Notes on Global Games

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Abstract

These are notes based on Morris, S. and Shin, H.S. 2003. Global games: theory and applications. In Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Societ y), ed. M. Dewatripont, L. Hansen and S. Turnovsky. Cambridge: Cambridge University Press. They review the basic logic of global games and of their possible applications.

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1 Introduction

Many social, political and economic problems are naturally modeled as a game of incomplete information, where a player's payoff depends on his own action, the actions of others, and **some unknown fundamentals**. For example, many accounts of currency attacks, bank runs, and liquidity crises give a central role to players' uncertainty about other players' actions and on the state of "fundamentals". Similarly, the outcome of many situations of regime changes depends on the decision of the other players, who, in turn, might depend on a common unknown fundamental.

Because other players' actions in such situations are motivated by their beliefs, the decision maker must take account of the beliefs held by other players, and we know from the classic game theory that rational behavior in such environments not only depends on economic agents' beliefs about economic fundamentals, but also depends on beliefs of higher-order – i.e., players' beliefs about other players' beliefs, players' beliefs about other players' beliefs about other players' beliefs, and so on. In principle, rational strategic behavior should be analyzed in the space of all possible infinite hierarchies of beliefs; however, such analysis is highly complex and is likely to prove intractable in general. It is therefore useful to identify strategic environments with incomplete information that are rich enough to capture the important role of higher-order beliefs in economic settings, but simple enough to allow tractable analysis. Global games represent one such environment. Uncertain economic fundamentals are summarized by a state θ and each player observes a different signal of the state with a small amount of noise. Assuming that the noise technology is common knowledge among the players, each player's signal generates beliefs about fundamentals, beliefs about other players' beliefs about fundamentals, and so on.

The global games approach open up other interesting avenues of investigation. One of them is the **importance of private and public information** in contexts where there is an element of coordination between the players. There is plentiful anecdotal evidence from a variety of contexts that public information has an apparently disproportionate impact relative to private information. Financial markets apparently "overreact" to announcements from central bankers that merely state the obvious, or reaffirm widely known policy stances; political behavior such as demonstrations may become suddenly salient because of mere public statement. More generally, many behavioral bubbles appears suddenly following apparently irrelevant public announcements. But a closer look at this phenomenon with the benefit of the insights given by global games makes such instances less mysterious. If participants are concerned about the reaction of other participants to the news, the public nature of the news conveys more information than simply the "face value" of the announcement. It conveys important strategic information on the likely beliefs of other market participants. In this case, the "bubble" would be entirely rational and determined by the type of equilibrium logic inherent in a game of incomplete information.

The purpose of these pages is to describe how such models work and in particular how global game reasoning can be applied to social, political and economic problems. This would allow to disentangle two properties of global games. The first property is that a unique outcome is selected in the game. A second, more subtle, question is how such a unique outcome depends on the underlying information structure and the noise in the players' signals. Although in some cases the outcome is sensitive to the details of the information structure, there are cases where a particular outcome is selected and where this outcome turns out to be robust to the form of the noise in the players' signals. The theory of "robustness to incomplete information" as developed by Kajii and Morris 1997 holds the key to this property.

This note is organized as follows.

2 Coordination Games and the Intuition behind Global Games

Coordination games make up a special but rich class of games, being a subset of the games with strategic complementarity or *supermodular games.*

Example 1 *To introduce this class, let we consider the simplest possible example:*

		2	
		L	R
1	U	0, 0	0, −1
	D	−1, 0	1, 1

Game 1

The interesting characteristics of game 1 are

1. the multiplicity of equilibria;

2. the Pareto ranking of these equilibria, where the Pareto inferior equilibrium is *risk dominant*[1] w.r.t to the Pareto efficient strategy profile;

3. the intuitive role of confidence and expectations as critical elements for determining players' rational behavior;

4. a natural propagation mechanism such that a change in the structural parameters that affect the payoff of one player lead to similar responses in the behavior of all agents, i.e. to positive comovements.

Suppose that it is possible to rank players' strategies such that

$$R > L \quad \text{and} \quad D > U$$

then coordination games display two further properties:

1. *strategic complementarity,* i.e. an "higher" choice by 2 increases the marginal return to "higher" choices by 1;

2. *positive spillovers, i.e.* the payoffs of a player increases as the choice of the other player increases.

The general point behind this trivial example is that complete information games often have multiple Nash equilibria, and game theorists have long been interested in finding good reasons to remove or to reduce that multiplicity. Morris and Shin 2000 have argued that the apparent indeterminacy of beliefs in many models with multiple equilibria, such as in game 1, can be seen as the consequence of two modeling assumptions introduced to simplify the theory. First, the fundamentals are assumed to be common knowledge. Second, agents are assumed to be certain about others' behavior in equilibrium. Both assumptions are made for the sake of tractability, but they do much more besides. They allow agents' actions and beliefs to be perfectly coordinated in a way that invites multiplicity of equilibria.

