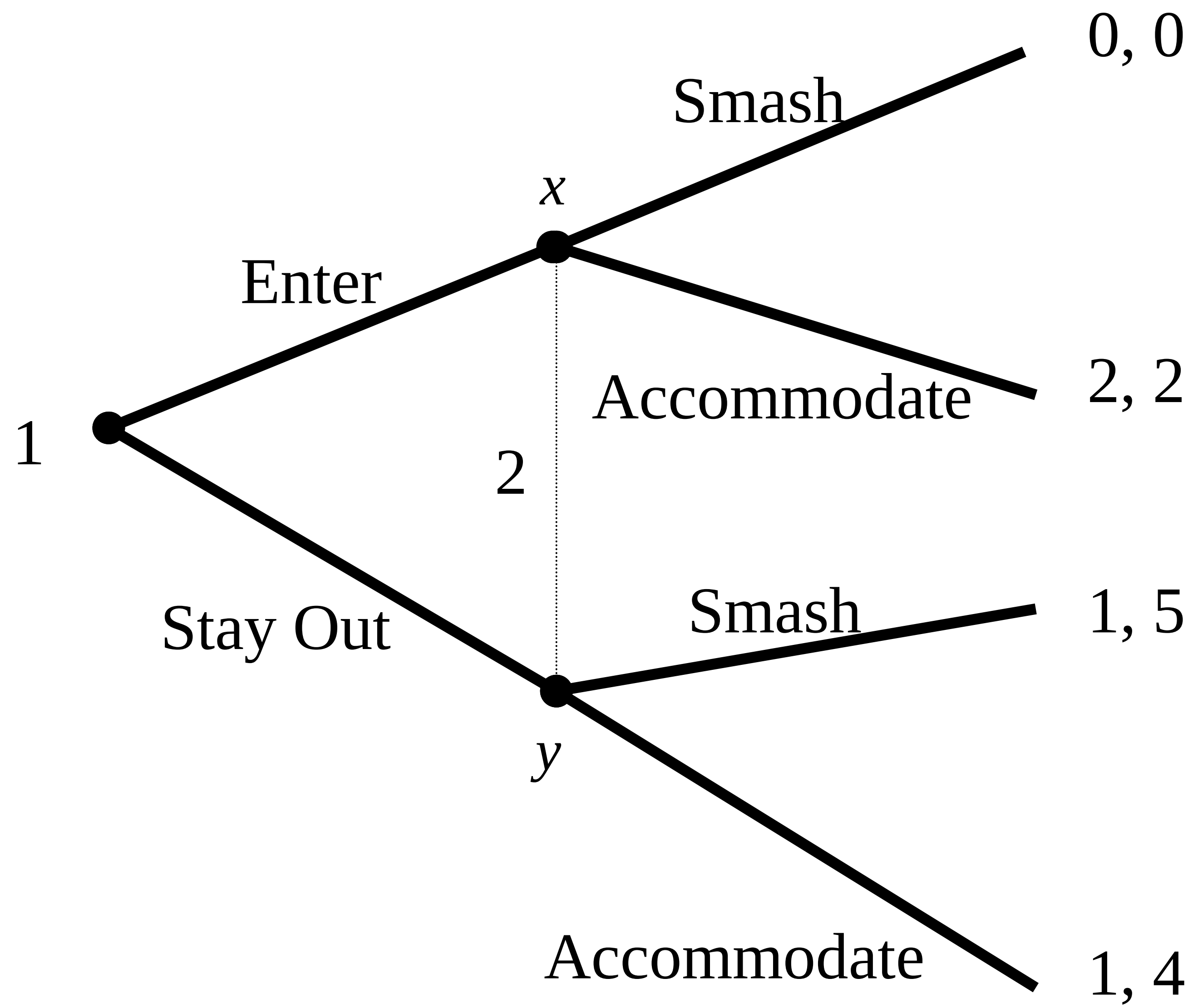
- **Sequential rationality:**
- apply **some notion of rational behavior** any time you face a well defined decision situation, i.e. in any information set
- This implies that players make threats and promises that they do have an incentive (according to that notion of rational behavior) to carry out, once the information set is reached, even if it had ex ante zero probability.

Sequential Rationality

1. **Bayesian rationality**
2. Bayesian updating

Sequential rationality in imperfect information games

- ***The idea of Sequential Rationality:***
  - **Every decision must be part of an optimal strategy for the remainder of the game**
- ***In games with imperfect information*:**
  - **At every decision situation (=information set) the player's subsequent strategy must be optimal with respect to some assessment of the probabilities of all uncertain events, including any preceding but unobserved choices made by other players (Bayesian rationality).**


0, 0
Smash
x
Enter
Accommodate
2, 2
1
2
Smash
1, 5
Stay Out
y
Accommodate
1, 4

Construction of a formal definition of sequential rationality

- Information possessed by the players in an extensive-form game is represented in terms of information sets.
- An information set *h(x)* for player i is a set of *i*'s decision nodes *x* among which *i* cannot distinguish. This implies that the same set of actions must be feasible at every node in an information set.
- Let this set of actions be denoted A(h). Also, let the set of player *i*'s information set be H_i and the set of all information sets be *H*.
- Restrict attention to games of perfect recall.

Construction of a formal definition of sequential rationality: notation

- A **behavior strategy** for player i is the collection

$$\pi_i \equiv \{\pi_h^i(a)\}_{h \in H_i}$$

where for each $h \in H_i$ and each $a \in A(h)$, $\pi_h{}^i(a) \geq 0$ and

$$\sum_{a \in A(h)} \pi_h^i(a) = 1.$$

- $\pi_h{}^i(a)$ is a probability distribution that describes i's behavior at information set h.
- $\pi = (\pi^1, \ldots, \pi^n)$
- $\pi^{-i} = (\pi^1, \ldots, \pi^{i-1}, \pi^{i+1}, \ldots, \pi^n)$.

Construction of a formal definition of sequential rationality: definitions - 1

- A ***system of beliefs*** μ is a specification $\mu_h(x)$ for each information set h, where
- **$\mu_h(x) \geq 0$ is the (conditional) probability player i assesses that a node $x \in h \in H_i$ has been reached, GIVEN $h \in H_i$.**
- Therefore $\sum_{x \in h} \mu_h(x) = 1 \quad \forall h \in H$

Construction of a formal definition of sequential rationality: definitions - 2

- An ***assessment*** *is* *a* beliefs-strategies pair (μ,π).

Definition of
SEQUENTIAL RATIONALITY
for imperfect information games

An assessment (μ,π) is ***sequentially rational*** if

- **given** the beliefs μ
- **each player's behavior strategy $\{\pi_h^i\}_h$ is a best response to (μ,π^{-i}) at *any* information set $h \in H_i$.**

Formal definition of
SEQUENTIAL RATIONALITY

An assessment (μ,π^*) is ***sequentially rational*** if

$$\forall i \in N, \quad \forall h \in H_i$$

$$\sum_{x\in h} \mu(x) \sum_{z\in Z(x)} v_i(z) P\left(z \mid \pi^*\right) \geq$$

$$\geq \sum_{x\in h} \mu(x) \sum_{z\in Z(x)} v_i(z) P\left(z \mid \pi_i', \pi_{-i}^*\right)$$

$$\forall \pi_i' \in \Pi_i$$

REMARK: ***sequential rationality requires players to use π^* to evaluate the "continuation" probability***

Effect of sequential rationality for imperfect information games

1. First, it eliminates strictly dominated actions from consideration off the equilibrium path.
2. Second, it **elevates beliefs to the importance of strategies**.

- **This provides a language — the language of beliefs — for discussing the merits of competing sequentially rational equilibria**.
    - ***Where these beliefs come from?***

beliefs are derived from the equilibrium strategies through Bayes' rule

$$\forall h(x) \text{ such that } \Pr(h(x) \mid \pi) > 0$$

$$\mu_{h(x)}(x) = \frac{\Pr(x \mid \pi)}{\Pr(h(x) \mid \pi)} \qquad \forall x \in h(x)$$

Sequential Rationality & Equilibrium as perfect forecast

1. Bayesian rationality
2. **Bayesian updating**

WEAK PERFECT BAYESIAN EQUILIBRIUM

Definition of WEAK PERFECT BAYESIAN EQUILIBRIUM

A ***Weak Perfect Bayesian equilibrium*** is an assessment (μ,π) such that

1. Each player is sequentially rational, i.e. each player's **behavior strategy is a best response at *any* information set $h \in H_i$, given her beliefs and opponents' behavior**, i.e.

$$\text{for any } h \in H_i,\ \pi_i(h) \in BR_i(\mu_h, \pi_{-i})$$

2. The **beliefs are derived from the equilibrium strategies** through Bayes' rule whenever possible, i.e.

$$\forall h(x) \text{ such that } \Pr(h(x) \mid \pi) > 0$$

$$\mu_{h(x)}(x) = \frac{\Pr(x \mid \pi)}{\Pr(h(x) \mid \pi)} \qquad \forall x \in h(x)$$

Nash equilibria in extensive form games

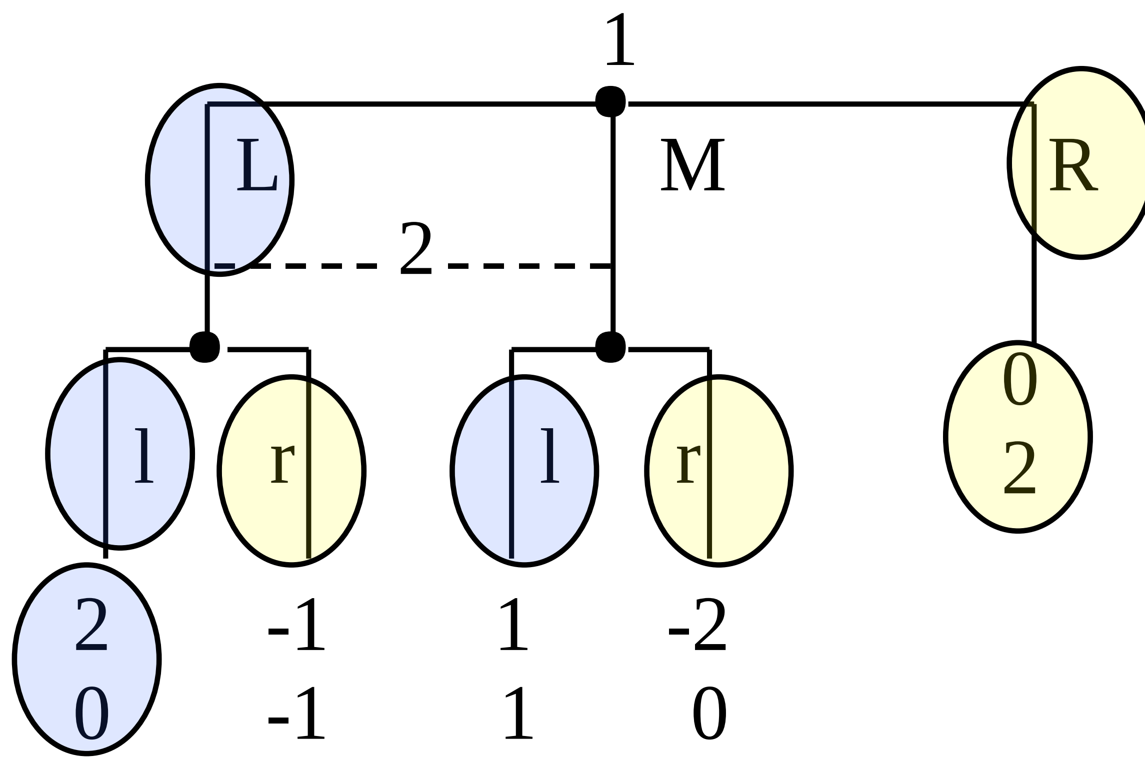
Calculate the Nash equilibria in pure strategies of the extensive form game

Nash equilibria in pure strategies: (L,l) and (R,r)

Normal Form

1 \ 2	l	r
L	$\underline{2}$, $\underline{0}$	-1,-1
M	1, $\underline{1}$	-2, 0
R	0, $\underline{2}$	$\underline{0}$, $\underline{2}$

Extensive Form



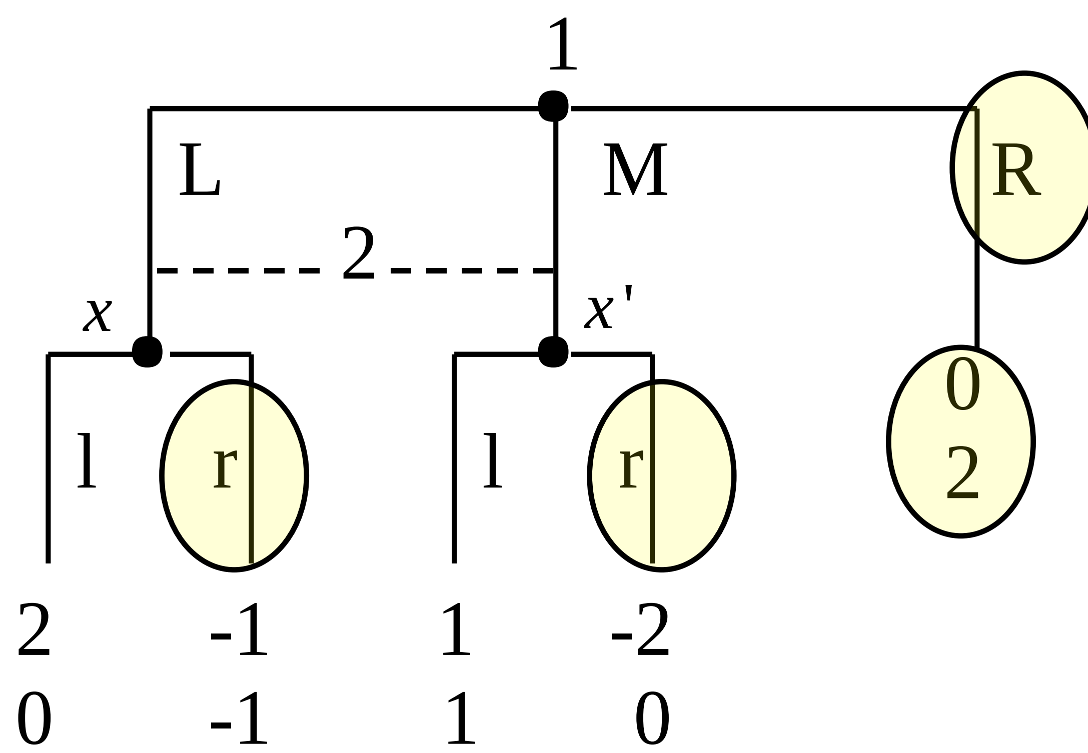
Problem with (R,r) : *r* is a strictly dominated action

- **(R,r) involves an non credible action by player 2**: if 2 gets the move, then r is a strictly dominated action for 2, so **no matter what player 1 did it is not in 2's interest to play r**. And yet, (R, r) is a NE.

