(Perfect) Complementarity: Bertrand vs Cournot

Paolo Bertoletti (Milan-Bicocca University)

15th of March, 2022 (a meeting of the Global Games group)

A partial equilibrium approach: demand

- Suppose that preferences (of a **representative consumer**) are quasi-linear:

$$U(\mathbf{q}, m) = u(\mathbf{q}) + m,$$

where m is the *numéraire*: $u(\mathbf{q})$ it is assumed to be differentiable and "strongly" concave (implying that its Hessian is a negative definite matrix).

- Assuming an interior solution, the FOCs for utility maximization do characterise the **inverse demand system** $\mathbf{p}(\mathbf{q})$:

$$\mathbf{p}(\mathbf{q}) = Du(\mathbf{q}),\ D\mathbf{p}(\mathbf{q}) = D^2 u(\mathbf{q}).$$

- The **direct demand system** satisfies:

$$\mathbf{q}(\mathbf{p}) = \mathbf{p}^{-1}(\mathbf{p}),\ D\mathbf{q}(\mathbf{p}) = \left[D^2 u(\mathbf{q}(\mathbf{p}))\right]^{-1}.$$

Substitutability

- Commodities can be (locally) stratified according to the sign of the derivatives $\frac{\partial p_i(\mathbf{q})}{\partial q_j}$ and $\frac{\partial q_i(\mathbf{p})}{\partial p_j}$.
- In particular, goods i and j are said to be **substitutes** if $\frac{\partial p_i(\mathbf{q})}{\partial q_j} < 0$ (or $\frac{\partial q_i(\mathbf{p})}{\partial p_j} > 0$) and **complements** if $\frac{\partial p_i(\mathbf{q})}{\partial q_j} > 0$ (or $\frac{\partial q_i(\mathbf{p})}{\partial p_j} < 0$).
- With more than 2 commodities the classifications made according to the direct and indirect demand systems need *not* agree.
- As an example, consider the quadratic, *symmetric* utility with 2 commodities:

$$u\left(\mathbf{q}\right) = aq_1 + aq_2 - \frac{1}{2}\left(bq_1^2 + bq_2^2 + 2dq_1q_2\right),$$

where $a > 0$ and $b > \left|d\right|$.

The linear (symmetric) demand system

- It is immediate to obtain (for *positive prices and quantities*):

$$\begin{aligned} p_i(\mathbf{q}) &= a - bq_i - dq_j, \\ q_i(\mathbf{p}) &= \widetilde{a} - \widetilde{b}p_i + \widetilde{d}p_j, \end{aligned}$$

  where $\widetilde{a} = \frac{a}{b+d}$, $\widetilde{b} = \frac{b}{b^2-d^2}$, $\widetilde{d} = \frac{d}{b^2-d^2}$, $i, j = 1, 2$, $i \neq j$.
- $\frac{d}{b} \in (-1, 1)$ is an **inverse measure of differentiation**: commodities are substitutes when d, $\widetilde{d} > 0$ and complements when d, $\widetilde{d} < 0$.
- In fact, commodities are **perfect** substitutes when $d = b > 0$, and independent when $d = 0$.
- The cross, inverse demand in the case of good complementarity is linear increasing (see Fig. 1).

Fig. 1: Cross inverse demand with complementarity in the linear model

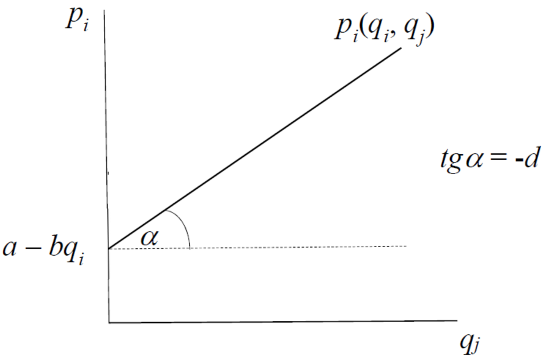

Figure: Cross inverse demand with complementarity in the linear demand system

Oligopoly: Cournot vs Bertrand

- In a **quantity-setting** (Cournot) oligopoly firms take as given the inverse demand system $\mathbf{p}(\mathbf{q})$, while in a **price-setting** (Bertrand) oligopoly firms take as given the direct demand system $\mathbf{q}(\mathbf{p})$.
- This difference does matter, as it is well known.
- In particular, in a *Cournotian setting* the payoff function is given by:

