Introduction to Global Games

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Introduction

- Many social, political and economic problems are naturally modeled as **games of incomplete information**, where a player's payoff depends on his own action, the actions of others, and some unknown fundamentals.
- In principle, rational strategic behavior should be analyzed in the space of all possible infinite hierarchies of beliefs; however, such analysis is highly complex and is likely to prove intractable in general.
- It is therefore useful to identify strategic environments with incomplete information that are rich enough to capture the important role of higher-order beliefs in economic settings, but simple enough to allow tractable analysis.
- **Global games** represent one such environment.
  - Uncertain economic fundamentals are summarized by a state θ and each player observes a different signal of the state with a small amount of noise.
  - Assuming that the noise technology is common knowledge among the players, each player's signal generates beliefs about fundamentals, beliefs about other players' beliefs about fundamentals, and so on.

Coordination Games and the Intuition behind Global Games - 1

- Coordination games make up a special but rich class of games, being a subset of the games with strategic complementarity or supermodular games.
- **Example:** simplest possible

		2: L	2: R
1	U	0, 0	0, −1
	D	−1, 0	1, 1

Game 1

- The interesting characteristics of game 1 are:
  1. the multiplicity of equilibria;
  2. the Pareto ranking of these equilibria, where the Pareto inferior equilibrium is risk dominant w.r.t to the Pareto efficient strategy profile;
  3. the intuitive role of confidence and expectations as critical elements for determining players' rational behavior;
  4. a natural propagation mechanism such that a change in the structural parameters that affect the payoff of one player lead to similar responses in the behavior of all agents, i.e. to positive comovements.

Coordination Games and the Intuition behind Global Games - 2

- **Example:**

		2: L	R
1	U	0, 0	0, −1
	D	−1, 0	1, 1

Game 1

Suppose that it is possible to rank players' strategies such that R>L and D>U, then coordination games display two further properties:

1. strategic complementarity, i.e. an "higher" choice by 2 increases the marginal return to "higher" choices by 1;
2. positive spillovers, i.e. the payoffs of a player increases as the choice of the other player increases.

- Carlsson and van Damme 1993a worked on the idea that because of the intuitive role of confidence and expectations as critical elements for determining players' rational behavior, and because complete information games are actually the limit of incomplete information games, the introduction of a bit of private information might be effective on this multiplicity.

Coordination Games and the Intuition behind Global Games - 3

- **Example:**

		2	
		L	R
1	U	0, 0	0, −1
	D	−1, 0	1, 1

Game 1

A complete information model entails the implicit assumption that there is common knowledge among the players of the payoffs of the game.

- Suppose that, instead of observing payoffs exactly, payoffs are observed with a small amount of continuous noise; and suppose that --- before observing their signals of payoffs --- there was an ex ante stage where any payoffs were possible. Based on the latter feature, Carlsson and van Damme 1993a dubbed such games `global games'.
- In contrast with the logic of coordination games with common knowledge of payoffs, global games allow theorists to escape the straitjacket of perfect coordination of actions and beliefs.

Coordination Games and the Intuition behind Global Games - 4

- **Example:** consider a parametric class of games, G(θ), of the following type

		2	
		Invest	Not Invest
1	Invest	θ, θ	$\theta - 1, 0$
	Not invest	$0, \theta - 1$	$0, 0$

Game $G(\theta)$

- when θ<0, then (Not invest, Not invest) is the unique dominant strategy profile;
- when θ>1, then (Invest, Invest) is the unique dominant strategy profile;
- when θ∈[0,1], then there are two Nash equilibria that can be Pareto ranked, such that investing is a risk dominant action if θ≥(1/2), not investing if θ≤(1/2).
- This class of games allows to introduce the class of global games

Coordination Games and the Intuition behind Global Games - 5

- suppose that
  1. the players do not exactly observe θ, but a signal $\theta_i = \theta + \sigma \varepsilon_i$
  2. ε_i is identically and independently normally distributed $\varepsilon_i \sim N(0,1)$
  3. each player believes that θ is uniformly distributed on the real line $\theta \sim U(R)$.
- **Remark**: The assumption that θ is uniformly distributed on the real line is nonstandard, but presents no technical difficulties:
  - Such "improper priors" (with an infinite mass) are well behaved, as long as we are concerned only with conditional beliefs and players will always condition on signals that generate 'proper' posteriors
  - it is possible to show that an improper prior can be seen as a limiting case either as the prior distribution of θ becomes diffuse or as the standard deviation of the noise σ becomes small
- θ_i is informative on the underlying state of nature θ and on the other players' signal θ_{-i} since these signals are correlate with θ. In particular

$$E(\theta|\theta_i)=\theta_i \qquad \theta|\theta_i \sim N(\theta_i\ , \sigma^2) \qquad \theta_{-i}|\theta_i \sim N(\theta_i, 2\sigma^2).$$

Bayes Nash Equiibria in the class of games G(θ) - 1

- A pure strategy for player *i* is a mapping s_i: R → {Invest, Not invest}.
- Suppose that player i assume that player -i plays a cutoff strategy such that

$$s^k_{-i}(\theta_{-i}) = \begin{cases} Invest & if \;\; \theta_{-i} > k \\ Not\ Invest & if \;\; \theta_{-i} \leq k. \end{cases}$$

- Therefore, according to i, player -i will not invest with probability

$$\Phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right)$$

- This implies that i's expected payoff from investing is

$$\theta_i\left[1-\Phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right)\right]+(\theta_i-1)\,\Phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right)=\theta_i-\Phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right)$$

- so that i best response to $s_{-i}^k(\theta_{-i})$ is

$$s_i^{br}\left(s^k_{-i}(\theta_{-i})\,|\theta_i\right) = \begin{cases} Invest & if \;\; \theta_i > \Phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right) \\ Not\ Invest & if \;\; \theta_i \leq \Phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right). \end{cases}$$