Normal Form

		2: l	2: r
1	L	$\underline{2}$, $\underline{0}$	-1,-1
	M	1, $\underline{1}$	-2, 0
	R	0, $\underline{2}$	$\underline{0}$, $\underline{2}$

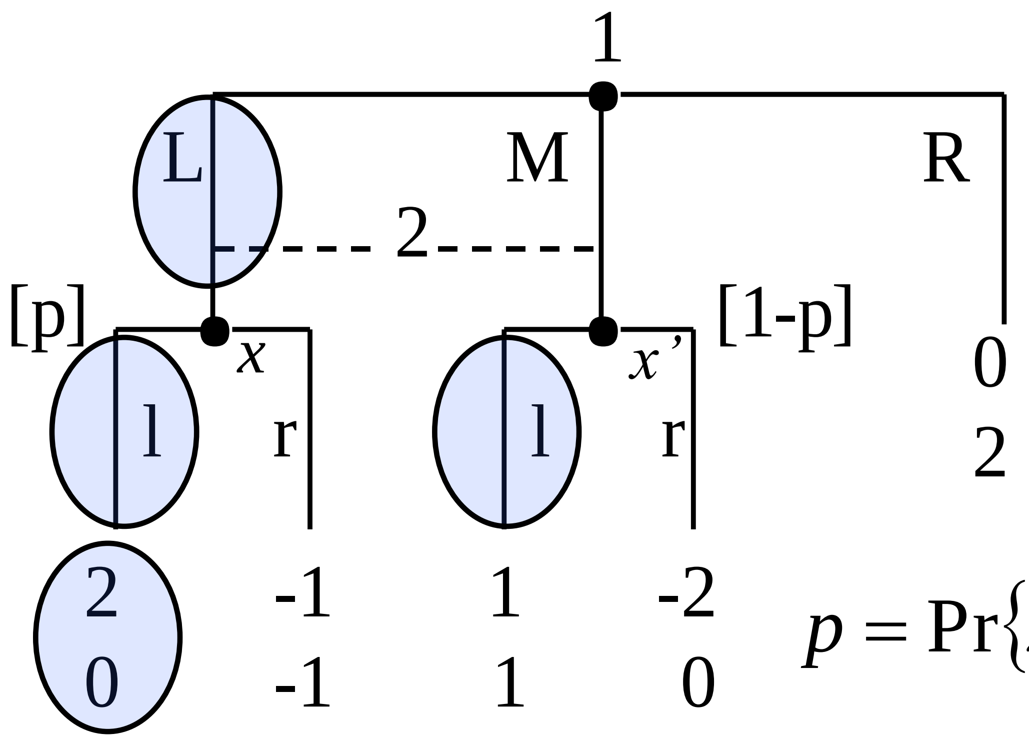
Extensive Form



Game 1: how to calculate WPBE.

Start with the first possible NE:

1. $\pi^1(L)=1, \pi^2(l)=1$



[.] denotes a system of beliefs μ.

$$p = \Pr\{x \mid \{x, x'\}\} = \frac{\Pr\{x \mid \hat{\pi}\}}{\Pr\{\{x, x'\} \mid \hat{\pi}\}} =$$

$$= \frac{\pi^1(L)}{\pi^1(L) + \pi^1(M)} = \frac{1}{1+0} = 1$$

Calculus of WPBE in Game 1:

- Strategy l is sequentially rational for *the* system of **belief derived from equilibrium strategies** using Bayes rule:

$$Eu_2(l \mid p=1) = 0\times 1 + 1\times 0 = 0 > Eu_2(r \mid p) = -1\times 1 + 0\times 0 = -1$$

And L is a best reply for player 1 to l

$$Eu_1(L,l) = 2 > Eu_1(M,l) = 1$$

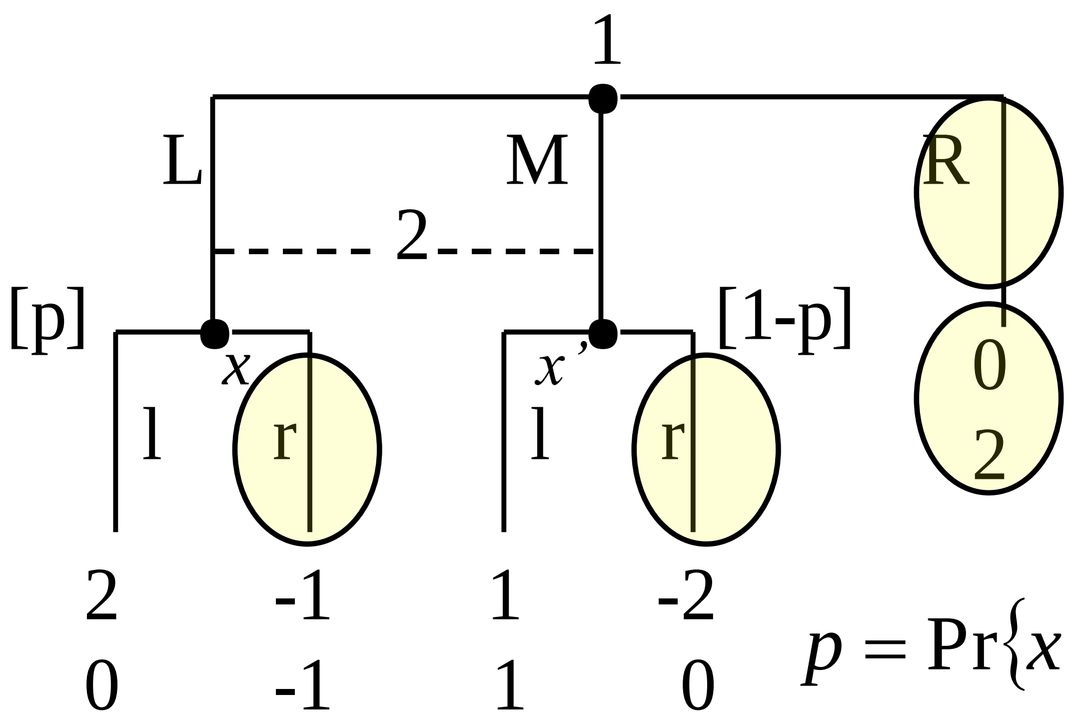
$$Eu_1(L,l) = 2 > Eu_1(R,l) = 0$$

Therefore (L,l),p=1 is a WPBE

Game 1: how to calculate WPBE.

Then consider the second possible NE:

2. $\pi^1(R)=1, \pi^2(r)=1$



[.] denotes a system of beliefs μ.

$$p = \Pr\{x \mid \{x, x'\}\} = \frac{\Pr\{x \mid \hat{\pi}\}}{\Pr\{\{x, x'\} \mid \hat{\pi}\}} =$$

$$= \frac{\pi^1(L)}{\pi^1(L) + \pi^1(M)} = \frac{0}{0+0} \in [0,1] = p$$

Game 1:

- Strategy r is not sequentially rational for ***any*** **possible system of belief:**

$$Eu_2(l \mid p) = 0 \times p + 1 \times (1 - p) > -1 \times p + 0 \times (1 - p) = Eu_2(r \mid p)$$

$$\Updownarrow$$

$$Eu_2(l \mid p) = (1 - p) > -p = Eu_2(r \mid p) \Leftrightarrow Eu_2(l \mid p) = 1 > 0 = Eu_2(r \mid p)$$

- This is how weak perfect Bayesian equilibrium prevents strictly dominated strategies from being used as threats off the equilibrium path: they are not sequentially rational for any possible system of beliefs.

Theorem

A strategy profile π is a Nash equilibrium of an EFG if and only if there exists a system of beliefs μ such that

1. The strategy profile π is sequentially rational given a belief system μ **at all information sets h such that** $\Pr(h \mid \pi) > 0$
2. The system of beliefs μ is derived from π through Bayes' rule whenever possible.

Hence:

$$WPBE_{\pi} \subseteq NE$$

Existence result

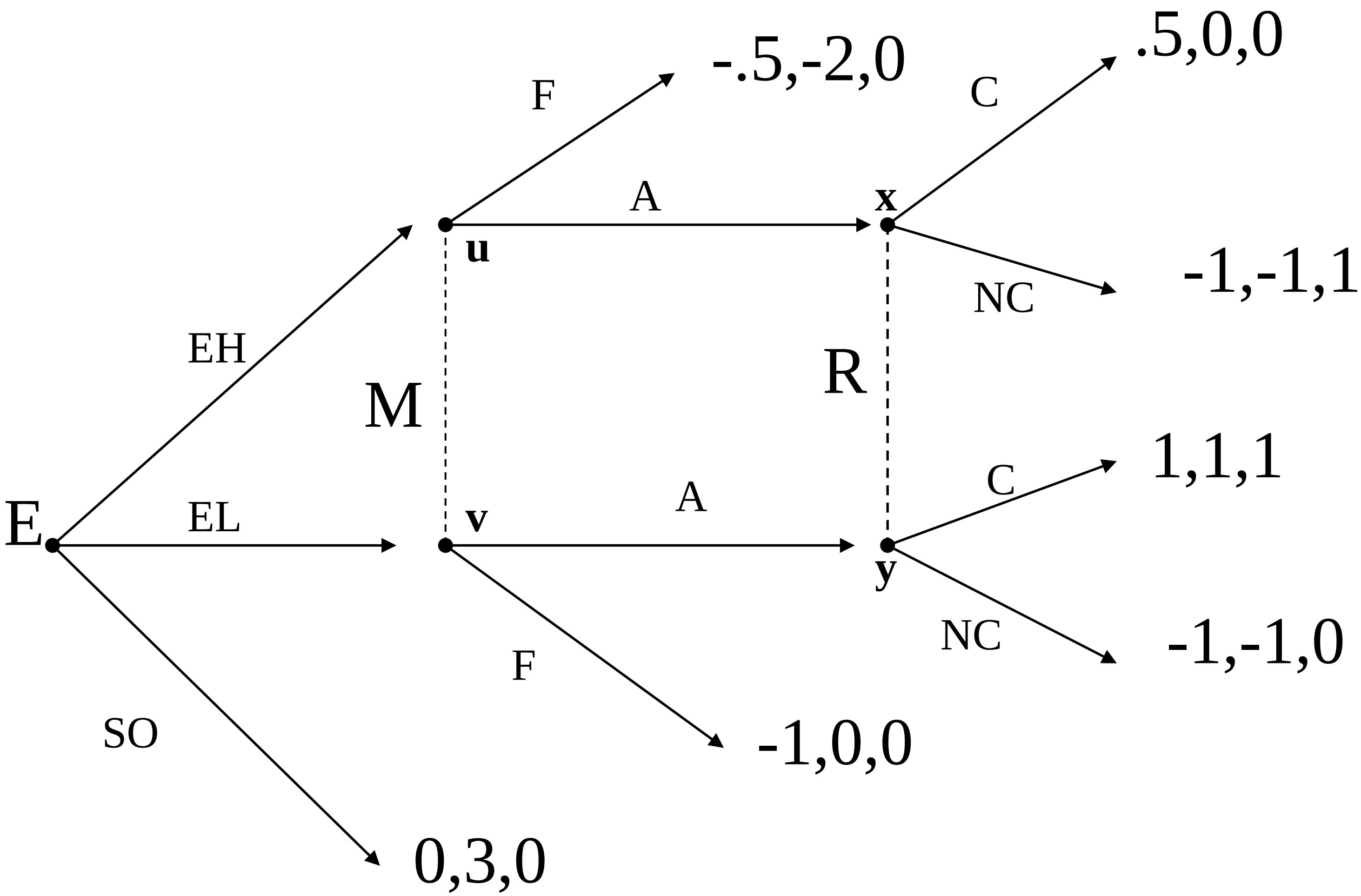
For every finite extensive-form game there exists at least one weak perfect Bayesian equilibrium.

AN EXAMPLE OF HOW TO CALCULATE WPBE

- MODIFIED ENTRY GAME:

- **Players**: Entrant, Monopolist and Regulator
- **Rules of the game**:
  - – E enter with high or low investment or stays out
  - – M cannot observe the amount of investment and have to decide whether to accomodate or fight
  - – If M accomodates, R, who is uninformed of the amount of investment, has to decides whether the market situation conforms to existing regulation or does not.

The extensive game



THE SET OF WPBE

First the set of Nash Equilibria since

$$WPBE_{\pi} \subseteq NE$$

EH

	C	NC
A	0.5, 0, 0	-1, -1, 1
F	-0.5, -2, 0	-0.5, -2, 0

EL

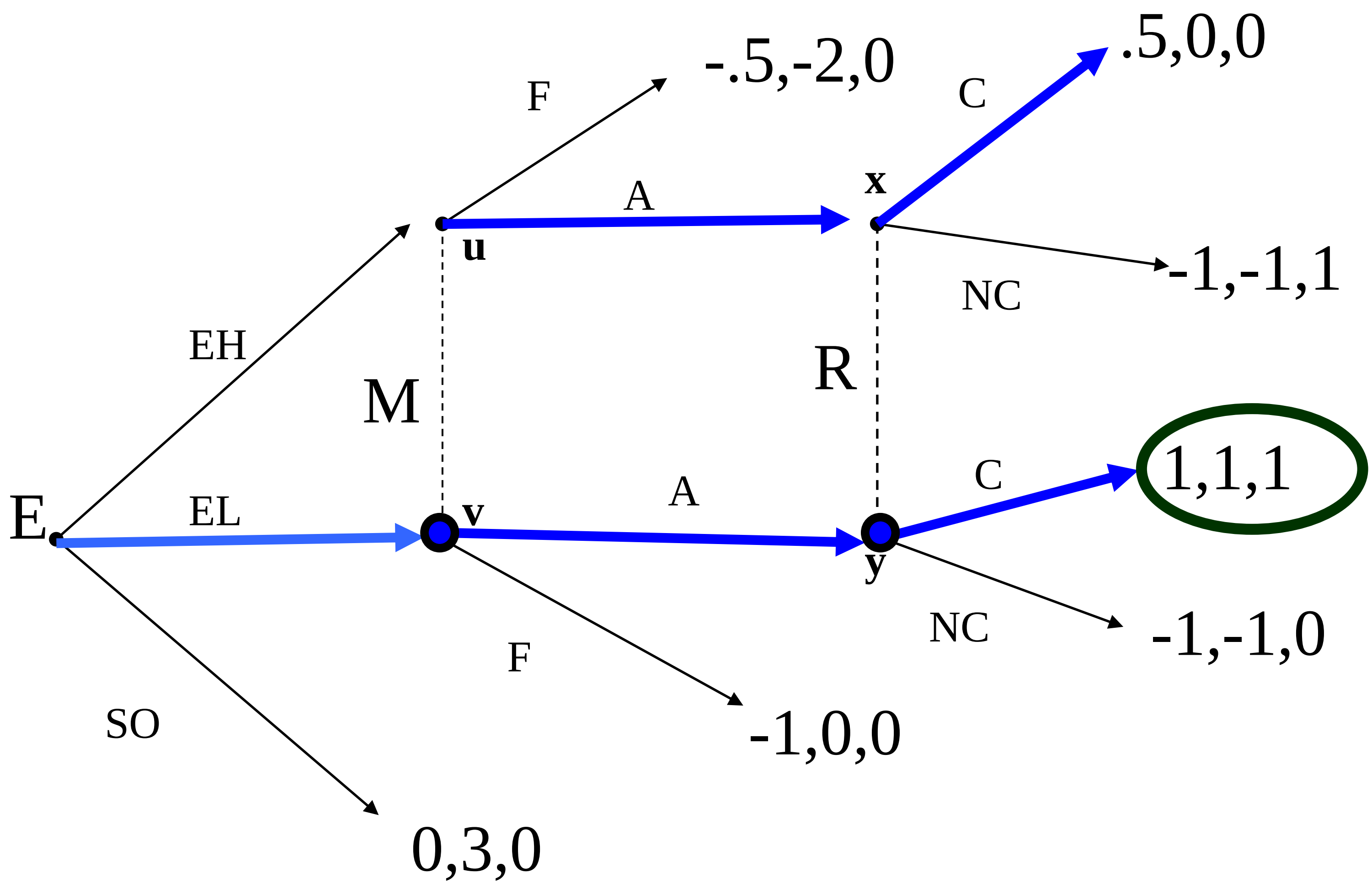
	C	NC
A	1, 1, 1	-1, -1, 0
F	-1, 0, 0	-1, 0, 0

SO

	C	NC
A	0, 3, 0	0, 3, 0
F	0, 3, 0	0, 3, 0

Four NE: (EL,A,C), (SO,A,NC), (SO,F,C), (SO,F,NC)

Is (EL,A,C) a WPBE?