$$\pi_i(\mathbf{q}) = p_i(\mathbf{q})\, q_i - C_i(q_i),$$

- with

$$\begin{aligned} \frac{\partial \pi_i(\mathbf{q})}{\partial q_i} &= \frac{\partial p_i(\mathbf{q})}{\partial q_i} q_i + p_i(\mathbf{q}) - C_i'(q_i), \\ \frac{\partial^2 \pi_i(\mathbf{q})}{\partial q_i \partial q_j} &= \frac{\partial^2 p_i(\mathbf{q})}{\partial q_i \partial q_j} q_i + \frac{\partial p_i(\mathbf{q})}{\partial q_j}. \end{aligned}$$

Strategic Complementarity

- Assuming that demand is twice differentiable, **strategic complementarity** arises in the Cournotian setting iff (everywhere) $\frac{\partial^2 \pi_i(\mathbf{q})}{\partial q_i \partial q_j} \geq 0$, i.e., iff

$$\frac{\partial^2 p_i(\mathbf{q})}{\partial q_i \partial q_j} q_i \geq -\frac{\partial p_i(\mathbf{q})}{\partial q_j},$$

(which is equivalent to $-\frac{\partial \ln \left\{ \frac{\partial p_i(\mathbf{q})}{\partial q_j} \right\}}{\partial \ln q_i} \leq 1$ **under good complementarity**).

- Notice in the case of the linear demand system (since $\frac{\partial^2 p_i(\mathbf{q})}{\partial q_i \partial q_j} = 0$) there is strategic complementarity if and only if goods are complements or independent.

Cournot-Nash equilibria with complementarity in the (symmetric) linear demand system

- Assuming for the sake of simplicity that $C_i = 0$, it is easy to see that the best reply functions are given by

$$q_i\left(q_j\right) = \frac{a - dq_j}{2b},$$

- and that in the unique NE:

$$q_i^C = \frac{a}{2b+d}, p_i^C = \frac{ab}{2b+d} = bq_i^C, \pi_i^C = \frac{a^2 b}{\left(2b+d\right)^2} = b\left(q_i^C\right)^2.$$

- It can also be proven that **Bertand prices are smaller than Cournotian prices** (in this *symmetric* setting), but profit are higher in the former case, since a (multiproduct) monopolist would choose even **smaller** prices (and **larger** quantities):

$$\begin{aligned} p_i^C &= \frac{ab}{2b+d} > \frac{a}{2} = p_i^m, \\ q_i^C &= \frac{a}{2b+d} < \frac{a}{2\left(b+d\right)} = q_i^m. \end{aligned}$$

Homogeneous products

- Products are **homogeneous** when $u(\mathbf{q}) = u(q)$ where $q = \sum_i q_i$ (for example, this is the case if $b = d > 0$ in the linear demand system).
- Then $p_i(\mathbf{q}) = P\left(\sum_i q_i\right)$ and, referring to the case of two commodities:

$$q_i(p_i, p_j) = \begin{cases} = 0 & \text{if } p_i > p_j \\ = \alpha_i(p_i)\, D(p_i) & \text{if } p_i = p_j \\ = D(p_i) & \text{if } p_i < p_j \end{cases} \tag{1}$$

where $\alpha_i(p)$, $i = 1, 2$ are arbitrary functions such that $0 \leq \alpha_i(p) \leq 1$, $\alpha_1(p) + \alpha_2(p) = 1$, reflecting the fact that demand functions are not uniquely defined when prices are identical, and $D(p) = P^{-1}(p)$. See Fig 2 and 3.