Bayes Nash Equiibria in the class of games G(θ) - 2

- Note that the equation $\theta_i - \Phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right) = 0$ as a unique solution in θ_i, *b(k)*
- Then, the best response of player 1 is to follow a cut-off strategy with threshold equal to b(k):

$$s_i^{br}\left(s_{-i}^k(\theta_{-i})\,|\theta_i\right) = \begin{cases} Invest & if \;\; \theta_i > b(k) \\ Not\; Invest\; if & \theta_i \leq b(k). \end{cases}$$

Note that

1. $$k \to -\infty \Rightarrow \Phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right) \to 0 \Rightarrow b(k) \to 0,$$

   i.e. if player 2 always invest, player 1 will invest if the signal θ_i is positive;

2. $$k \to \infty \Rightarrow \Phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right) \to 1 \Rightarrow b(k) \to 1,$$

   i.e. if player 2 never invest, player 1 will invest if the signal θ_i is greater than 1;

Bayes Nash Equiibria in the class of games G(θ) - 3

3.

$$k = \frac{1}{2} \Rightarrow b(k) = \frac{1}{2}$$

since when $\theta_i = \frac{1}{2}$ then $E(\theta|\theta_i) = \frac{1}{2}$, which implies that according to i's beliefs, player $-i$ will not invest with probability $\frac{1}{2}$:

4. by total implicit differentiation

$$b'(k) = \frac{1}{1 + \frac{\sqrt{2\pi}}{\phi\left(\frac{1}{\sqrt{2\sigma}}(k-\theta_i)\right)}} \in (0,1)$$

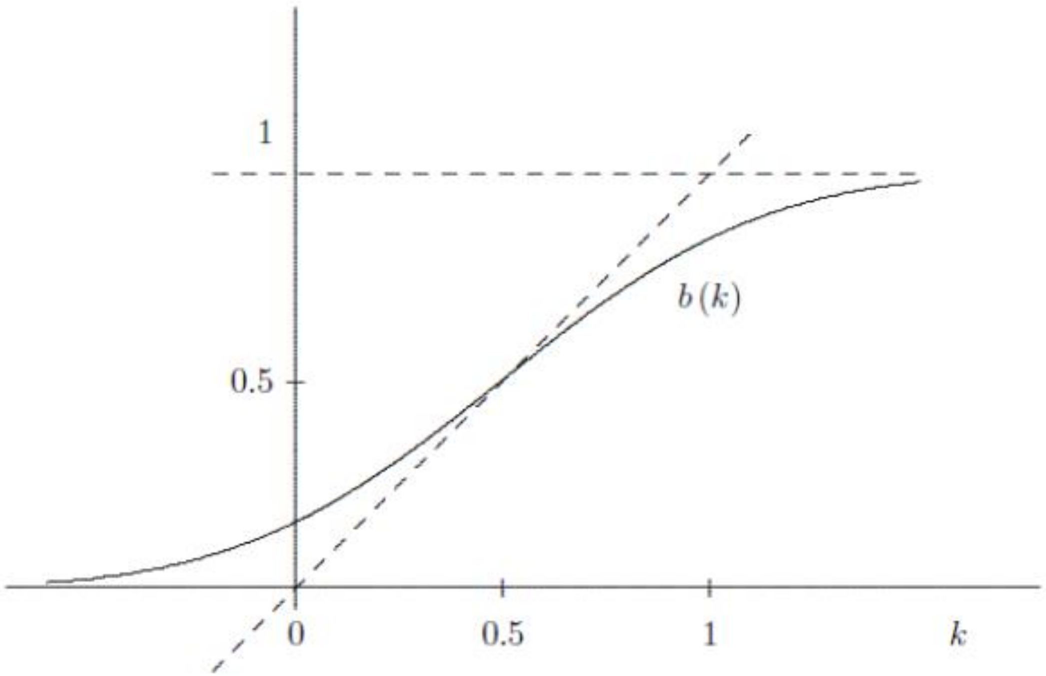


Figure 1: The function $b(k)$

Bayes Nash Equiibria in the class of games G(θ) - 4

- **Result:** The unique equilibrium has both players investing only if they observe a signal greater than 1/2.
- Actually, the strategy with threshold (1/2) is in fact the unique strategy surviving iterated deletion of (interim) strictly dominated strategies, since a strategy s_i survives n rounds of iterated deletion of strict dominated strategies if and only if

$$s_i(\theta_i) = \begin{cases} Invest \quad if \;\; \theta_i > b^n(1) \\ Not\ Invest\, if \;\; \theta_i \leq b^n(0) \end{cases}$$

- where $b^n(k) = \overbrace{b(b(...b(k)))}^{n \text{ times}}.$
- **Proof:** by induction or using Milgrom and Roberts 1990 argument

Bayes Nash Equiibria in the class of games G(θ) - 5

- **Intuition:** the uniform prior assumption ensures that each player, whatever his signal, attaches probability (1/2) to his opponent having a higher signal and probability to him having a lower signal. This property remains true no matter how small the noise is, but breaks discontinuously in the limit: when noise is zero, he attaches probability 1 to his opponent having the same signal.
- The striking feature of this result is that no matter how small σ is, players' behavior is influenced by the existence of the ex ante possibility that their opponent has a dominant strategy to choose each action. Thus, a "grain of doubt" concerning the opponent's behavior has large consequences.
- The probability that either individual invests is $\Phi\left(\frac{\frac{1}{2}-\theta}{\sigma}\right)$

  and, conditional on θ, their investment decisions are independent.