$$\pi_E(EL) = 1, \pi_M(A) = 1, \pi_R(C) = 1$$

The first possible WPBE

- The following assessment is a WPBE:

$$\pi_E(EL)=1, \pi_M(A)=1, \pi_R(C)=1$$

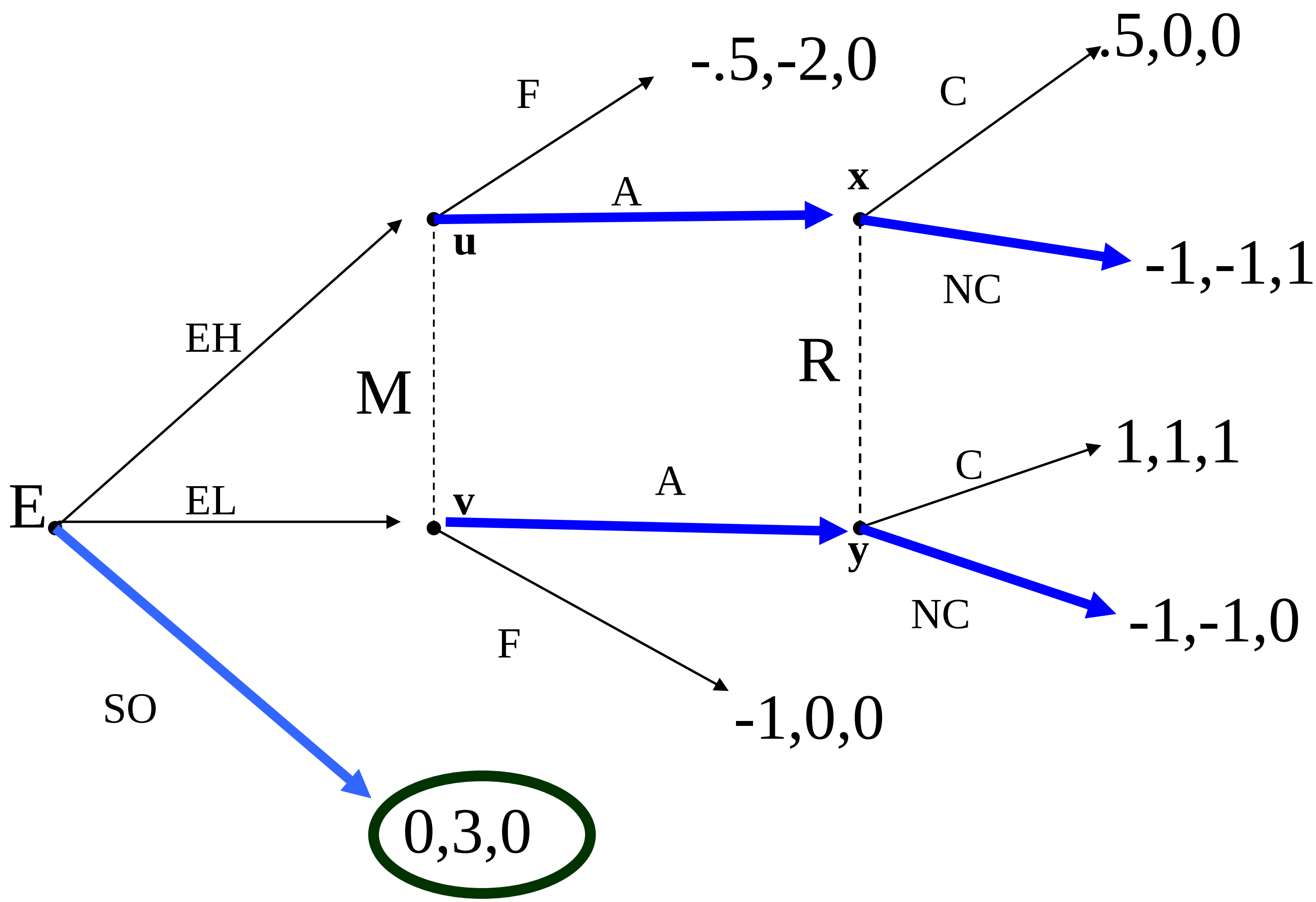
$$\mu\left(u\middle|\{u,v\}\right)=\mu\left(x\middle|\{x,y\}\right)=0$$

- Since C is a best reply to μ(y)=1 and A is a best reply to μ(v)=1 & to C.
- Check if beliefs can be derived by Bayes rule

$$\mu\left(u\middle|\{u,v\}\right)=\frac{\Pr\left(\{u\}\mid\pi\right)}{\Pr\left(\{u,v\}\mid\pi\right)}=\frac{\pi_E(EH)}{\pi_E(EL)+\pi_E(EH)}=\frac{0}{1+0}=0$$

$$\mu\left(x\middle|\{x,y\}\right)=\frac{\Pr\left(\{x\}\mid\pi\right)}{\Pr\left(\{x,y\}\mid\pi\right)}=\frac{\pi_E(EH)\times\pi_M(A)}{\pi_M(A)[\pi_E(EL)+\pi_E(EH)]}=\frac{0\times 1}{1[1+0]}=0$$

Is (SO, A, NC) a WPBE?



$$\pi_E(SO) = 1, \pi_M(A) = 1, \pi_R(NC) = 1$$

A second WPBE

- The following assessment is a WPBE:

$$\pi_E(SO)=1, \pi_M(A)=1, \pi_R(NC)=1$$

$$\mu\left(u\middle|\{u,v\}\right)=\mu\left(x\middle|\{x,y\}\right)\geq 1/2$$

- Since
  - SO is best reply to A&NC
  - NC is a best reply to $\mu(x) \geq ½ \Leftrightarrow \mu(x|\{x,y\}) \geq 1- \mu(x)$
  - A is a best reply to $\mu(u) \geq ½$ & to NC $\Leftrightarrow$

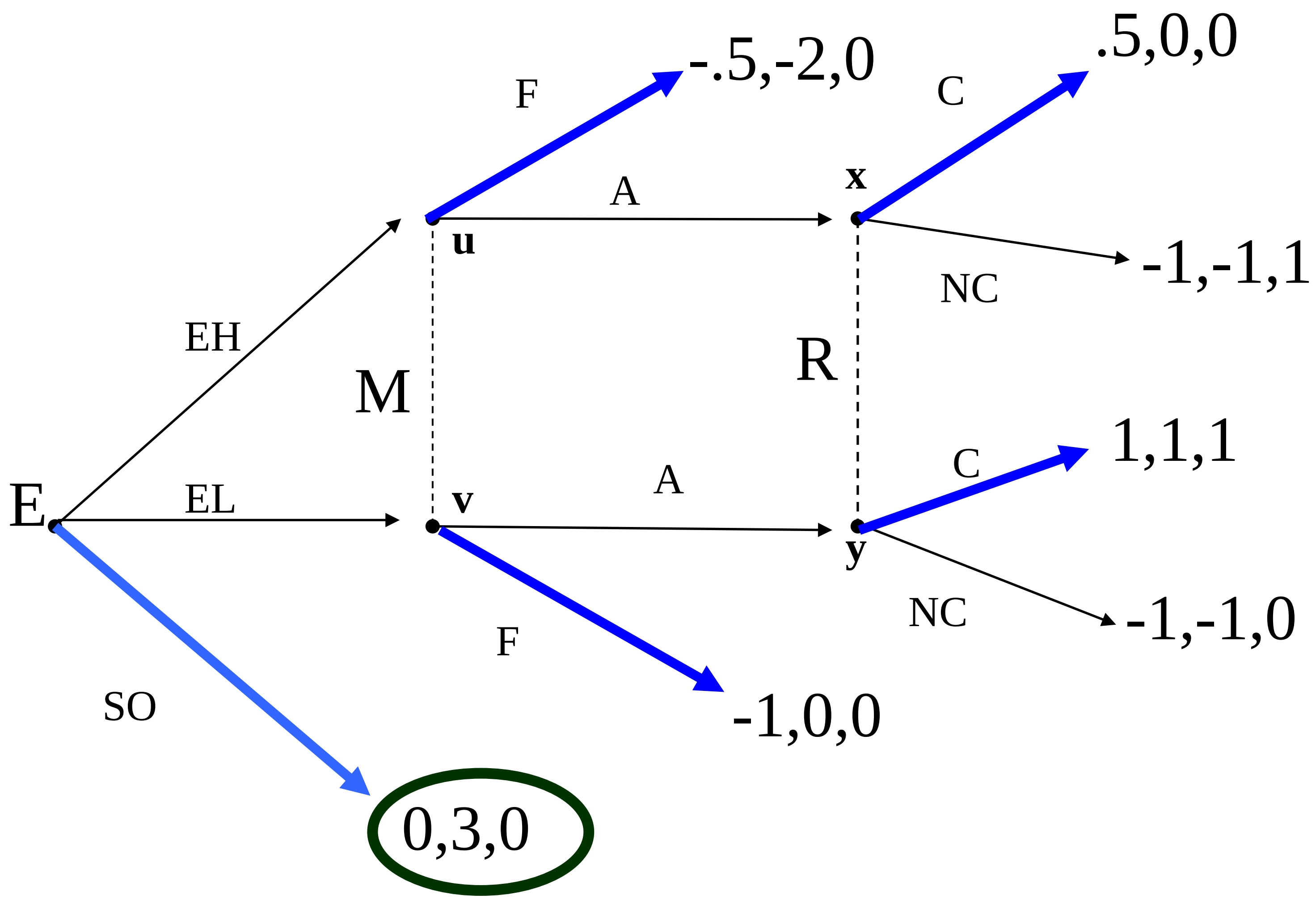
$$\Leftrightarrow \; - \mu(u|\{u,v\})-(1- \mu(u|\{u,v\})) \geq -2\mu(u)$$

- check if beliefs can be derived by Bayes rule

$$\mu\left(u\middle|\{u,v\}\right)=\frac{\Pr\left(\{u\}\,|\,\pi\right)}{\Pr\left(\{u,v\}\,|\,\pi\right)}=\frac{\pi_E(EH)}{\pi_E(EL)+\pi_E(EH)}=\frac{0}{0+0}\in[0,1]$$

$$\mu\left(x\middle|\{x,y\}\right)=\frac{\Pr\left(\{x\}\,|\,\pi\right)}{\Pr\left(\{x,y\}\,|\,\pi\right)}=\frac{\pi_E(EH)\times\pi_M(A)}{\pi_M(A)[\pi_E(EL)+\pi_E(EH)]}=\frac{\pi_E(EH)}{\pi_E(EL)+\pi_E(EH)}=\mu(u)=\frac{0}{0+0}\in[0,1]$$

Is (SO, F, C) a WPBE?



$$\pi_E(SO) = 1, \pi_M(F) = 1, \pi_R(C) = 1$$

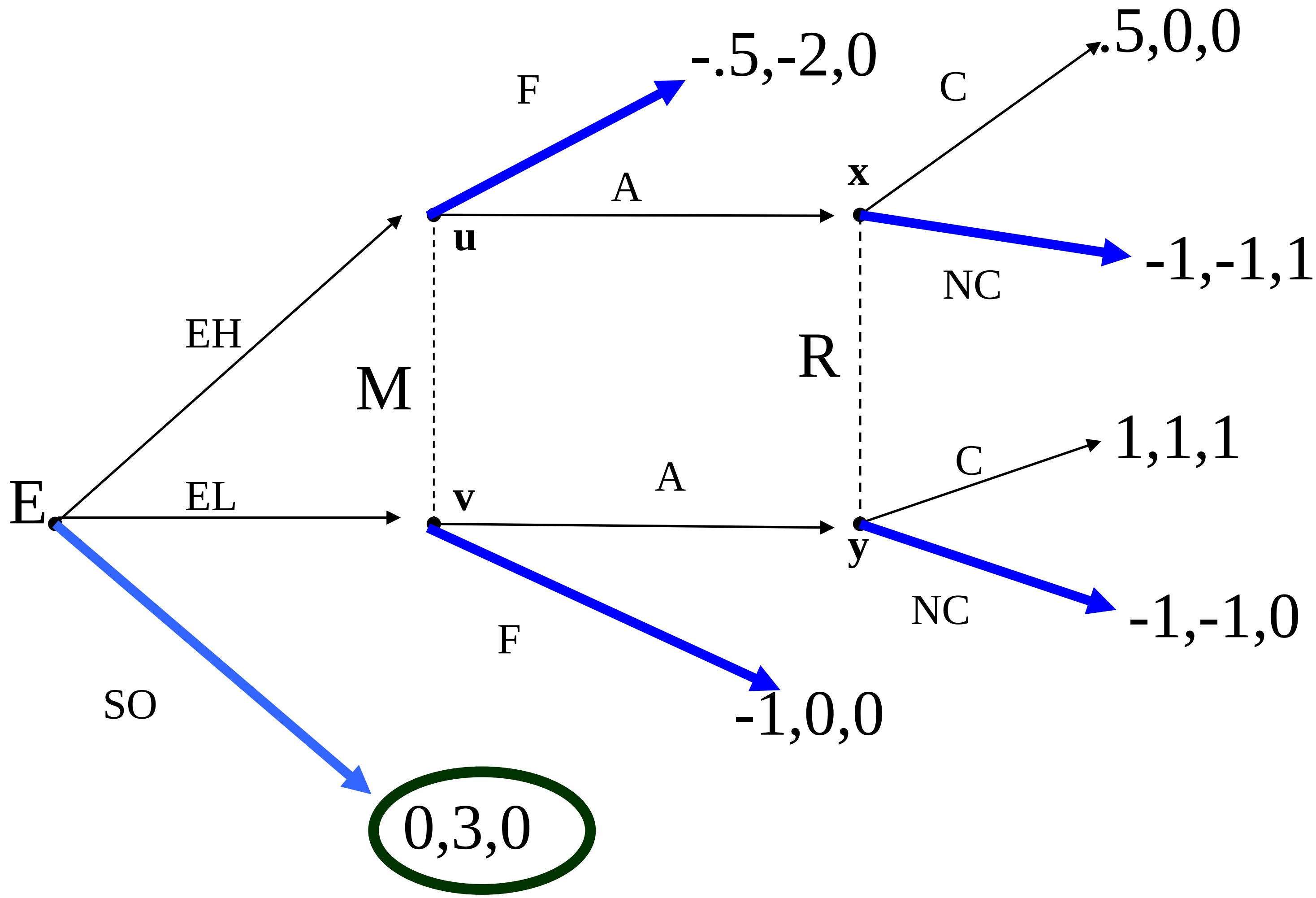
The third NE is not a WPBE

- The following strategy profile is not part of a WPBE:

$$\pi_E(SO) = 1, \pi_M(F) = 1, \pi_R(C) = 1$$

- Since
  - SO is best reply to A & C
  - C is a best reply to $\mu(x) \le ½ \Leftrightarrow 1\text{-}\mu(x) \ge \mu(x)$
  - F is never a best reply to any $\mu(u) \in [0,1]$ & to C since $-2\mu(u) \ge 1- \mu(u)$ is never satisfied
- Sequential rationality for player M is not satisfied
- Hence it is not a SE too.