Fig. 2: Direct demand with 2 homogeneous products

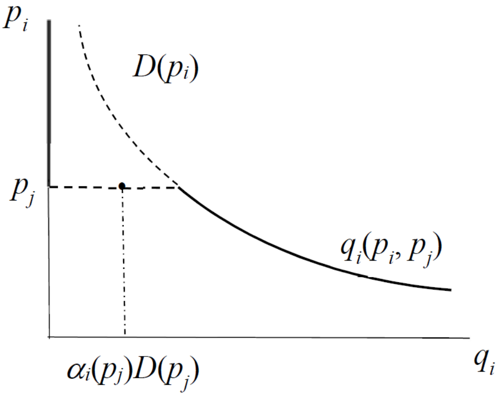

Figure: Direct demand with 2 homogeneous products

Fig. 3: Cross direct demand with 2 homogeneous products

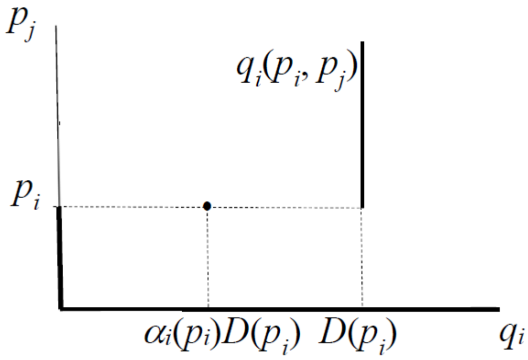

Figure: Cross direct demand with 2 homogeneous products

Perfect complements

- Products are **perfect complements** when they can only be consumed in fixed proportions. For example, suppose that they must be consumed in a one-to-one ratio: then $u(\mathbf{q}) = u(\min\{q_1, .., q_n\})$ (this is not captured by the linear demand system).
- Then $q_i(\mathbf{p}) = D\left(\sum_i p_i\right)$ and, referring to the case of two commodities:

$$p_i(q_i, q_j) = \begin{cases} = 0 & \textit{if } q_i > q_j \\ = \widetilde{\alpha}_i(q_i)\, P(q_i) & \textit{if } q_i = q_j \\ = P(q_i) & \textit{if } q_i < q_j \end{cases} \tag{2}$$

where $\widetilde{\alpha}_i(q_i)$, $i = 1, 2$ are such that $0 \leq \widetilde{\alpha}_i(\widetilde{q}) \leq 1$, $\widetilde{\alpha}_1(\widetilde{q}) + \widetilde{\alpha}_2(\widetilde{q}) = 1$, and $P(\widetilde{q})$ for the common quantity $\widetilde{q} = q_1 = q_2$ is such that $D(p) = P^{-1}(p)$. See Fig 4 and 5.