Two by Two General Global Games - 1

- The previous result can be generalized to the class of two-player, two-action games.
- Consider a generic 2x2 game:

		2 L	R
1	U	θ_1, θ_2	θ_3, θ_4
	D	θ_5, θ_6	θ_7, θ_8

Game 3

- Thus a vector $\theta \in R^8$ describes the payoffs of the game
- For a generic choice of θ, there are three possible configurations of Nash equilibria:
  1. there is a unique Nash equilibrium with both players using strictly mixed strategies;
  2. there is a unique strict Nash equilibrium with both players using pure strategies;
  3. there are two pure strategy strict Nash equilibria and one strictly mixed strategy Nash equilibrium.
- Suppose that (U,L) and (D,R) are strict Nash equilibria of the above game, which requires $\theta_1 > \theta_5$, $\theta_7 > \theta_3$, $\theta_2 > \theta_4$, $\theta_8 > \theta_6$.

Two by Two General Global Games - 2

- Now consider the following incomplete information game G(σ).
- Each player i observes a signal $\theta_i = \theta + \sigma\,\varepsilon_i$ where the θ_i, θ and ε_i are eight-dimensional random variables distributed as in G(θ).
- Thus we have a class of incomplete information games parameterized by σ>0.
- In the incomplete information game G(σ), a strategy for player i is a map of the following type $s_i: R^8 \rightarrow A_i$, where $A_1=\{U,D\}$, $A_2=\{L,R\}$.
- Then, Carlsson and van Damme (1993) proves the following Theorem
- **Theorem:** For any sequence of games $G(\sigma^n)$ where $\sigma^n \rightarrow 0$ and any sequence of equilibria of those games, the average play converges at almost all payoff realizations to the unique Nash equilibrium (if there is one) and to the risk dominant Nash equilibrium (if there are multiple Nash equilibria).
- **Summary:** Carlsson and van Damme 1993 named their perturbed games for the two player, two action case "global games" because all possible payoff profiles were possible. They showed that there was a general way of adding noise to the payoff structure such that, as the noise went to zero,
  1. there was a unique action surviving iterated deletion of (interim) dominated strategies, a limit uniqueness result
  2. the action that got played in the limit was independent of the distribution of noise added, a noise independent selection result.

Comments - 1

- The original analysis of Carlsson and van Damme 2003 relaxed the assumption of common knowledge of payoffs in a particular way: they assumed that there was a common prior on payoffs and that each player observes a small conditionally independent signal of payoffs.
- when the noise is small one can show that types in the perturbed game are close to common knowledge types in the product topology on the universal type space
- Thus the `discontinuity' in equilibrium outcomes in global games when noise goes to zero is illustrating the same sensitivity to higher order beliefs of the famous example of the electronic mail game by Rubinstein 1989.
- **Question:**
- how general is the phenomenon that Rubinstein 1989 and Carlsson and van Damme 1993 identified?
- That is, for which games and actions is it the case that, under common knowledge, the action is **part of an equilibrium** (and thus survives iterated deletion of strictly dominated strategies) but for a type `close' to common knowledge of that game, that action is the **unique** action surviving iterated deletion of strictly dominated strategies.

Comments - 2

- **Answer:**
- Weinstein and Yildiz 2007 shows that this is true for every action surviving iterated deletion of strictly dominated strategies in the original game.
- **Implication:** the selections that arise in standard global games arise not just because one relaxes common knowledge, but because it is relaxed in a particular way:
- the common prior assumption is maintained and outcomes are analyzed under that common prior, and
- the noisy signal technology ensures particular properties of higher-order beliefs, that is, that each player's beliefs about how other players' beliefs differ from his is not too dependent on the level of his beliefs.

Comments on Strategic Uncertainty - 1

- In global games, the importance of the noisy observation of the underlying state lies in the fact that it generates strategic uncertainty, that is, uncertainty about others' behavior in equilibrium, because of players' uncertainty about other players' payoffs.
- Thus, understanding global games involves understanding how equilibria depend on players' uncertainty about other players' payoffs.
- But, clearly, it is not going to be enough to know each player's beliefs about other players' payoffs.
- We must also take into account each player's beliefs about other players' beliefs about his payoffs, and further such higher-order beliefs.
- Players' payoffs and higher-order beliefs about payoffs are the true primitives of a game of incomplete information, not the asymmetric information structure.
- In these introductory examples, we told an asymmetric information story about how there is a true state of fundamentals θ drawn from some prior and each player observes a signal of θ generated by some technology.
- But, our analysis of the resulting game implicitly assumes that there is common knowledge of the prior distribution of θ and of the signaling technologies.

Comments on Strategic Uncertainty - 2

- The classic arguments of Harsanyi (1967--1968) and Mertens and Zamir (1985) tell us that we can assume common knowledge of some state space without loss of generality.
- But such a common knowledge state space makes sense with an **incomplete information interpretation** where a player's "type" is a description of his higher-order beliefs about payoffs, but not with an a**symmetric information interpretation** where a player's "type" is a signal drawn according to some ex ante fixed distribution.
- Thus, the noise structures analyzed in global games are interesting because they represent a tractable way of generating a rich structure of higher-order beliefs.
- The analysis of global games represents a natural vehicle to illustrate the power of higher-order beliefs at work in applications.
- But, then, the natural way to understand the "trick" to global games analysis is to go back and understand what is going on in terms of higher-order beliefs..

Global Games with Many Players - 1

- **Example:**
- There is a continuum of players $i \in [0,1]$
- who should decide whether to invest or not.
- The payoff is

$$U_i(L,l) = \begin{cases} \theta + l - 1 & L = Invest \\ 0 & L = Not \;\; Invest \end{cases}$$

where l is the proportion of other players choosing to invest.