Is (SO, F, NC) a WPBE?



$$\pi_E(SO)=1, \pi_M(F)=1, \pi_R(NC)=1$$

A fourth WPBE

- The following assessment is a WPBE:

$$\pi_E(SO)=1, \pi_M(F)=1, \pi_R(NC)=1$$

$$\mu(u)\leq 1/2 \,\&\, \mu(x)\geq 1/2$$

- Since
  - SO is best reply to A&NC
  - NC is a best reply to μ(x) ≥ ½ ⇔ μ(x) ≥1- μ(x)
  - F is a best reply to μ(u) ≤ ½ & to NC ⇔ -2μ(u) ≥ -1
- check if beliefs can be derived by Bayes rule

$$\mu\left(u\,\middle|\,\{u,v\}\right)=\frac{\pi_E(EH)}{\pi_E(EL)+\pi_E(EH)}=\frac{0}{0+0}\in[0,1]$$

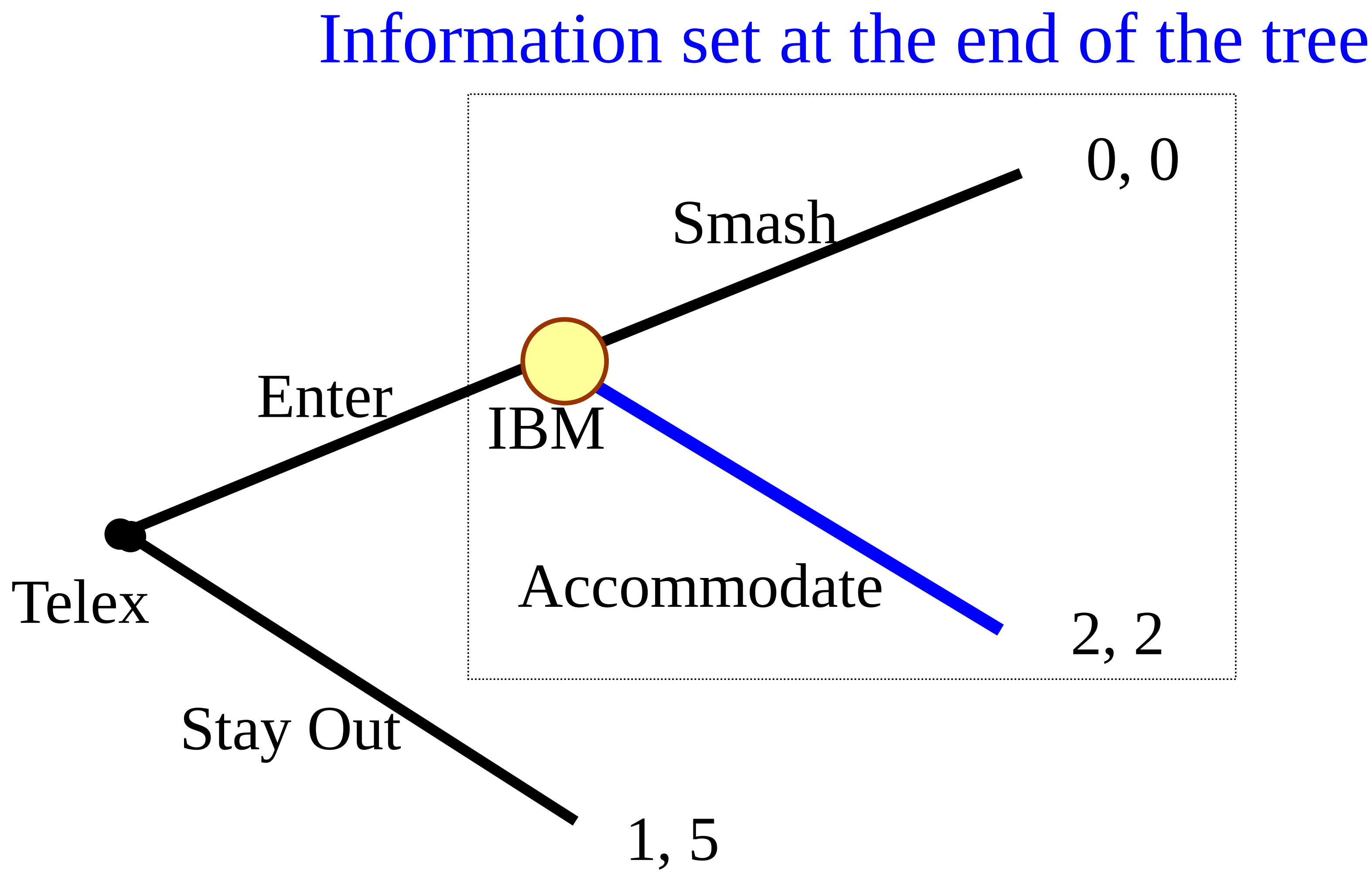
$$\mu\left(x\,\middle|\,\{x,y\}\right)=\frac{\pi_E(EH)\times\pi_M(A)}{\pi_M(A)[\pi_E(EL)+\pi_E(EH)]}=\frac{0\times 0}{0[0+0]}\in[0,1]$$

WPBE in Perfect Information Games:
Backward Induction

Backward Induction

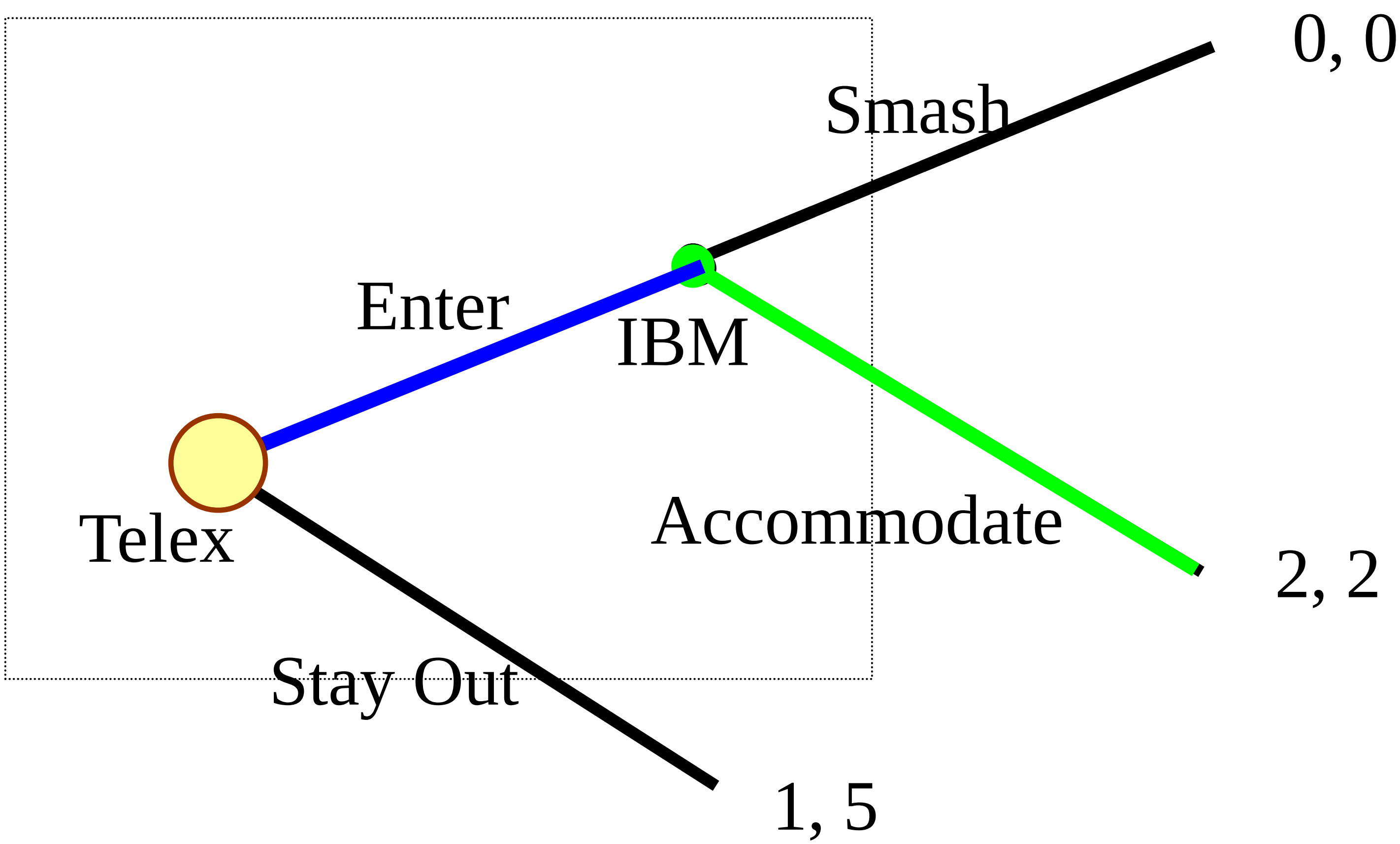
- **<u>Backward Induction</u>** if
  1. **Rationality means to avoid strictly dominated actions**, and
  2. **<u>Sequential Rationality is common knowledge</u>**
- Practically **<u>Backward induction</u>** is the process of analyzing a game from back to front, from information sets at the end of the tree to information sets at the beginning
- At each information set, one strikes from considerations actions that are dominated, **given the terminal nodes that can be reached and that will be reached according to backward induction.**
- **B.I works well in PERFECT INFORMATION GAMES**

Applying backward induction to the entry game



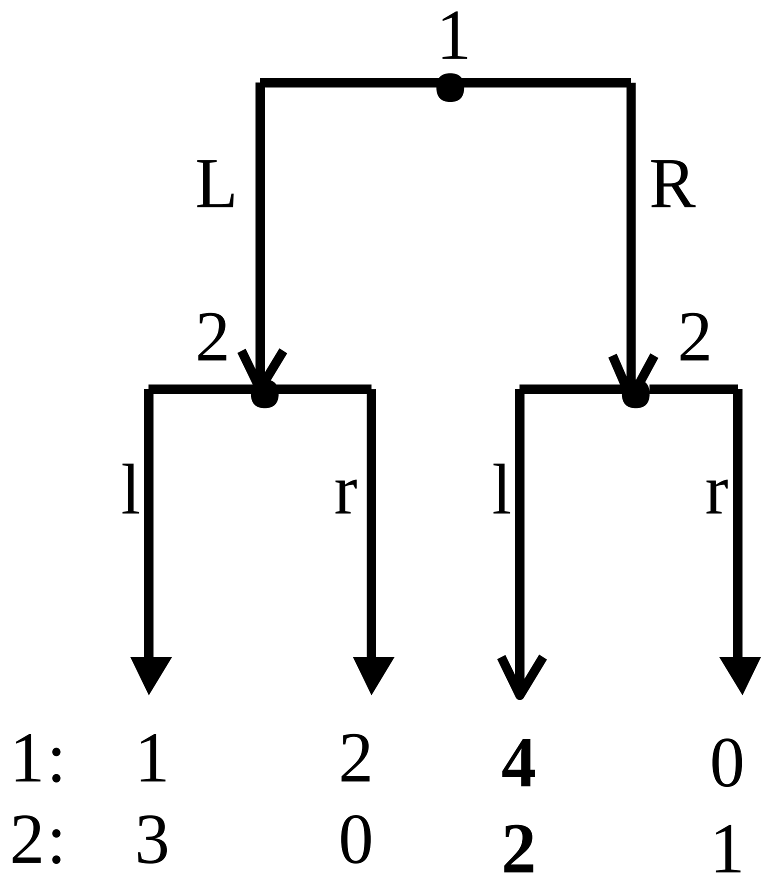
Applying backward induction to the entry game

Working back on the tree, given common knowledge of backward induction



Nash Equilibria
and
Backward Induction

Example 2:
backward induction as a refinement of Nash equilibria

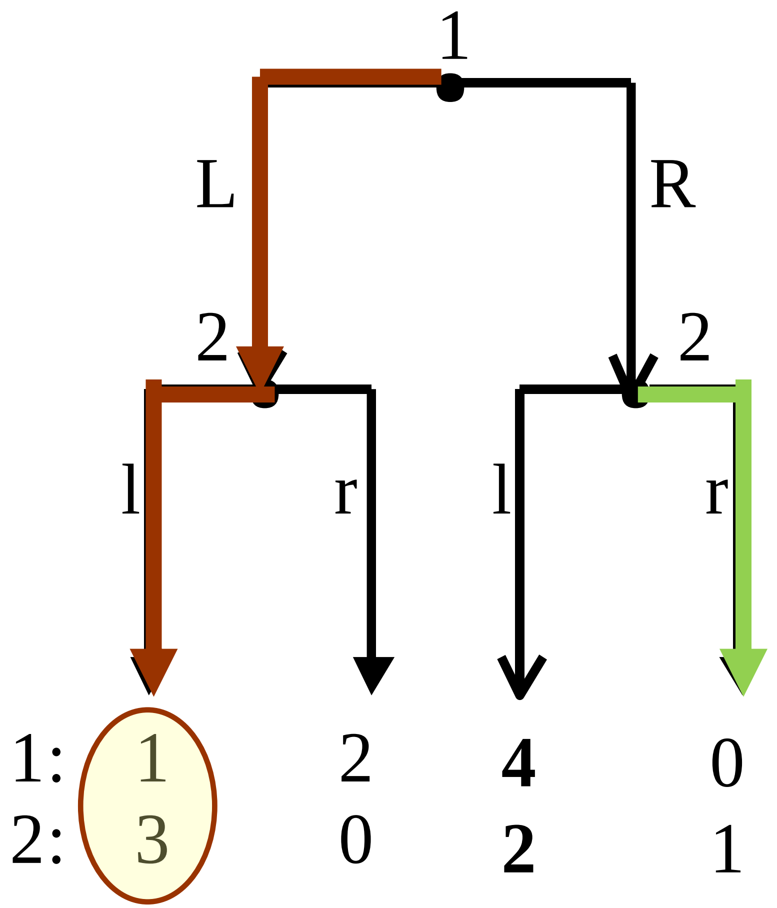


Example 2 in Normal Form

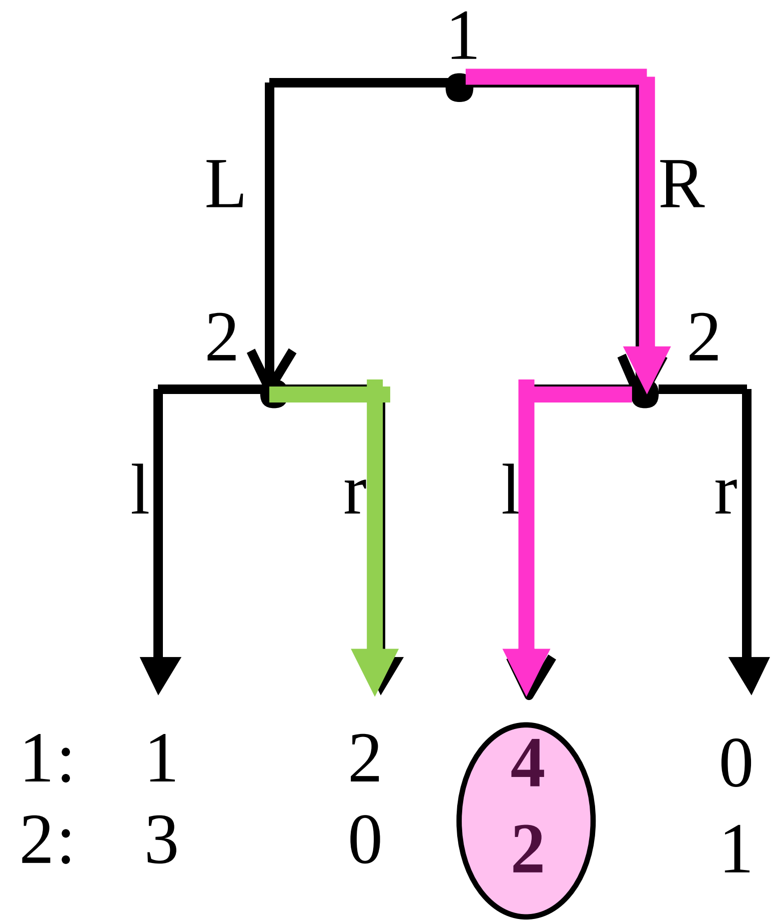
		2			
		ll	lr	rl	rr
1	L	1, $\underline{3}$	$\underline{\mathbf{1}}$, $\underline{\mathbf{3}}$	2, 0	$\underline{2}$, 0
	R	$\underline{\mathbf{4}}$, $\underline{\mathbf{2}}$	0, 1	$\underline{\mathbf{4}}$, $\underline{\mathbf{2}}$	0, 1

- Three Nash equilibria in pure strategies:
  - {R,ll}, {L,lr}, and {R,rl}.