Fig. 4: Inverse demand with 2 perfect complements

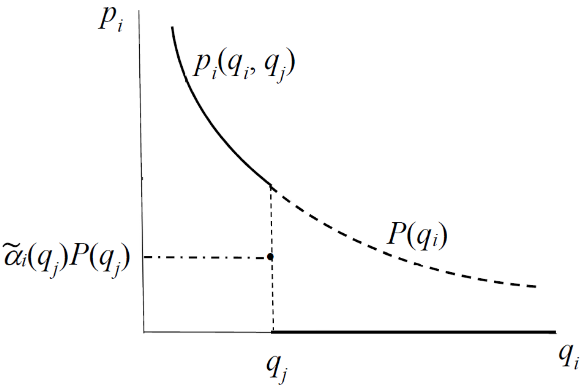

Figure: Inverse demand with 2 perfect complements

Fig. 5: Cross inverse demand with 2 perfect complements

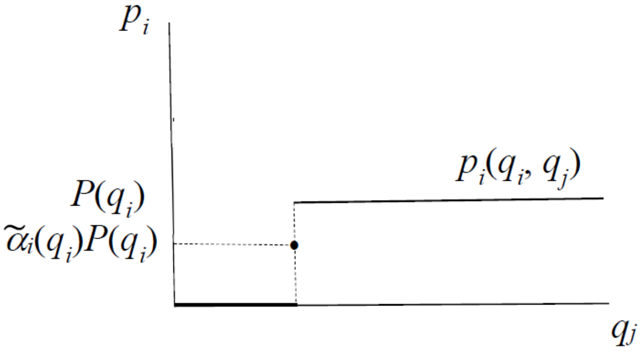

Figure: Cross demand with 2 perfect complements

"Duality" in oligopoly theory

- Sonnenschein (1968) noted that **Cournot's duopoly with homogeneous products is dual to Bertrand's duopoly with perfect complements**, since revenue functions are given respectively by $\widetilde{R}_i\left(q_1, q_2\right) = q_i P\left(q_1 + q_2\right)$ and $R_i\left(p_1, p_2\right) = p_i D\left(p_1 + p_2\right)$.
- Sonnenschein (1968) used this fact to extend to the latter model a well-known criticism of the former: "each duopolist can obtain a greater revenue by reducing his price a little and selling the quantity that clears the market (provided, of course, the other duopolist does not change his price)".
- A second "duality" seems to have gone unnoticed: **Cournot's duopoly with perfect complements is dual to Bertrand's duopoly with homogeneous products**, as illustrated above.

A simple result

- *Proposition 1.* Suppose that the duopolists' cost functions $C_i(q_i)$ are non-decreasing and differentiable, and that it exists a finite $P(0)$ such that $P(0) > C_i'(0)$, $i = 1, 2$: then in the Cournot duopoly game with perfect complements and simultaneous moves there exists a unique Nash equilibrium (in pure strategy) in which both quantities are null.
- Implication: *the provision of perfectly complementary goods might actually be impossible, under general cost conditions, if the market is not either perfectly competitive or monopolized*.
- That an imperfectly competitive market might find difficult to provide complementary goods was long ago suggested by Spence (1976), who noted that some good may not be produced at all. Similar results have been proved more recently by using the theory of supermodular games: Vives (1999).

A simple example

- Suppose $u(\mathbf{q}) = a \min\{q_1, q_2\} - \frac{b}{2}(\min\{q_1, q_2\})^2$ and $C_i(q_i) = cq_i$, with $a > 2c$.
- Then $P(\widetilde{q}) = a - b\widetilde{q}$ and $D(p_1 + p_2) = \frac{a-(p_1+p_2)}{b}$, and quantities are null in the **unique** Cournotian equilibrium $q_i^C = 0$.
- A perfectly competitive market would provide the **Pareto-efficient** quantities $q_i{}^\circ = \frac{a-2c}{b}$ at prices equal to the marginal cost c.
- In the **unique** Bertrand equilibrium $p_i^B = \frac{a+c}{3}$ and $q_i^B = \frac{a-2c}{3b}$. But the equilibrium profit $\pi_i^B = \frac{(a-2c)^2}{9b}$ is smaller that the profit that each duopolist could get by reducing slightly his quantity and selling it at the market clearing price, provided that the other duopolist does not change his quantity: Sonnenschein (1968).
- A monopolist would sell quantities $q_i^m = \frac{a-2c}{2b}$ at prices p_i^m such that $p_1^m + p_2^m = \frac{a+2c}{2}$, with $q_i{}^\circ > q_i^m > q_i^B > q_i^C = 0$ and $2c < p_1^m + p_2^m < p_1^B + p_2^B = \frac{2(a+c)}{3} < p_1^C + p_2^C = a$.

References

Bertoletti, Paolo (2022) "The dual of Bertrand with homogenous products is Cournot with perfect complements", DEMS Working Paper Series No. 490, University of Milan-Bicocca.

Sonnenschein, Hugo (1968) "The Dual of Duopoly Is Complementary Monopoly: or, Two of Cournot's Theories Are One", *Journal of Political Economy* 76 (2), 316-8.

Spence, Michael (1976) Production Selection, Fixed Costs, and Monopolistic Competition, *Review of Economic Studies* 43 (2), 217-35.

Vives, Xavier (1999) *Oligopoly Pricing. Old Ideas and New Tools*, MIT Press, Cambridge (USA).