- The **information structure** is standard:
- each player i observes a private signal $\theta_i = \theta + \sigma\varepsilon_i$
- where ε_i is identically and independently normally distributed $\varepsilon\_i \sim N(0,1)$
- $\theta \sim U(R)$.

Global Games with Many Players - 2

- **Result:**
- As for the two player game, the unique strategy surviving iterated deletion of strictly dominated strategies is

$$s_i(\theta_i) = \begin{cases} Invest & if \;\; \theta_i > \frac{1}{2} \\ Not\ Invest & if \;\; \theta_i \leq \frac{1}{2} \end{cases}$$

- **Proof:** Consider a player i who has observed signal θ_i and thinks that all his opponents are following the threshold strategy with cutoff point k. Since $E(\theta|\theta_i) = \theta_i$, i will assign probability $\Phi\left(\frac{k-\theta_i}{\sqrt{2}\sigma}\right)$

to any given opponent observing a signal less than k, which is also i's expectation of the proportion of players who observe a signal less than k, because the realization of the signals are independent conditional on θ. Thus, i's expected payoff to investing will be

$$\theta_i - \Phi\left(\frac{k-\theta_i}{\sqrt{2}\sigma}\right)$$

as in the previous 2×2 example, and all the previous arguments go through.

Global Games with Many Players - 3

- **Remark:** the equilibrium outcome is also consistent with a "Laplacian" procedure that places far less demands on the capacity of the players, and that seems to be far removed from equilibrium of any kind.
- **Algorithm:** The "Laplacian" procedure has the following three steps: 1.
  1. Estimate θ from the signal θ_i
  2. Postulate that l is distributed uniformly on the unit interval [0,1]
  3. Take the optimal action.
- **Proof:** Since $E(\theta|\theta_i)=\theta_i$, the expected payoff to investing if l is uniformly distributed is θ_i - (1/2), whereas the expected payoff to not investing is zero. Thus, a player following this procedure will choose to invest if and only if and only if θ_i > (1/2), which is identical to the unique equilibrium strategy previously outlined.

A General Approach to Symmetric Binary Action Global Games with a Continuum of Players

Uniform Prior and Private Values - 1

- **Example:**

1. There is a continuum of players $i \in [0,1]$
2. Each player has to choose an action $a \in \{0,1\}$
3. All players have the same payoff function $u : \{0,1\} \times [0,1] \times \mathbb{R} \to \mathbb{R}$ where $u(a, l, \theta_i)$ is i's player's payoff if she chooses action a, a proportion *l* of the other players choose action 1, and her "private signal" is θ_i.

- **Information structure:**

1. $\theta \sim U(R)$
2. $\theta_i = \theta + \sigma\varepsilon_i$ with $\sigma > 0$
3. ε_i is a noise distributed on R according to a continuous density $f(\cdot)$, possibly non symmetric and with mean different from 0.

- **Result**: the density of $\theta | \theta_i$ is well defined and is $\left(\frac{1}{\sigma}\right) f\left(\frac{\theta_i - \theta}{\sigma}\right)$

Uniform Prior and Private Values - 2

- **Remark: since** i's payoff is independent of which of the opponents choose action 1, to analyze best responses, it is enough to know the payoff gain from choosing one action rather than the other. Thus, the utility function is parameterized by a function

$$\pi : [0,1] \times \mathbb{R} \to \mathbb{R} \textit{ such that } \pi(l, \theta_i) \equiv u(1, l, \theta_i) - u(0, l, \theta_i).$$

- **Definition:** An action is the Laplacian action if it is a best response to a uniform prior over the opponents' choice of action.

Uniform Prior and Private Values - 3

Assumption 2 *The players' payoffs satisfy the following properties*

P.1 ***Action Monotonicity:*** *$\pi(l, \theta)$ is nondecreasing in l, i.e. the players actions are strategic complements;*

P.2 ***State Monotonicity:*** *$\pi(l, \theta)$ is nondecreasing in θ, i.e. a player's optimal action is increasing in the unknown state;*

P.3 ***Strict Laplacian State Monotonicity:*** *there exists a unique θ^* solving $\int_{l=0}^{1} \pi(l, \theta^*)dl = 0$, i.e. there is at most one crossing for a player with Laplacian beliefs;*

P.4 ***Limit Dominance:*** *there exist $\underline{\theta} \in \mathbb{R}$ and $\overline{\theta} \in \mathbb{R}$, such that*

(a) *$\pi(l, \theta_i) < 0$ for all $l \in [0, 1]$ and for all $\theta_i \leq \underline{\theta}$, i.e. action $a = 0$ is a dominant strategy for sufficiently low signals;*

(b) *$\pi(l, \theta_i) > 0$ for all $l \in [0, 1]$ and for all $\theta_i \geq \overline{\theta}$, i.e. action $a = 1$ is a dominant strategy for sufficiently high signals;*

P.5 ***Continuity:*** *$\int_{l=0}^{1} g(l) \pi(l, \theta_i)dl$ is continuous with respect to θ_i and density g with respect to the weak topology, therefore the payoff function might be discontinuous at one value of l.*

Uniform Prior and Private Values - 4

- **Definition**: Let define the game satisfying these assumptions as $G^*(\sigma)$.
- **Proposition:** In game $G^*(\sigma)$there is essentially a unique iterated strictly undominated strategy profile $(s_i^*)_{i\in[0,1]}$ such that

$$\forall i \in [0,1] \qquad s_i^*(\theta_i) = \begin{cases} 0\, if & \theta_i < \theta^* \\ 1\, if & \theta_i > \theta^* \end{cases} \quad where \;\; \theta^* \;\; satisfies \; \int_0^1 \pi(l, \theta^*)\, dl = 0.$$

- **Sketch of the proof:** The key idea of the proof is that, with a uniform prior on θ, observing θ_i gives no information to a player on her ranking within the population of signals. Thus, she will have a uniform prior belief over the proportion of players who will observe higher signals.