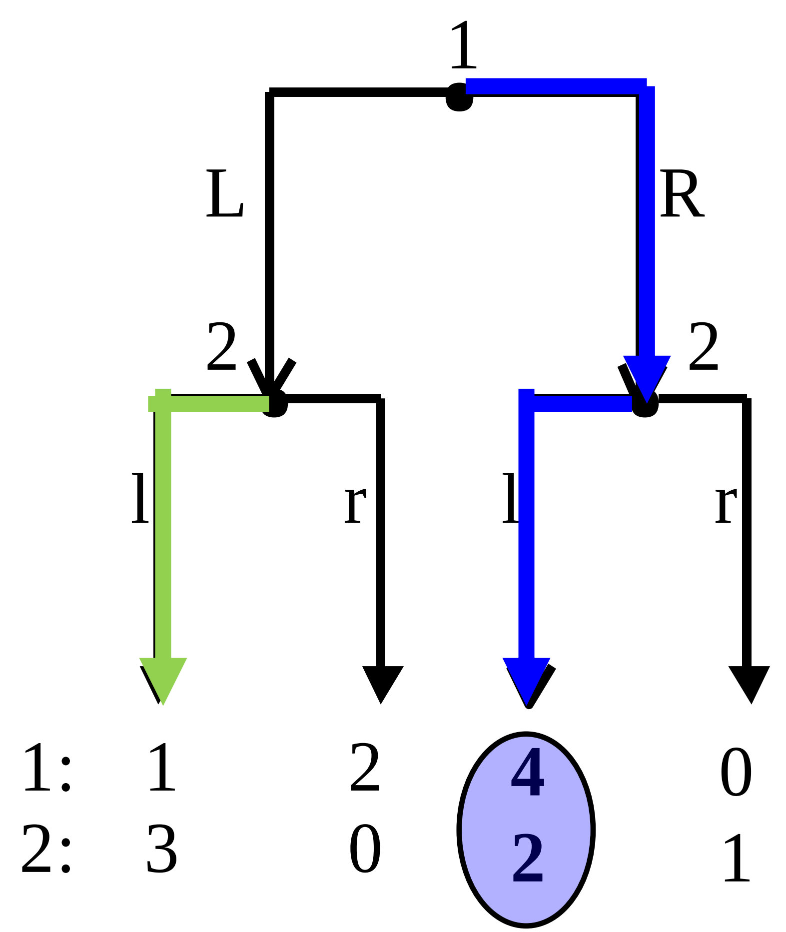
Example 2:
backward induction as a refinement of Nash equilibria: NE (L,lr) is not BI



Example 2:
backward induction as a refinement of Nash equilibria: NE (R,rl) is not BI



Example 2:
backward induction as a refinement of Nash equilibria: NE (R,ll) is BI



Example 2 in Normal Form

		2			
		ll	lr	rl	rr
1	L	1, <u>3</u>	<u>1</u>, <u>3</u>	2, 0	<u>2</u>, 0
	R	<u>4</u>, <u>2</u>	0, 1	<u>4</u>, <u>2</u>	0, 1

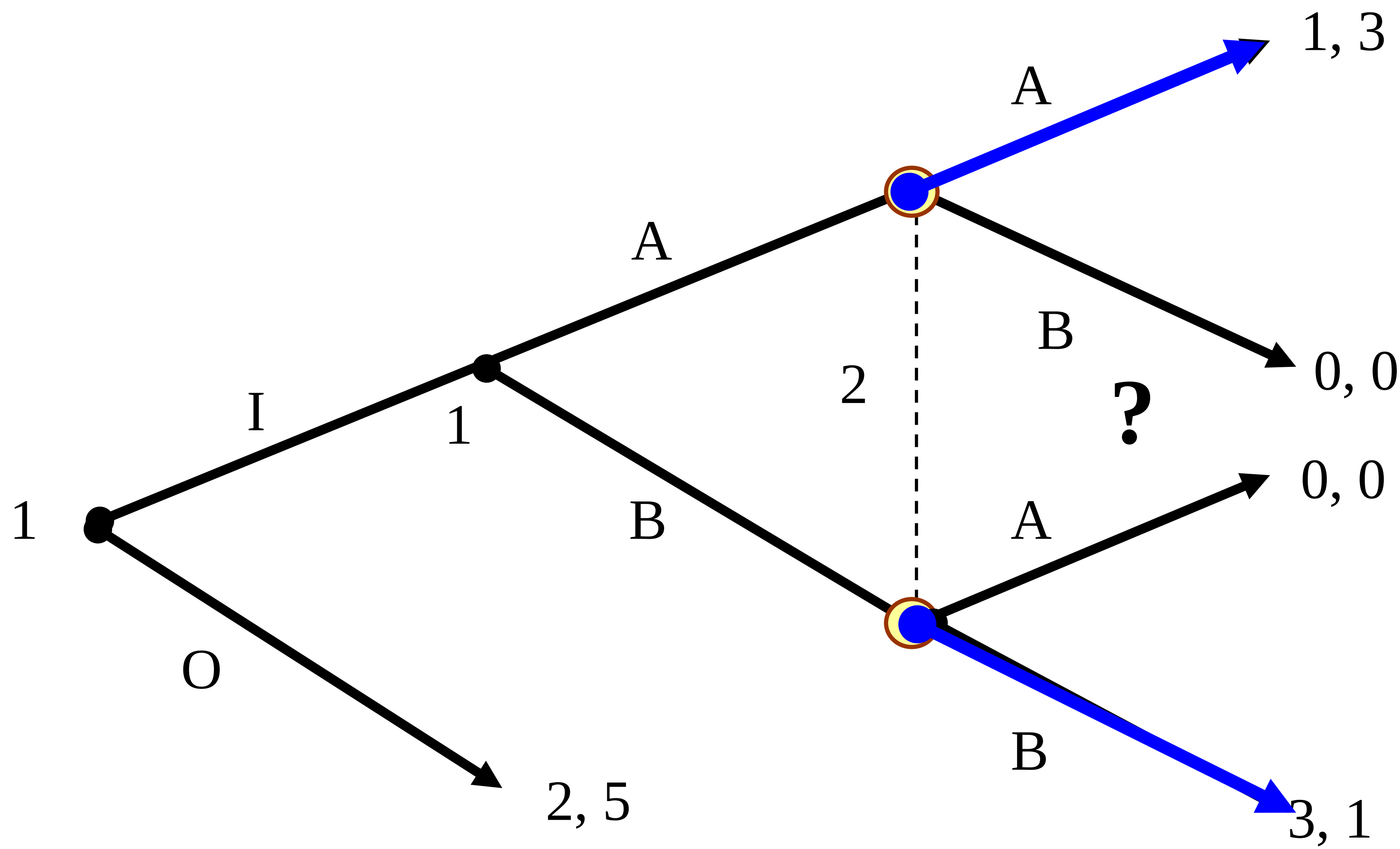
- Three Nash equilibria in pure strategies: {R,ll}, {L,lr}, and {R,rl}.
- {L,lr}, and {R,rl} involve non credible threats
- The unique NE compatible with BI is {R,rl}, this NE is called **perfect**

Backward induction in perfect information games

- In perfect information games
  - best responses/deletion of strictly dominated actions
- are played at each decision node
- If there are no ties in the payoffs, then b.i. completely solves the game: b.i. identifies a single rational strategy profile for the players
- **B.I. solution are Nash equilibria, since no player has an incentive to deviate at any information set**
- **<u>RESULT</u>:**
  1. **Almost every finite game with perfect information has a pure-strategy Nash equilibrium**
  2. **Almost always B.I. identifies one equilibrium.**

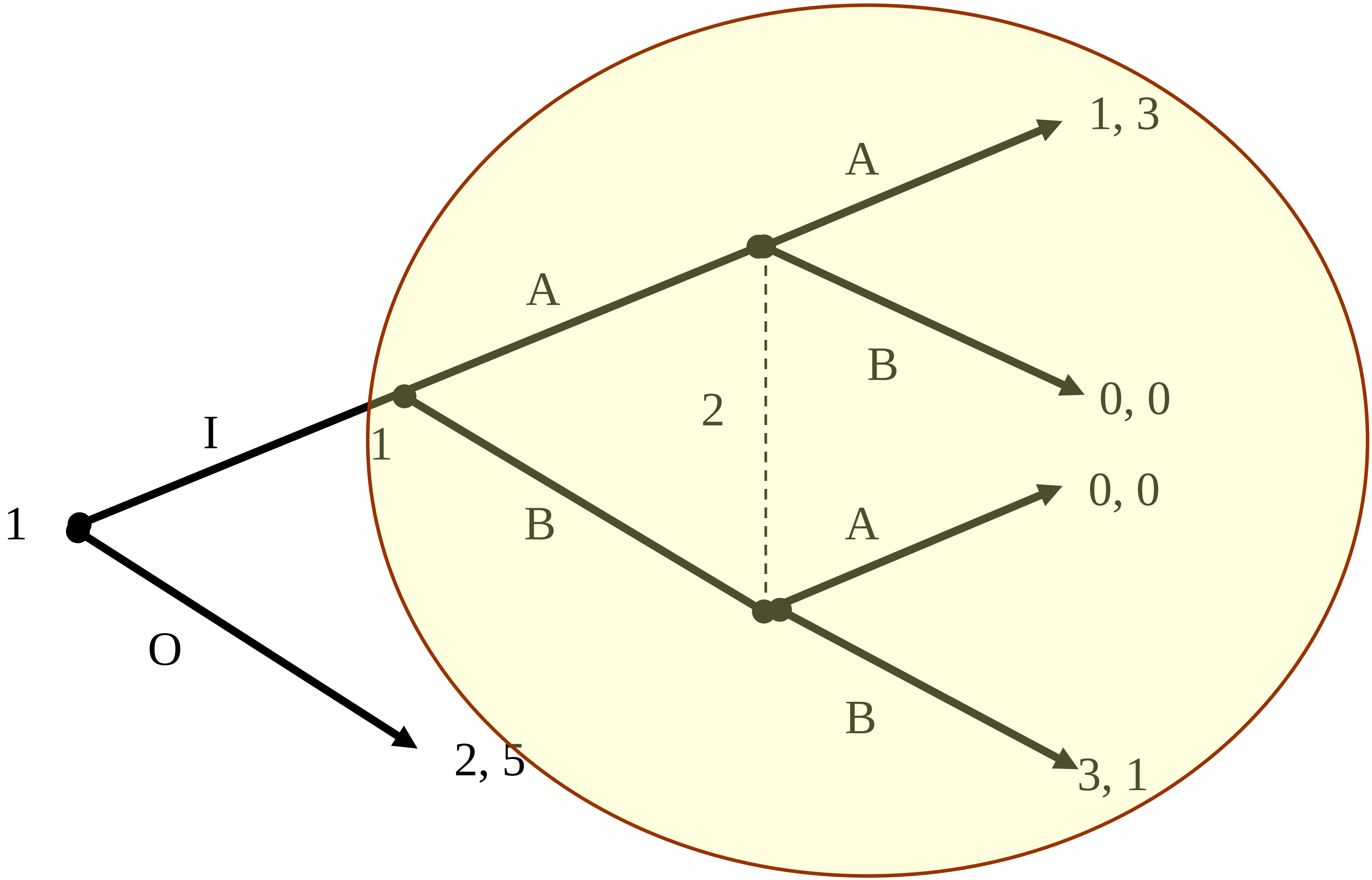
A PROBLEM WITH BACKWARD INDUCTION IN IMPERFECT INFORMATION GAMES

EXAMPLE WHERE BACKWARD INDUCTION DOESN'T WORK



Subgame Perfection

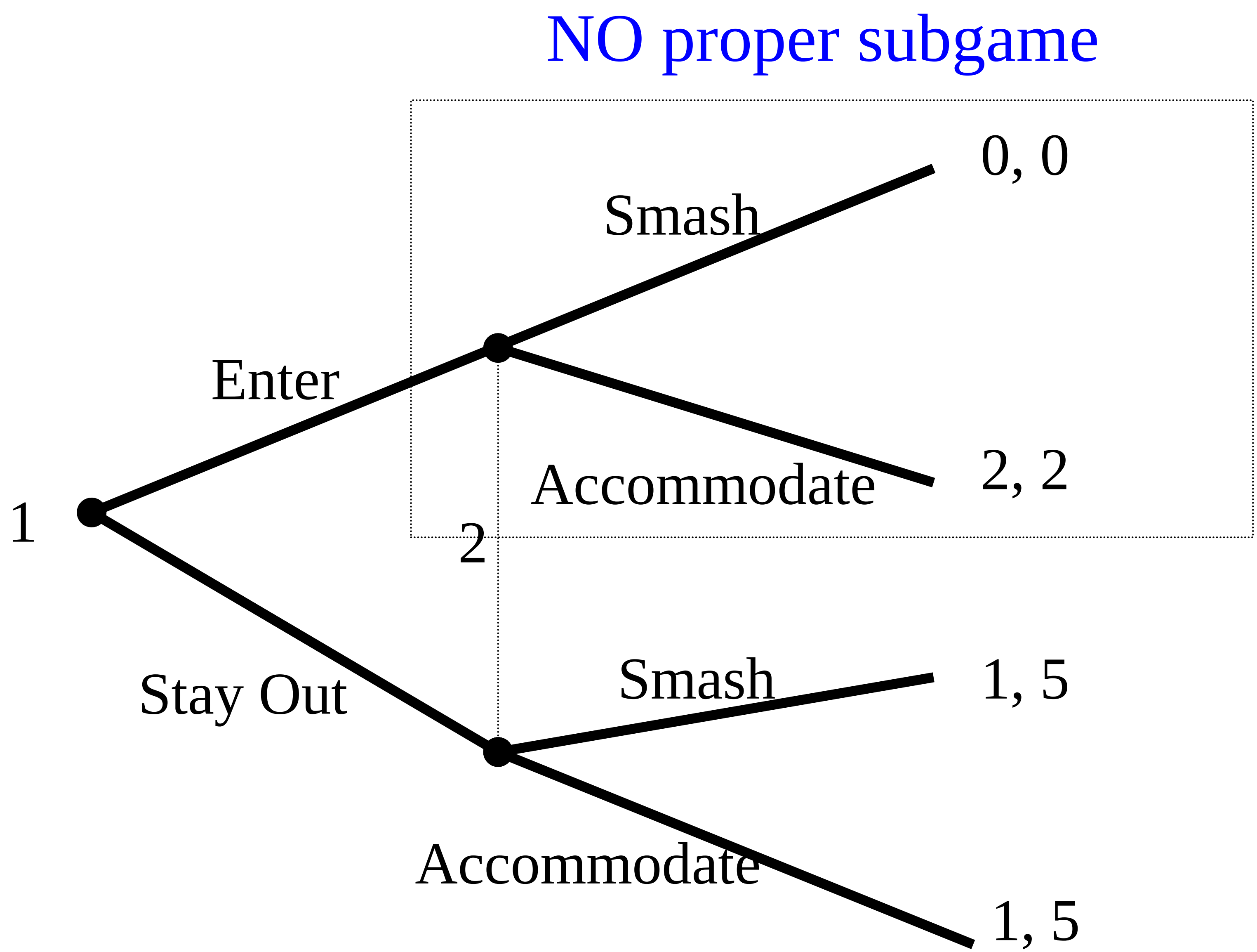
WHEN BACKWARD INDUCTION DOESN'T WORK USE SUBGAMES



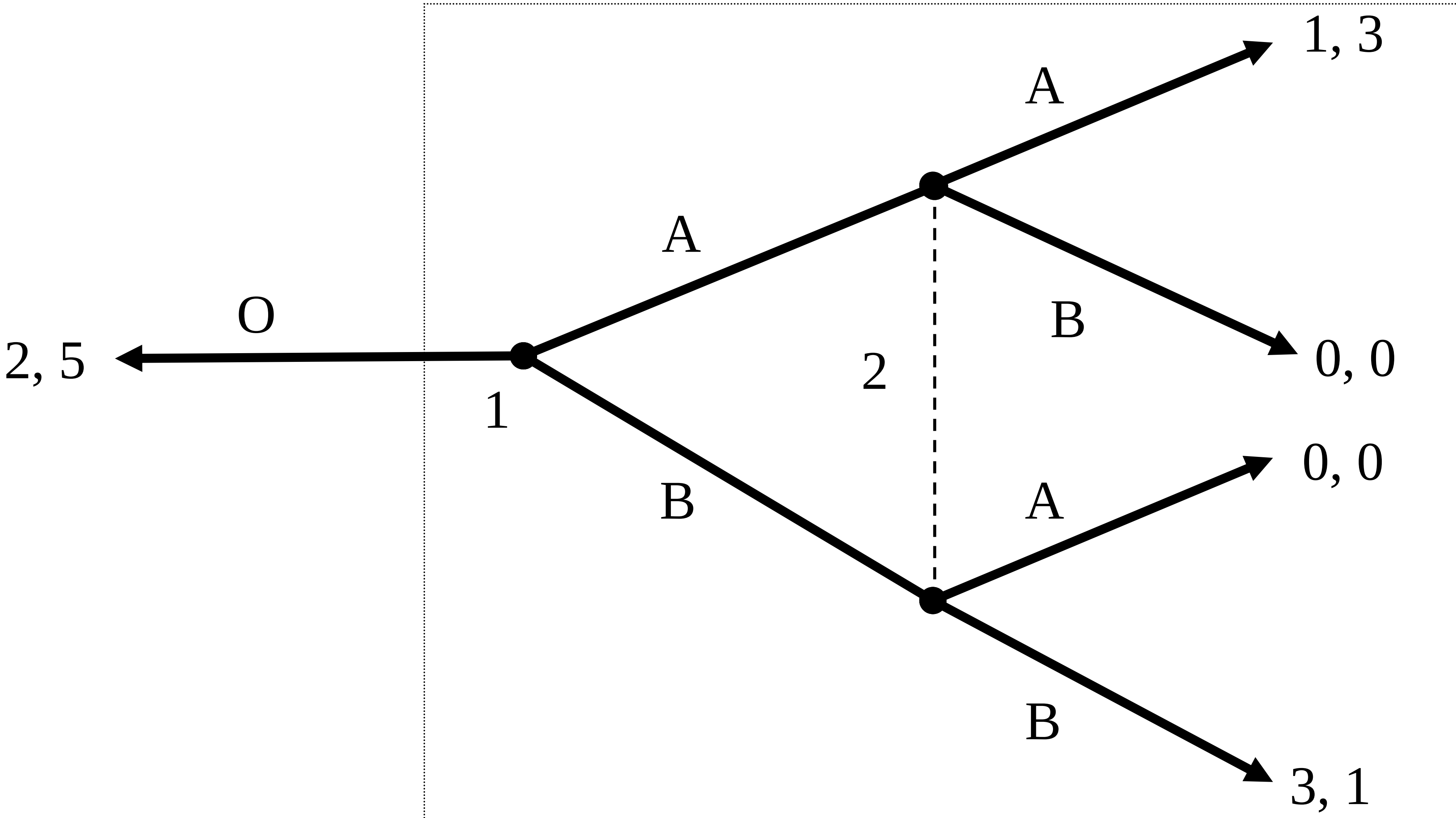
Formal definition of Proper Subgame

- Consider a game Γ consisting of a tree T linking the information sets $h \in H$ and payoffs at each terminal node of T.
- A *proper subtree* T_h is the tree
  1. beginning at a singleton information set h such that
  2. it includes all information sets and outcomes following h, and
- a ***proper subgame*** $\boldsymbol{\Gamma_h}$ is the subtree T_h and the payoffs at each terminal node of T_h.