General Prior and Common Values - 1

- **Example:**

1. There is a continuum of players $i \in [0,1]$
2. Each player has to choose an action $a \in \{0,1\}$
3. All players have the same payoff function $u : \{0, 1\} \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ where $u(a, l, \theta)$ is i's player's payoff if she chooses action a, a proportion l of the other players choose action 1, and the realized state is θ.

- **Information structure:**

1. $\theta \sim p(R)$ where p(R) is a a continuously differentiable strictly positive density on the real line R.
2. $\theta_i = \theta + \sigma\varepsilon_i$ with $\sigma > 0$
3. ε_i is a noise distributed on R according to a continuous density $f(\cdot)$, possibly non symmetric and with mean different from 0.
4. $\int_{-\infty}^{\infty} zf(z)dz$ is well defined.

General Prior and Common Values - 2

- We must impose two extra technical assumptions:

Assumption 3 *P.4* Uniform Limit Dominance: there exist* $\underline{\theta} \in \mathbb{R}$, *a* $\overline{\theta} \in \mathbb{R}$, *and a strictly positive* $\epsilon \in \mathbb{R}_{++}$ *such that*

$$(a)\ \pi(l, \theta) < -\epsilon \ \text{ for all } l \in [0,1] \ \text{ and } \ \text{for all } \theta \leq \underline{\theta};$$

$$(b)\ \pi(l, \theta) > \epsilon \ \text{ for all } l \in [0,1] \ \text{ and } \ \text{for all } \theta \geq \overline{\theta}.$$

- **Remark**: Assumption P.4* strengthens assumption P.4 of Limit Dominance by requiring that the payoff gain to choosing action 0 is uniformly negative for sufficiently low values of θ, and the payoff gain to choosing action 1 is uniformly positive for sufficiently high values of θ.

General Prior and Common Values - 3

- **Definition**: Let define the game satisfying assumptions P.1, P.2, P.3, P.4*, P.5 and I.6 as G(σ).
- **Proposition:** Let θ^* be defined solving $\int_0^1 \pi(l,\theta^*)dl=0$. For any $\delta>0$, there exists $\underline{\sigma}>0$ such that for all $\sigma \geq \underline{\sigma}$, if strategy s_i survives iterated deletion of strictly dominated strategies in the game G(σ), then

$$\forall i \in [0,1] \qquad s_i(\theta_i) = \begin{cases} 0 \; if \quad \theta_i < \theta^* - \delta \\ 1 \; if \quad \theta_i > \theta^* + \delta. \end{cases}$$

Comments

- Assumptions P.1 and P.2 represent very strong monotonicity assumptions:
- P.1 requires that each player's utility function is supermodular in the action profile,
- P.2 requires that each player's utility function is supermodular in his own action and the state.
- Vives (1990) showed that the supermodularity property P.2 of complete information game payoffs is inherited by the incomplete information game.
- Thus, the existence of a largest and smallest strategy profile surviving iterated deletion of dominated strategies when payoffs are supermodular, noted by Milgrom and Roberts (1990), can be applied also to the incomplete information game.
- The state monotonicity assumption P.2 implies, in addition, that the largest and smallest equilibria consist of cutoff strategies.
- Once we know that we are interested in cutoff strategies, the very weak assumption P.3 is sufficient to ensure the equivalence of the largest and smallest equilibria and thus the uniqueness of equilibrium.

Inefficiency of Equilibrium Outcomes in Global Games

- **Result:** In general, in global games the equilibrium outcomes are not efficient.
- **Proof:** In the limit, in equilibrium all players will be choosing action 1 when the state is θ if $\int_0^1 \pi(l,\theta)\,dl > 0.$
- On the other hand, efficiency requires to choose action 1 at state θ if u(1,1,θ) > u(0,0,θ), and these conditions will not coincide in general. For example, in the investment example, we had π(l,θ) = u(1,l,θ) - u(0,l,θ) = θ+l-1 that implies

$$\int_0^1 \pi(l,\theta)\,dl = \int_0^1 (\theta + l - 1) = \theta - \frac{1}{2}.$$

so that both players will be investing if the state θ is at least (1/2), although it is efficient for them to be investing if the state is at least 0.

Public and Private Signals - 1

- To understand the effects of public signals for global games, consider the Investment Game with a continuum of players previously analyzed with private information only.
- **Example:**
- There is a continuum of players $i \in [0,1]$
- who should decide whether to invest or not.
- The payoff is

$$U_i(L,l) = \begin{cases} \theta + l - 1 & L = Invest \\ 0 & L = Not \;\; Invest \end{cases}$$

where l is the proportion of other players choosing to invest.

- The **information structure** is:
- each player i observes a private signal $\theta_i = \theta + \sigma\varepsilon_i$
- where ε_i is identically and independently normally distributed $\varepsilon_i \sim N(0,1)$
- $\theta \sim N(y, \tau)$ where y is a public signal.