An example of no proper subgame



Another example of no proper subgame



NO proper subgame

Subgame Perfection - 1
(Selten, 1965)

- The concept of sequential rationality can be expanded to cover general extensive form games:
  - Apply Nash equilibrium any time you face a well defined strategic situation
  - The notion of subgame is the formal translation of "a well defined strategic situation"

Definition of **Subgame Perfect Equilibrium**

A Nash equilibrium of Γ is *subgame perfect* if

1. it specifies Nash equilibrium strategies

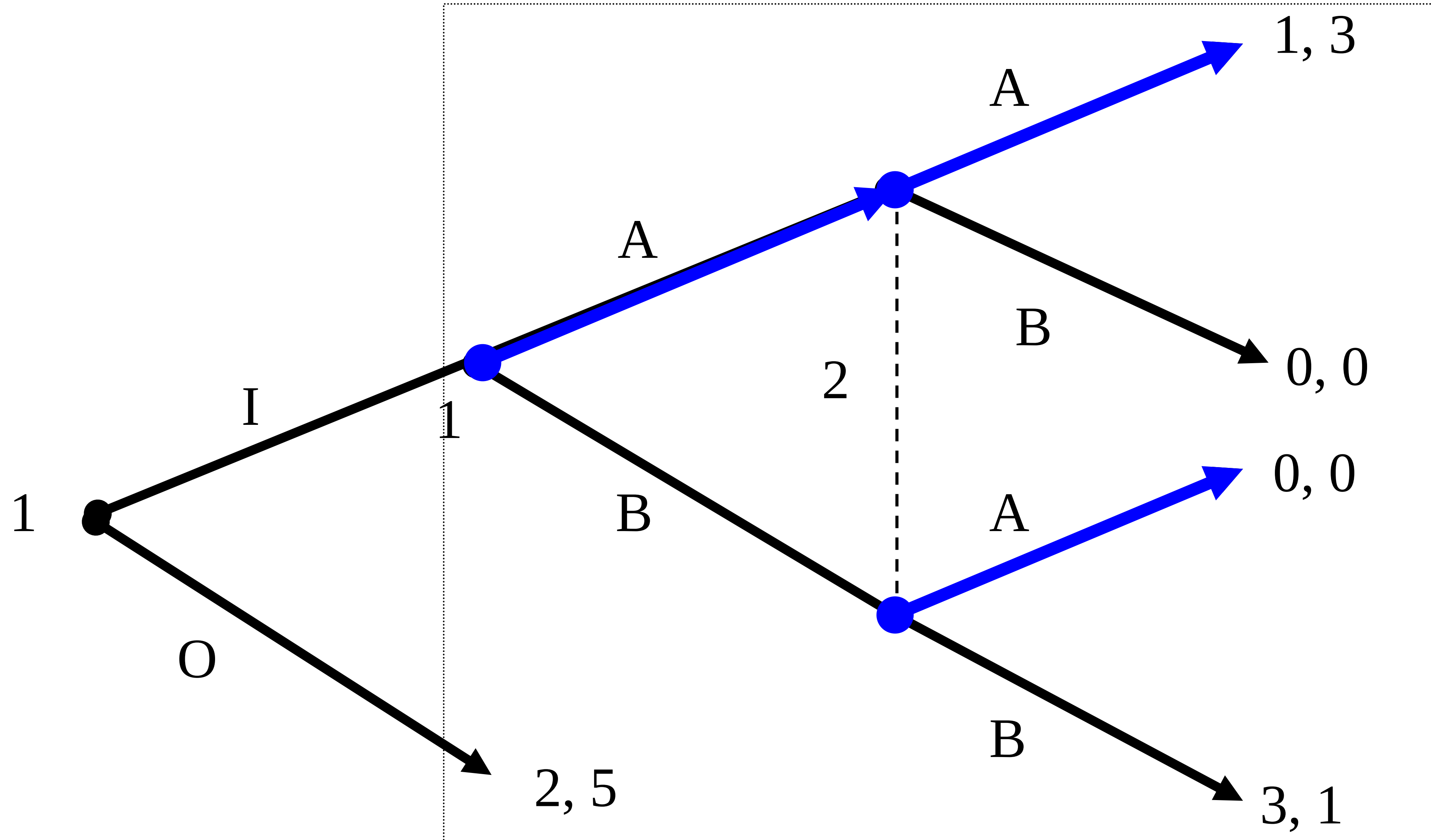
2. in every proper subgame of Γ

In other words, the players act "optimally"

(i.e. Nash Equilibrium)

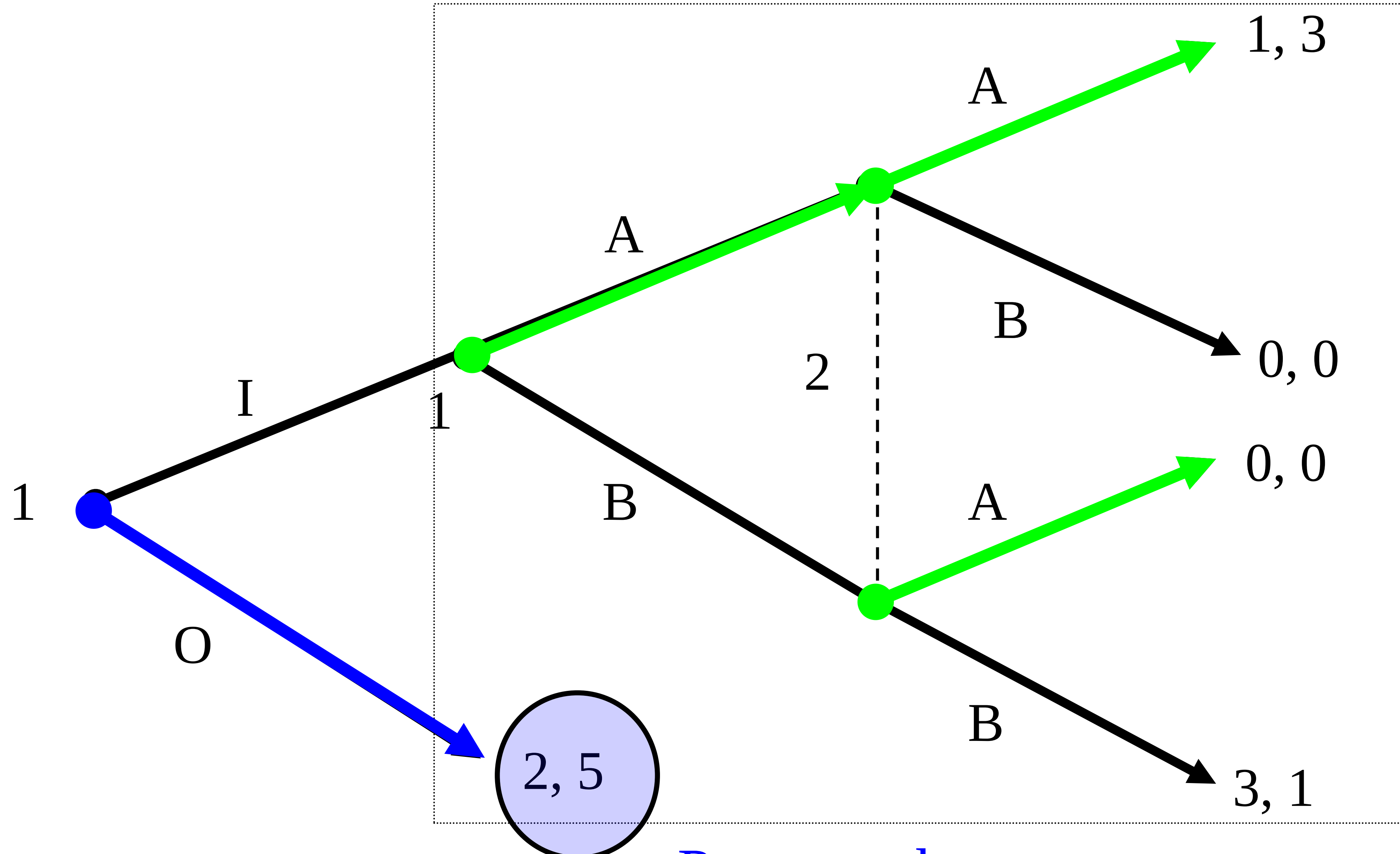
at every point during the game.

A Pure Subgame Perfect equilibria



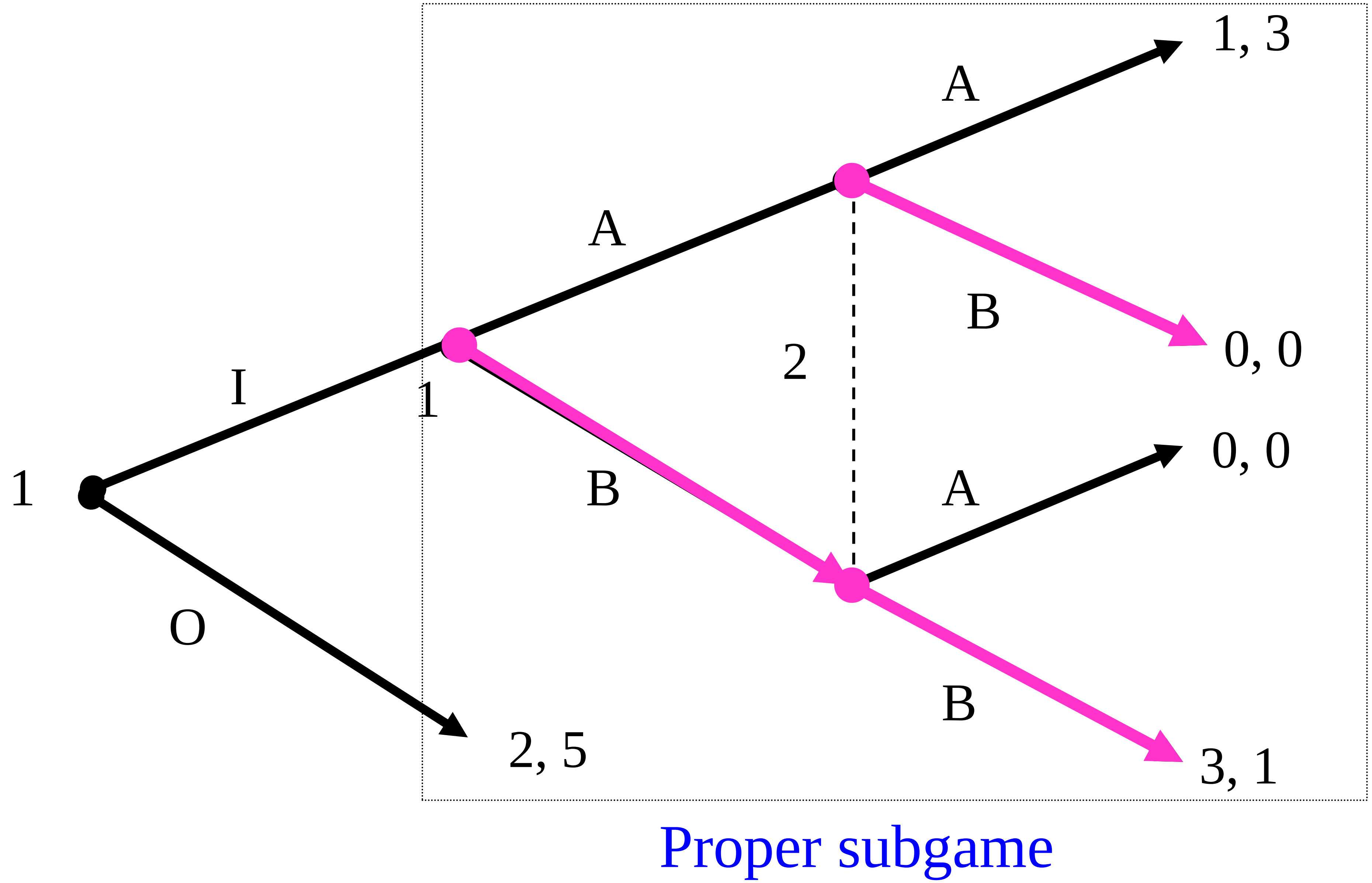
Proper subgame

A Pure Subgame Perfect equilibria (OA,A)

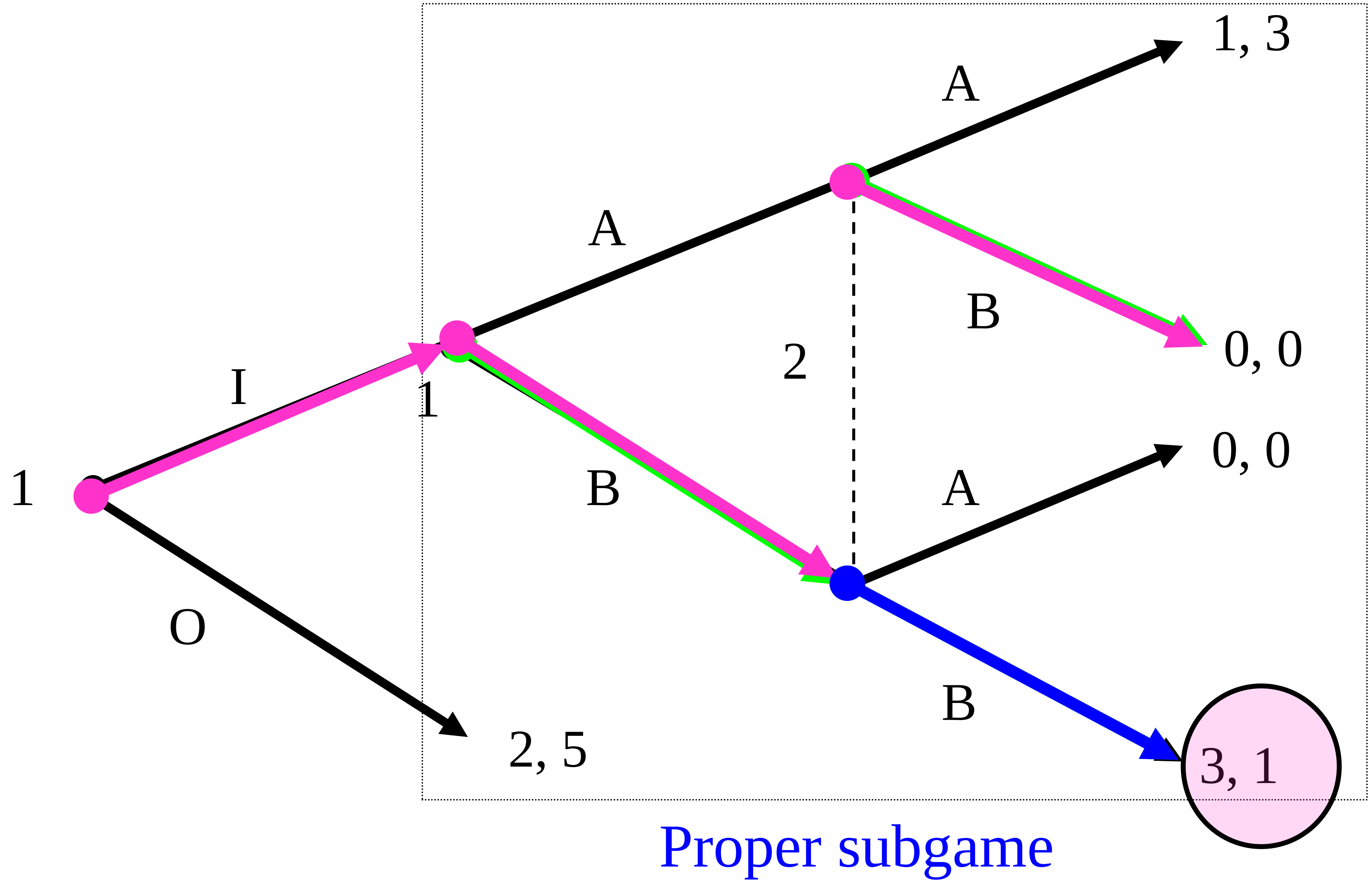


Proper subgame

Another Pure Subgame Perfect equilibria



Another Pure Subgame Perfect equilibria (IB,B)



Subgame Perfect Equilibrium
and
Nash Equilibria

The normal form game

	A	B
OA	2, 5	2, 5
OB	2, 5	2, 5
IA	1, 3	0, 0
IB	0, 0	3, 1

The reduced strategic form game

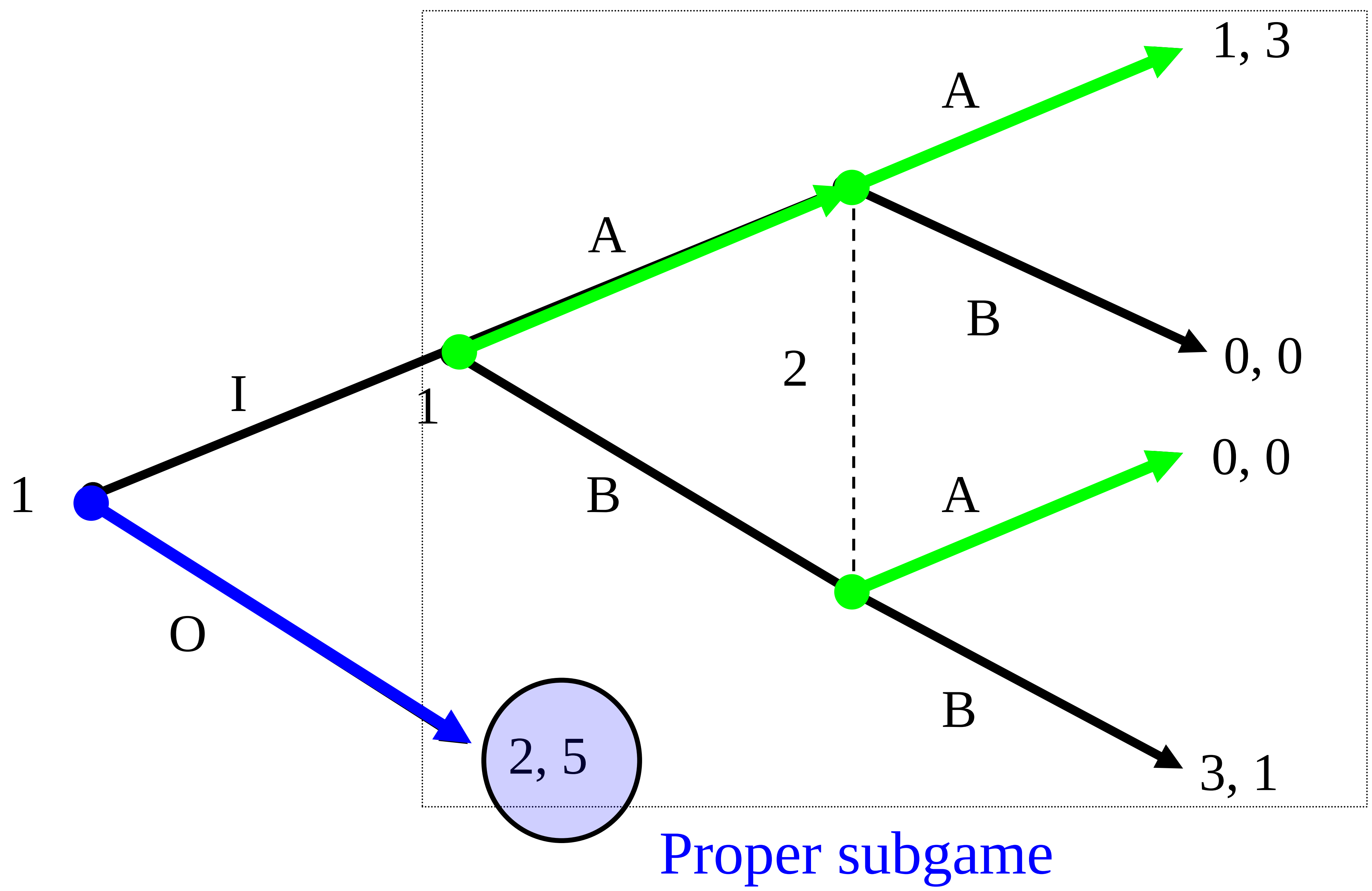
	A	B
O	2, 5	2, 5
IA	1, 3	0, 0
IB	0, 0	3, 1

Three PURE STRATEGY Nash Equilibria

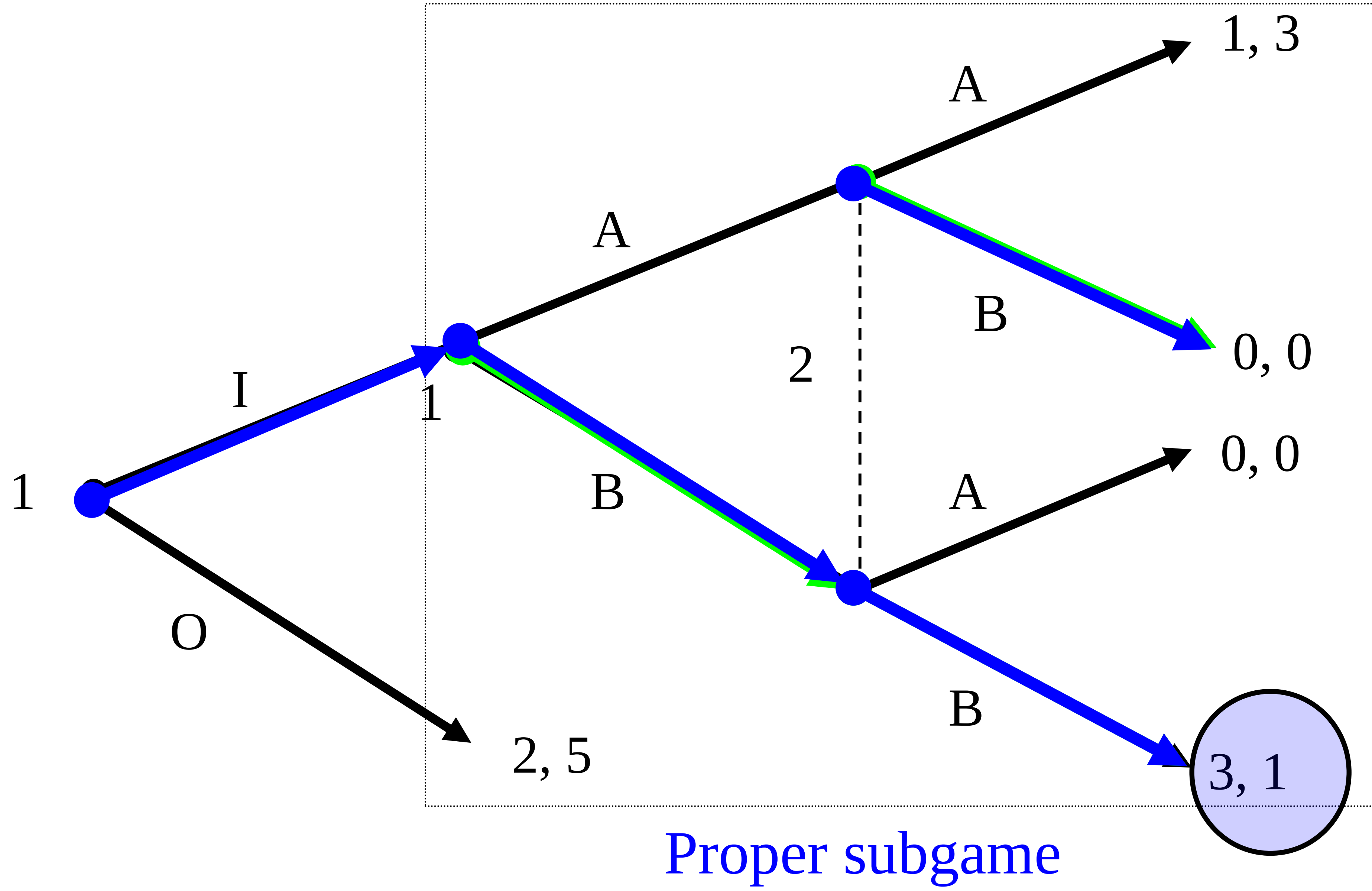
(OA,A) & (OB,A) &(IB,B)

	A	B
O	**2**, **5**	2, **5**
IA	1, **3**	0, 0
IB	0, 0	**3**, **1**

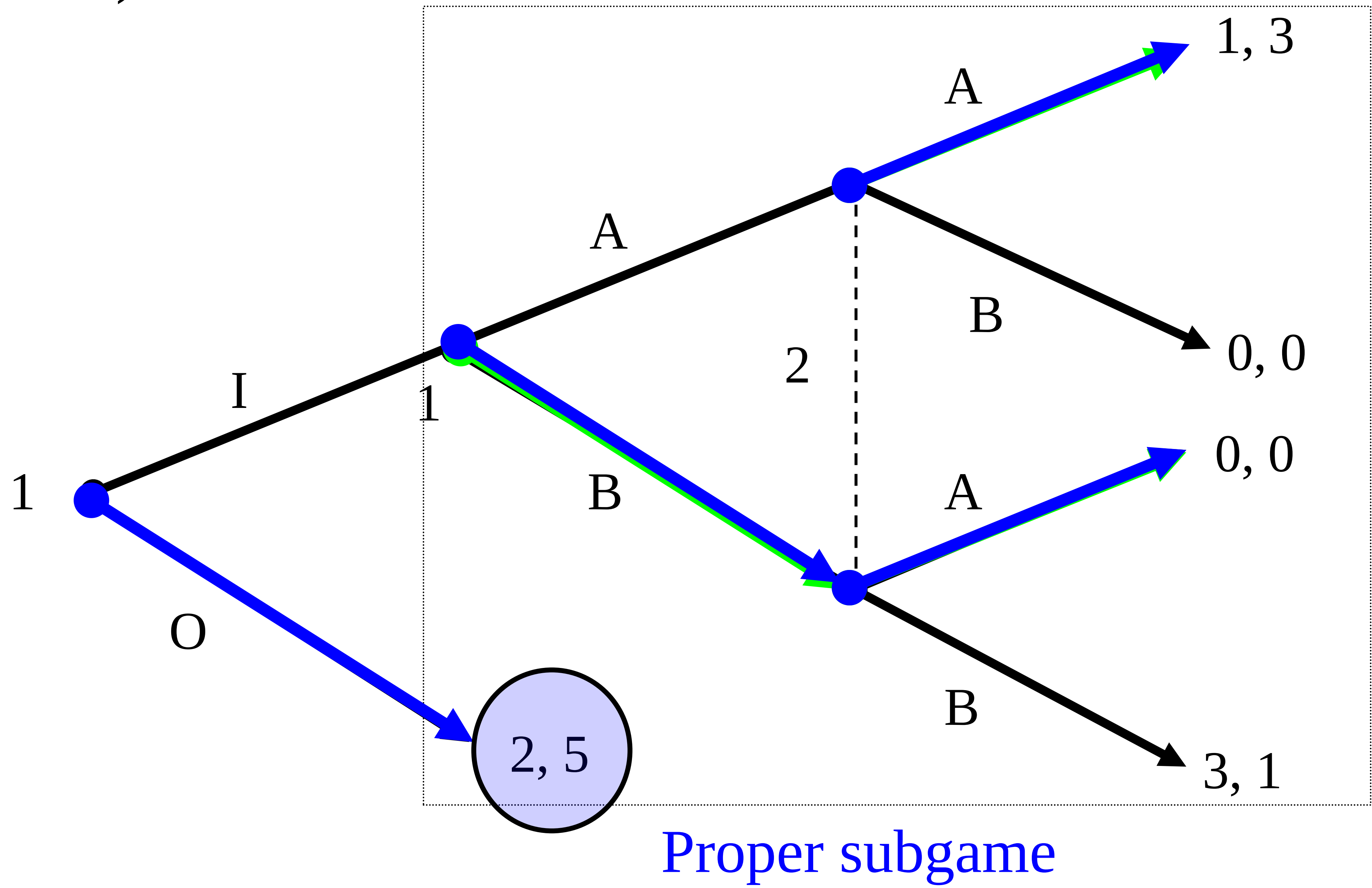
(OA,A) is Nash and Subgame Perfect



(IB,B) is Nash and Subgame Perfect



(OB,A) IS NASH BUT NOT SUBGAME PERFECT



RESULTS

1. *A subgame perfect equilibrium is a Nash equilibrium.*

   This implies that SGPE are a **refinement** of NE

2. *Given a finite extensive-form game, there exists a subgame-perfect Nash equilibrium.*