Public and Private Signals - 2

- From standard statistics, we get the following result.
- **Result:**

$$E(\theta|\theta_i) = \frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2}$$

- Consider the following cutoff strategy

$$s(E(\theta|\theta_i)) = \begin{cases} Invest & \text{if } E(\theta|\theta_i) > \kappa \\ Not\ \ Invest & \text{if } E(\theta|\theta_i) \leq \kappa. \end{cases}$$

- Let define

$$\widetilde{\gamma}(\sigma, \tau) = \frac{\sigma^2}{\tau^4}\left(\frac{\sigma^2 + \tau^2}{\sigma^2 + 2\tau^2}\right).$$

Public and Private Signals - 3

- Morris and Shin (2006) prove the following result.
- **Proposition:** The game has a symmetric switching strategy equilibrium with cutoff κ if κ solves the equation

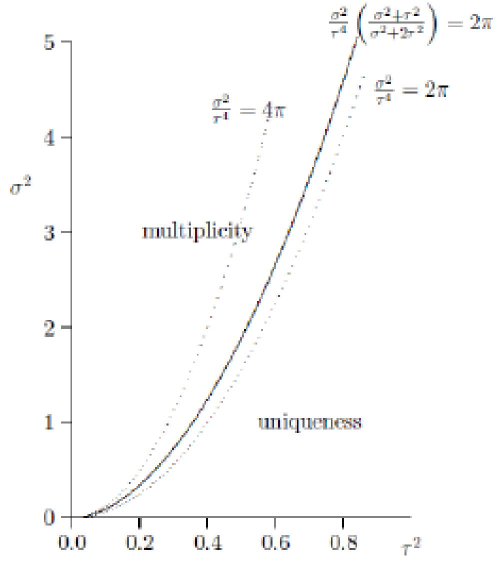
$$\kappa = \Phi\left(\sqrt{\tilde{\gamma}}\,(\kappa - y)\right);$$

then

1. if γ(σ,τ)≤2π, there is a unique value of κ solving the previous equation and the strategy with cutoff κ is the essentially unique strategy surviving iterated deletion of strictly dominated strategies;
2. if γ(σ,τ)>2π,, then (for some values of y) there are multiple values of κ solving the previous equation and multiple symmetric cutoff strategy equilibria.

Public and Private Signals - 4

- The following picture plots the two regions in the space (τ^2, σ^2):



- **Corollary:** Suppose $\gamma(\sigma,\tau) \leq 2\pi$, then

1. if $\theta < 0$, in equilibrium for any y it is optimal not to invest;
2. if $\theta > 1$, in equilibrium for any y it is optimal to invest;
3. if $\theta \in [0,1]$, then in equilibrium the higher y, the more likely it is optimal to invest. Thus, the players will always invest for sufficiently high y, and not invest for sufficiently low y. This implies that **changing y has a larger impact on a player's action than changing his private signal (controlling for the informativeness of the signals), the "publicity" effect**.

The Role of Public and Private Information - 1

- To explore the strategic impact of public information, we examine how much a player's private signal must adjust to compensate for a given change in the public signal. Consider the cutoff equation

$\kappa = \Phi\left(\sqrt{\tilde{\gamma}}\,(\kappa - y)\right)$ when $\kappa = \bar{\theta} = \frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2}$, which implies $\frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2} = \Phi\left(\sqrt{\tilde{\gamma}}\left(\frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2} - y\right)\right)$

- Considering this equation as an implicit function $\theta_i(y)$, we have

$$\frac{d\theta_i(y)}{dy} = -\frac{\frac{\sigma^2}{\tau^2} + \sqrt{\tilde{\gamma}}\phi(\cdot)}{1 - \sqrt{\tilde{\gamma}}\phi(\cdot)}$$

- which measures how much the private signal would have to change to compensate for a change in the public signal leaving the player indifferent between investing or not investing.

The Role of Public and Private Information - 2

- On the other hand, if we ignore strategic effect of a change in y, the private signal has a different substitution ratio that can be derived maintaining constant

  $\bar{\theta} = \frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2}$ so that $\frac{d\theta_i}{dy} = -\frac{\sigma^2}{\tau^2}.$

- The ratio between these two substitution ratio defines the **publicity multiplier**

$$\zeta = \frac{1 + \frac{\tau^2}{\sigma^2}\sqrt{\tilde{\gamma}}\phi(\cdot)}{1 - \sqrt{\tilde{\gamma}}\phi(\cdot)}$$

- which is **increasing in gamma tilde**, which is intuitive since it is directly related to the informativeness of the public versus the private signal.
- However, remember that gamma tilde is bounded above by 2π, otherwise we go into the regions of multiple equilibria.

Applications:

a quick survey of some simple basic models

Pricing Debt - 1

- This application refers to Morris and Shin (2004). Consider the following simple model.
  1. There are two periods:
     1. in period 1, a continuum of investors hold collateralized debt that will pay
        - 1 in period 2 if it is rolled over and if an underlying investment project is successful;
        - 0 in period 2 if the project is not successful;
        - $\kappa \in (0,1)$, the value of the collateral, if an investor does not roll over his debt.
  2. The success of the project depends on
     - the proportion l of investors who do not roll over and
     - the state of the economy, θ, which is distributed according to a continuum density $p(\cdot)$
     - Specifically, the project is successful if the proportion of investors not rolling over is less than θ/z.

Pricing Debt - 2

3. Write a=1 for the action "roll over" and a=0 for the action "do not roll over", then the payoffs can be written as follows:

$$u(a, l, \theta) = \begin{cases} 1 \text{ if } \; a = 1 \text{ and } \frac{\theta}{z} \geq 1 - l \\ 0 \text{ if } \; a = 1 \text{ and } \frac{\theta}{z} < 1 - l \\ \kappa \qquad \text{if } \; a = 0 \end{cases}$$

or alternatively

		Percentage of investors rolling over $1 - l \leq \frac{\theta}{z}$	$1 - l > \frac{\theta}{z}$
$i \in [0, 1]$	$a = 1$	1	0
	$a = 0$	κ	κ

Payoff structure

Hence,

$$\pi(l, \theta) = u(1, l, \theta) - u(0, l, \theta) = \begin{cases} 1 - \kappa \text{ if } \; \frac{\theta}{z} \geq 1 - l \\ -\kappa \; \text{ if } \; \frac{\theta}{z} < 1 - l \end{cases}$$

so that

$$\int_0^1 \pi(l, \theta)\, dl = \begin{cases} -\kappa \qquad \text{if } \; \theta \leq 0 \\ \frac{\theta}{z} - \kappa \text{ if } \; 0 \leq \theta \leq z \\ 1 - \kappa \quad \text{if } \; \theta \geq z. \end{cases}$$

Pricing Debt - 3

Remark: The game representing the model satisfies assumptions P.1* and P.2, and therefore Proposition 3 holds.