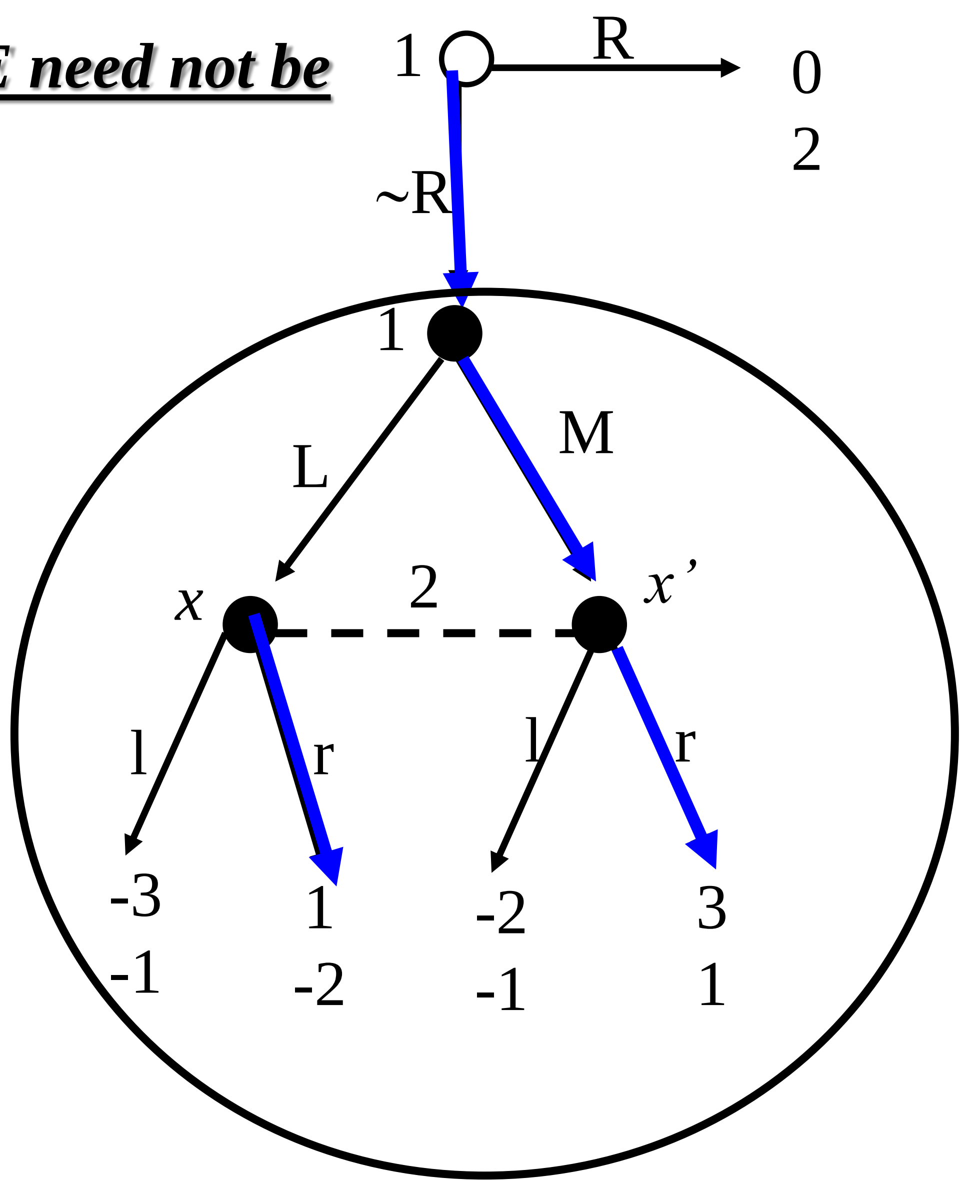
3. ***For games with perfect information, B.I. yields SGPE.***

- **A PROBLEM WITH SUBGAME PERFECTION IN IMPERFECT INFORMATION GAMES**

Applying SPE

A unique SPE (~RM,r),

Problem: ***A WPBE need not be subgame perfect***



Subgame perfection and WPBE

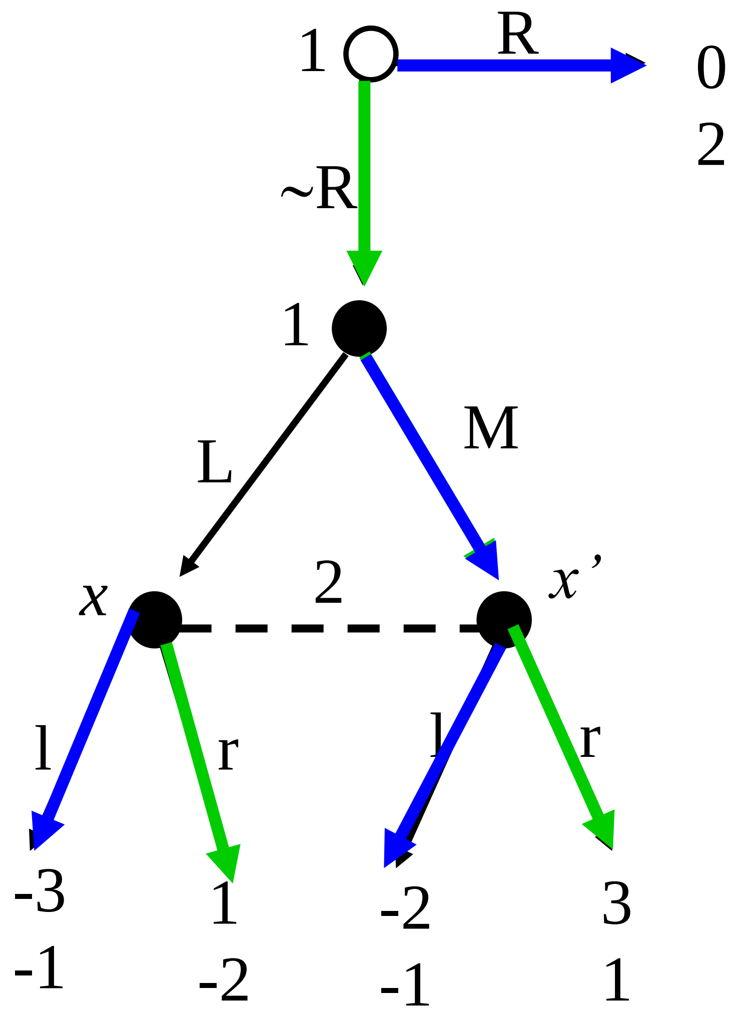
Problem: ***A WPBE need not be subgame perfect***

Two WPBE:

1. (~RM,r), which is SPE with $\mu(x'|h(x))=1$

2. (RM,l) which is not SPE but is WPBE with

$$\mu(x|h(x))=1$$



Game 2: Subgame perfection and WPBE

Deriving beliefs through

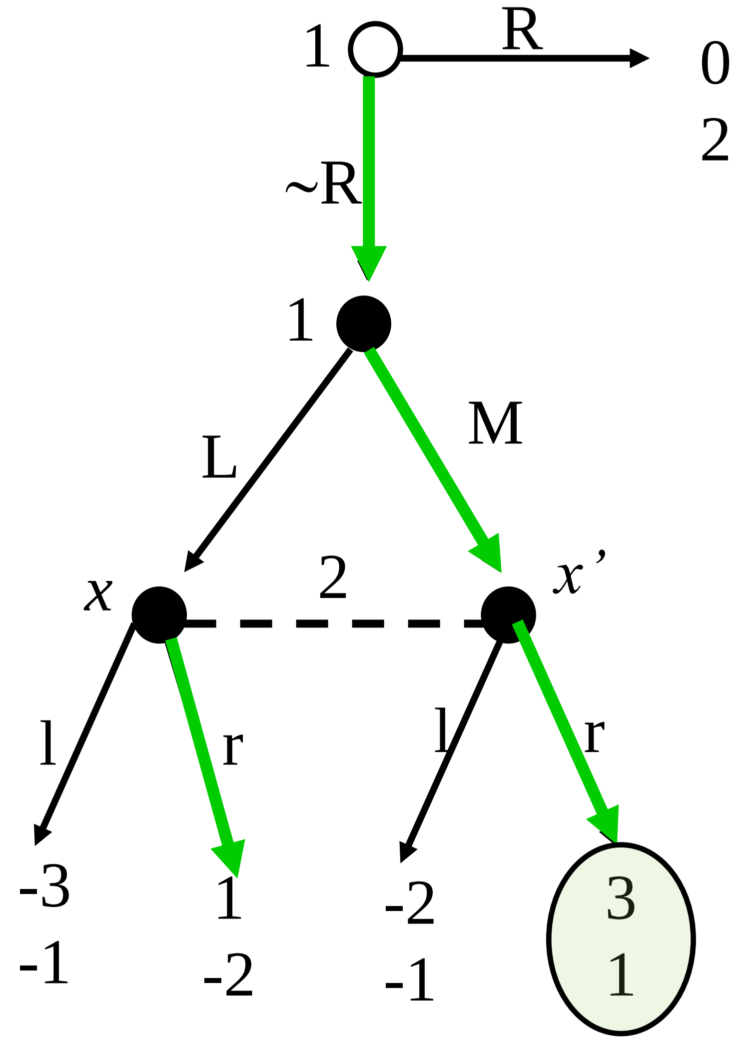
Bayesian rule from playing $\neg$R :

$$\mu(x \mid h(x)) = \frac{\Pr(x \mid \pi)}{\Pr(h(x) \mid \pi)} =$$

$$= \frac{\pi_1(\neg R) \times \pi_1(L)}{\pi_1(\neg R) \times \pi_1(L) + \pi_1(\neg R) \times \pi_1(M)} =$$

$$= \frac{1 \times 0}{1 \times 0 + 1 \times 1} = 0$$

$\therefore \mu(x' \mid h(x)) = 1$, then r is a best reply



Game 2: deriving beliefs for a WPBE

Deriving beliefs through

Bayesian rule from playing R :

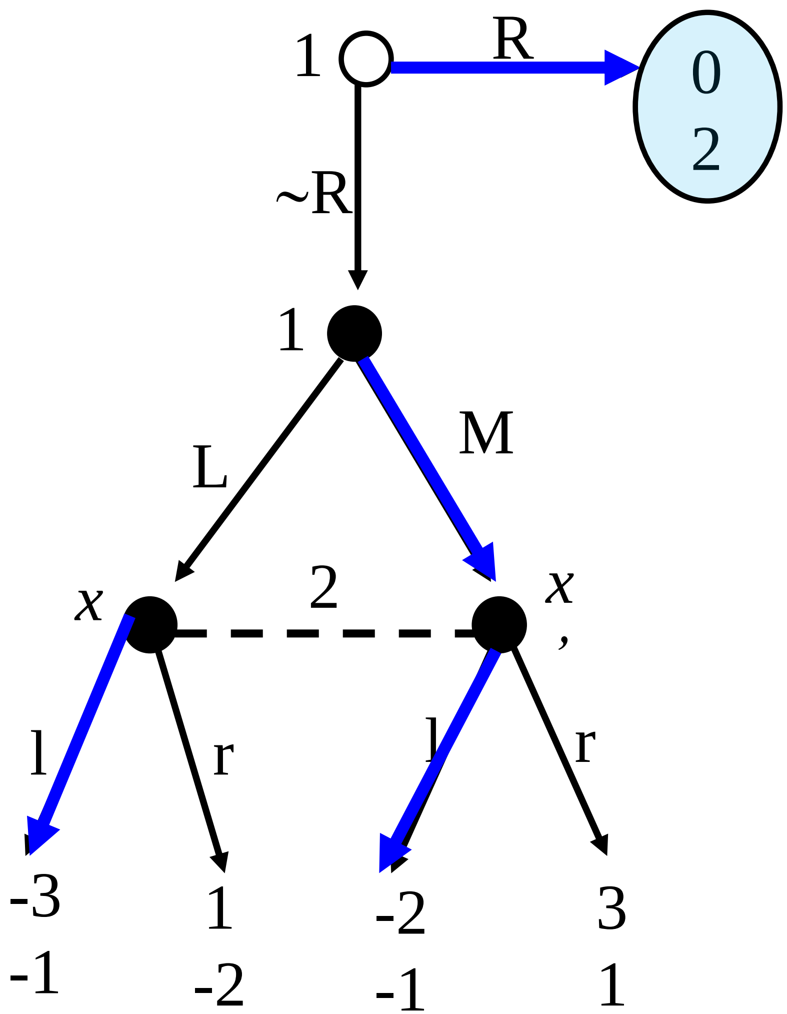
$$\mu(x \mid h(x)) = \frac{\Pr(x \mid \pi)}{\Pr(h(x) \mid \pi)} =$$

$$= \frac{\pi_1(\neg R) \times \pi_1(L)}{\pi_1(\neg R) \times \pi_1(L) + \pi_1(\neg R) \times \pi_1(M)} = \frac{0}{0}$$

$$\therefore \mu(x \mid h(x)) \in [0,1]$$

Suppose $\mu(x \mid h(x)) = 1$, then

M, l are best replies



Refining the notion of Weak Perfect Bayesian Equilibrium

- To solve the previous problem we try to refine the notion of WPBE, using **totally mixed strategies** and defining **SEQUENTIAL EQUILIBRIA.**
- A strategy profile π is *totally mixed* if it assigns strictly positive probability to each action $a \in A(h)$ for each information set $h \in H$.

Definition: **Consistency**

Definition:

- An assessment (μ,π) is ***consistent*** if

1. there exists a sequence of totally mixed behavioral strategies π_n and
2. corresponding beliefs μ_n derived from Bayes' rule

such that

$$\lim_{n\to\infty}(\mu_n,\pi_n) = (\mu,\pi).$$

Definition of
SEQUENTIAL EQUILIBRIUM

- A ***sequential equilibrium*** is an assessment (μ,π) that is both

1. *sequentially rational* and

2. *consistent*.

Game 2: deriving beliefs with consistency

Deriving consistent beliefs through Bayesian rule from playing RM,l:

$$\mu(x \mid h(x)) = \frac{\Pr(x \mid \pi)}{\Pr(h(x) \mid \pi)} =$$

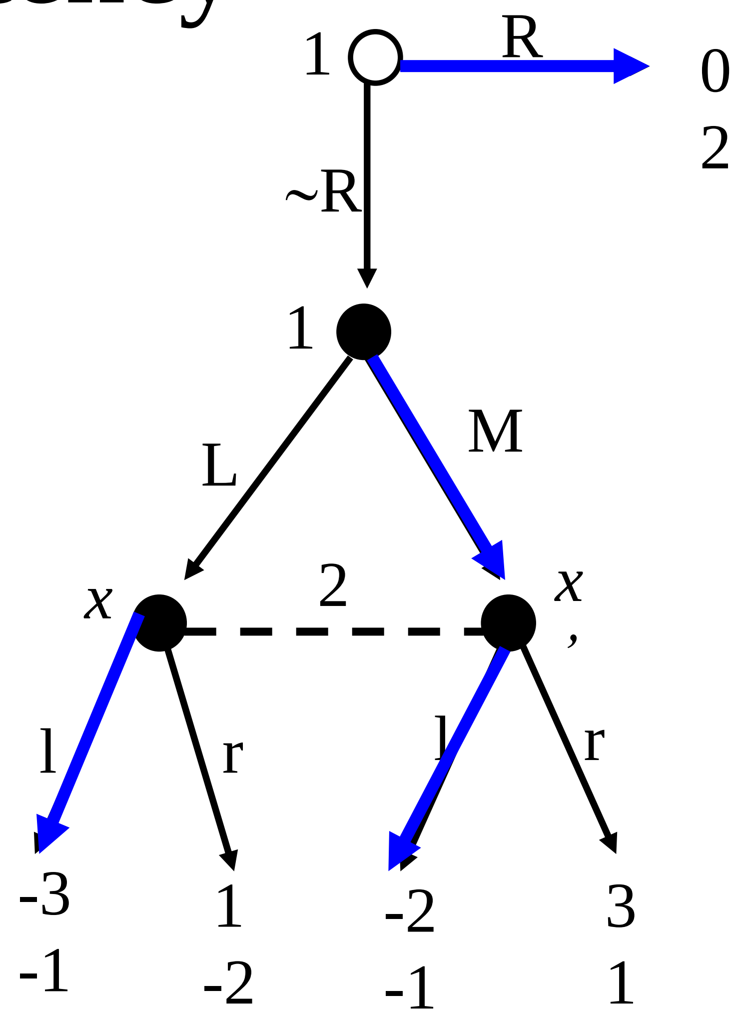
$$= \frac{\pi_1(\neg R) \times \pi_1(L)}{\pi_1(\neg R) \times \pi_1(L) + \pi_1(\neg R) \times \pi_1(M)} =$$

$$\frac{\varepsilon \times \eta}{\varepsilon \times \eta + \varepsilon \times (1-\eta)} = \frac{\eta}{\eta + 1 - \eta} \underset{\eta \to 0}{\to} 0$$

$$\therefore \mu(x \mid h(x)) = 0$$

then M, l are NOT best replies

the unique SE in pure strategies is $(\neg RM, r)$ which is Subgame Perfect



Meaning of SEQUENTIAL EQUILIBRIA

- In a SE any equilibrium strategy is approximated by a totally mixed strategy
- Because of this, any information set is reached with strictly positive probability possibly vanishing
- This means that out of equilibrium information sets are reached with small vanishing probabilities, i.e. **by mistakes**:

impossible events are explained as due to trembling hands.

Theorem

For every finite extensive-form game there exists at least one sequential equilibrium. Also, if (μ,π) is a sequential equilibrium then π is a subgame-perfect Nash equilibrium.

$$SE_{\pi} \subseteq WPBE_{\pi} \subseteq NE$$

Moreover

$$SE \neq \emptyset$$