Thus, we can state the following result.

Result: Let $\theta^* = z\kappa$, then the game has a unique (symmetric) cutoff strategy equilibrium, such that

$$\forall i \in [0,1] \qquad s_i(\theta_i) = \begin{cases} 0 \; if \quad \theta_i \le \theta^* \\ 1 \; if \quad \theta_i > \theta^*. \end{cases}$$

Remark: In other words, if private information about θ among the investors is sufficiently accurate, the project will collapse exactly if $\theta \le z\kappa$.

Question: We can now ask how debt would be priced ex ante in this model, i.e. before anyone observed private signals about θ.

Recalling that p(·) is the density of the prior on θ, and writing P(·) for the corresponding cdf, the value of the collateralized debt will be

$$V(\kappa) \equiv \kappa P(z\kappa) + 1 - P(z\kappa) = 1 - (1-\kappa) P(z\kappa)$$

that implies

$$\frac{dV(\kappa)}{d\kappa} = P(z\kappa) - z(1-\kappa)\,p(z\kappa).$$

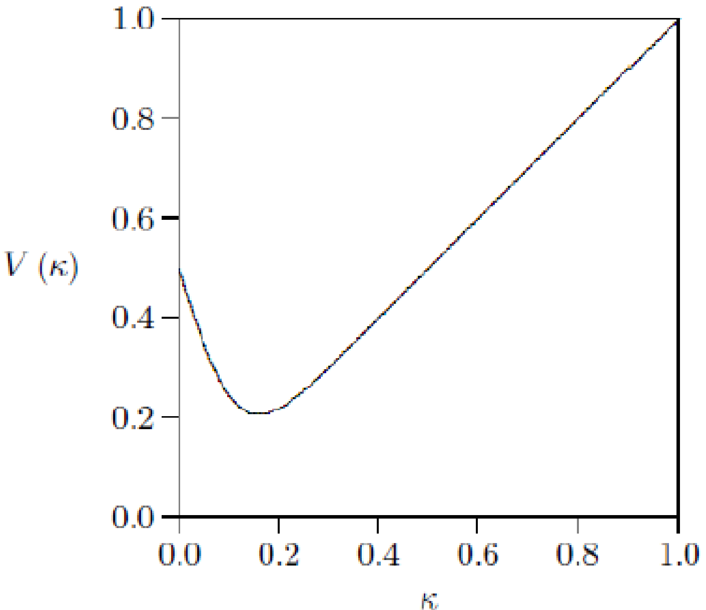
Pricing Debt - 4

Since $$\frac{dV(\kappa)}{d\kappa} = P(z\kappa) - z(1-\kappa)\,p(z\kappa).$$

Then increasing the value of collateral has two effects:

1. it increases the value of debt in the event of default (the direct effect);
2. it increases the range of θ at which default occurs (the strategic effect).

- For small κ, the strategic effect outweighs the direct effect, whereas for large κ, the direct effect outweighs the strategic effect.
- The following figure plots V(κ) for the case where z=10 and p(·) is the standard normal density.



Currency Crises - 1

- This application refers to Morris and Shin (1998). Consider the following simple model.
  1. There is a continuum of speculators must decide whether to attack a fixed--exchange rate regime by selling the currency short.
  2. Each speculator may short only a unit amount.
  3. There is a fixed transaction cost t of attacking, that can be interpreted as an actual transaction cost or as the interest rate differential between currencies.
  4. The current value of the currency is e^*;
  5. the monetary authority may defend or not the currency
     1. if the monetary authority does not defend the currency, the currency will float to the shadow rate $\zeta(\theta)$, where θ is the state of fundamentals, so that $\zeta(\theta)$ is increasing in θ. Assume $\zeta(\theta) \leq e^* - t$ for all θ;
     2. if the monetary authority does defend the currency, its value remains at e^*
     3. The monetary authority defends the currency if the cost of doing so is not too large, where the costs of defending the currency are increasing in the proportion of speculators who attack and decreasing in the state of fundamentals.
     4. Hence, there will be a critical proportion of speculators, $b(\theta)$, increasing in θ, who must attack in order for a devaluation to occur.

Currency Crises - 2

6. Write a=1 for the action "not attack" and a=0 for the action "attack", then the payoffs can be written as follows:

$$u(a,l,\theta) = \begin{cases} 0 & \text{if } a = 1 \\ e^* - \zeta(\theta) - t & \text{if } a = 0 \text{ and } 1 - l \geq b(\theta) \\ -t & \text{if } a = 0 \text{ and } 1 - l < b(\theta) \end{cases}$$

or alternatively

		Percentage of attacking speculators	
		$1 - l < b(\theta)$	$1 - l \geq b(\theta)$
$i \in [0,1]$	$a = 1$	0	0
	$a = 0$	$-t$	$e^* - \zeta(\theta) - t$

Payoff structure

Hence,

$$\pi(l,\theta) = u(1,l,\theta) - u(0,l,\theta) = \begin{cases} \zeta(\theta) + t - e^* & \text{if } l \leq 1 - b(\theta) \\ t & \text{if } l \leq 1 - b(\theta). \end{cases}$$

Currency Crises - 3

- **Result:** Suppose θ is common knowledge, then
  1. if $\theta < b^{-1}(0)$, then there is unique equilibrium in dominant strategies, $a^* = 0$ for all $i \in [0,1]$;
  2. if $b^{-1}(0) \leq \theta \leq b^{-1}(1)$, then there two equilibria such that $a^* = 0$ for all $i \in [0,1]$ and $a^{**} = 1$ for all $i \in [0,1]$;
  3. if $\theta > b^{-1}(1)$, then there is unique equilibrium in dominant strategies, $a^* = 1$ for all $i \in [0,1]$.
- On the other hand, if θ is observed with noise, we can apply the previous results, because the previous assumptions are satisfied. In particular

$$\int_0^1 \pi(l,\theta)\,dl = [1 - b(\theta)][\zeta(\theta) + t - e^*] + b(\theta)t$$

which implies $\int_0^1 \pi(l,\theta^*)\,dl = 0 \Leftrightarrow [1 - b(\theta^*)][\zeta(\theta^*) - e^*] = t.$ Thus, we can state the following result

- **Result**: The game representing our model with private information on θ has a unique (symmetric) cutoff strategy equilibrium

$$\forall i \in [0,1] \qquad s_i(\theta_i) = \begin{cases} 0 \; if & \theta_i \leq \theta^* \\ 1 \; if & \theta_i > \theta^*. \end{cases}$$

Bank Runs - 1

- Consider the following simple model by Goldstein and Pauzner (2005), who add noise to the classic bank runs model of Diamond and Dybvig (1983).
  1. There are two periods, 1 and 2
  2. There is a continuum of depositors $i \in [0,1]$ (with total deposits normalized to 1)
  3. Each depositor must decide whether to withdraw their money from a bank at period 1, denoted $a=0$, or at period 2, denoted by $a=1$.
  4. The withdrawn resources are entirely used for consumption that gives utility $U(\cdot)$.
  5. A proportion λ of depositors will have consumption needs only in period 1 and will thus have a dominant strategy to withdraw, thus we are concerned with the game among the proportion $1-\lambda$ of depositors.

Bank Runs - 2

6. The monetary payoffs are:
   - r>1 if the depositors withdraw their money in period 1 and there are enough resources;
   - $\frac{1-\lambda r}{(1-l)(1-\lambda)}$ if there are not enough resources to fund all those who try to withdraw, i.e. the remaining cash 1-λr is divided equally among early withdrawers. This happens when

$$\lambda r + (1-l)(1-\lambda) r \geq 1 \Leftrightarrow l \leq \frac{r-1}{(1-\lambda) r};$$

   - max{0,1-λr+(1-l)(1-λ)r}R(θ) ≥ 0 in period 2 for those who chose to wait until period 2 to withdraw their money, i.e. any remaining money after period 1 withdraws, max{0,1-λr+(1-l)(1-λ)r}, earns a total return R(θ)>0 in period 2, which is increasing in θ, and it is divided equally among those who chose to wait until period 2 to withdraw their money, l(1-λ).

Bank Runs - 3

Then the consumers monetary payoffs can be written as follows:

		Percentage of late consumers		
		$l \leq \frac{r-1}{(1-\lambda)r}$	$l \geq \frac{r-1}{(1-\lambda)r}$	
$i \in [0,1]$	$a = 0$	$\frac{1-\lambda r}{(1-l)(1-\lambda)}$	r	
	$a = 1$	0	$\left[\frac{1-\lambda r+(1-l)(1-\lambda)r}{l(1-\lambda)}\right]$	$R(\theta)$

Monetary payoff structure

Thus, the utilities of late consumers are

$$u(a,l,\theta) = \begin{cases} U\left(\frac{1}{1-l(1-\lambda)}\right) & \text{if } a=0 \text{ and } l \leq \frac{r-1}{(1-\lambda)r} \\ U(r) & \text{if } a=0 \text{ and } l \geq \frac{r-1}{(1-\lambda)r} \\ U(0) & \text{if } a=1 \text{ and } l \leq \frac{r-1}{(1-\lambda)r} \\ U\left(\left[\frac{1-\lambda r+(1-l)(1-\lambda)r}{l(1-\lambda)}\right] R(\theta)\right) & \text{if } a=0 \text{ and } l \geq \frac{r-1}{(1-\lambda)r} \end{cases}$$

Hence,

$$\pi(l,\theta) = u(1,l,\theta) - u(0,l,\theta) = \begin{cases} U(0) - U\left(\frac{1}{1-l(1-\lambda)}\right) & \text{if } l \leq \frac{r-1}{(1-\lambda)r} \\ U\left(\left[\frac{1-\lambda r+(1-l)(1-\lambda)r}{l(1-\lambda)}\right] R(\theta)\right) - U(r) & \text{if } l \geq \frac{r-1}{(1-\lambda)r} \end{cases}$$

Bank Runs - 4

- **Result:** Suppose θ is common knowledge, then for late consumers
    1. if θ is small so that also R(θ) is small, then there is unique equilibrium in dominant strategies, $a^* = 0$;
    2. if θ is intermediate so that also R(θ) is intermediate, then there are two equilibria, $a^* = 0$ and $a^{**} = 1$ for all $i \in [0,1]$;
    3. if θ is large so that also R(θ) is large, then there is unique equilibrium in dominant strategies, $a^* = 1$.
- On the other hand, if θ is observed with noise, we can apply the previous results, because the previous assumptions are satisfied.
- **Remark:** the game representing the model satisfies assumptions P.1 and P.2, and therefore Proposition 2 holds. In particular θ^* is defined by the following equation
- **Result:** The game representing our model with private information on θ has a unique (symmetric) cutoff strategy equilibrium, such that

$$s_i(\theta_i) = \begin{cases} 0 \ if \ \ \theta_i \leq \theta^* \\ 1 \ if \ \ \theta_i > \theta^*. \end{cases}$$