To remove such multiplicity, Carlsson and van Damme 1993a worked on the idea that because of the intuitive role of confidence and expectations as critical elements for determining players' rational behavior, and because complete information games are actually the limit of incomplete information games, the introduction of a bit of private information might be effective on this multiplicity. As well known, a complete information model entails the implicit assumption that among the players there is

[1] Harsanyi and Selten 1988.

common knowledge of the payoffs of the game. Suppose that, instead of observing payoffs exactly, payoffs are observed with a small amount of continuous noise; and suppose that — before observing their signals of payoffs — there was an ex ante stage where any payoffs were possible. Based on the latter feature, Carlsson and van Damme 1993a dubbed such games '**global games**'. In contrast with the logic of coordination games with common knowledge of payoffs, global games allow theorists to model information in a more realistic way, and thereby escape the straitjacket of perfect coordination of actions and beliefs. It turns out that there is a unique equilibrium in the global game with a small amount of noise. This uniqueness remains no matter how small the noise is and is independent of the distribution of the noise.

As well as any theoretical satisfaction at identifying a unique outcome in a game, there are more substantial issues at stake. Global games allow us to capture the idea that economic agents may be pushed into taking a particular action because of their belief that others are taking such actions. Thus, inefficient outcomes may be forced on the agents by the external circumstances even though they would all be better off if everyone refrained from such actions. We can draw the important distinction between whether there can be inefficient equilibrium outcomes and whether there is a unique outcome in equilibrium. Global games, therefore, are of more than purely theoretical interest. They allow more enlightened debate on substantial social, political and economic questions.

The following example from Carlsson and van Damme 1993b illustrates the basic ideas of global games.

Example 2 *Let consider a parametric class of games, $G(\theta)$, of the following type:*

		2	
		Invest	Not Invest
1	Invest	θ, θ	$\theta - 1, 0$
	Not invest	$0, \theta - 1$	$0, 0$

Game $G(\theta)$

Note that

1. when $\theta < 0$, then (*Not invest*, *Not invest*) is the unique dominant strategy profile;

2. when $\theta > 1$, then $(Invest,\ Invest)$ is the unique dominant strategy profile;

3. when $\theta \in [0,1]$, then there are two Nash equilibria that can be Pareto ranked, such that investing is risk dominant if $\theta \geq \frac{1}{2}$, not investing if $\theta \leq \frac{1}{2}$.

This class of games allows to introduce the class of global games: the basic idea of Carlsson and van Damme 1993 is that to select one equilibrium in the class $G(\theta)$ it is useful to consider an equilibrium of $G(\theta)$ as the limit of an equilibrium of an nearby incomplete information game as the amount of incomplete information on θ goes to zero. In particular suppose that

Hypothesis 1

1. *the players do not exactly observe θ, but a signal*

$$\theta_i = \theta + \sigma\varepsilon_i$$

2. *ε_i are identically and independently normally distributed with mean 0 and standard deviation 1:*

$$\varepsilon_i \sim N(0,1).$$

3. *Suppose for convenience that each player believes that θ is uniformly distributed on the real line (thus there is an 'improper' prior with infinite mass: this does not cause any technical or conceptually difficulties as players will always condition on signals that generate 'proper' posteriors):*

$$\theta \sim U(\mathbb{R}).$$

Remark 1 *The assumption that θ is uniformly distributed on the real line is nonstandard, but presents no technical difficulties. Such "improper priors" (with an infinite mass) are well behaved, as long as we are concerned only with conditional beliefs.*[2] *In particular, it is possible to show that an improper prior can be seen as a limiting case either as the prior distribution of θ becomes diffuse or as the standard deviation of the noise σ becomes small.*

Note that θ_i is informative on the underlying state of nature θ and on the other players' signal θ_{-i} since these signals are correlate with θ.

[2] See Hartigan 1983 for a discussion of improper priors.

Result 1 *These hypothesis imply that*

1.
$$E(\theta|\theta_i) = \theta_i$$

2.
$$\theta|\theta_i \sim N\left(\theta_i, \sigma^2\right)$$

3.
$$\theta_{-i}|\theta_i \sim N\left(\theta_i, 2\sigma^2\right).$$

Now, let elaborate on the idea that the dominant strategy aspect of the game when $\theta < 0$ and $\theta > 1$ will spill over to generate a unique outcome when $\theta \in [0, 1]$ and the signal is very informative.

In this class of games of incomplete information, a pure strategy for player i is a mapping from signals to actions:

$$s_i : \mathbb{R} \to \{Invest,\ Not\ invest\}.$$

Suppose that player i assume that player $-i$ plays a cutoff strategy such that

$$s_{-i}^k(\theta_{-i}) = \begin{cases} Invest & if \ \ \theta_{-i} > k \\ Not\ Invest & if \ \ \theta_{-i} \leq k. \end{cases}$$

Therefore, according to i, player $-i$ will not invest with probability

$$\mathbb{P}(\theta_{-i} \leq k|\theta_i) = \Phi\left(\frac{1}{\sqrt{2}\sigma}(k - \theta_i)\right)$$

where Φ is the cumulative distribution of the standard normal. This implies that $i's$ expected payoff from investing is

$$\theta_i\left[1 - \Phi\left(\frac{1}{\sqrt{2}\sigma}(k - \theta_i)\right)\right] + (\theta_i - 1)\,\Phi\left(\frac{1}{\sqrt{2}\sigma}(k - \theta_i)\right) = \theta_i - \Phi\left(\frac{1}{\sqrt{2}\sigma}(k - \theta_i)\right)$$

so that i best response to $s_{-i}^k(\theta_{-i})$ is

$$s_i^{br}\left(s_{-i}^k(\theta_{-i})\,|\theta_i\right) = \begin{cases} Invest & if \ \ \theta_i > \Phi\left(\frac{1}{\sqrt{2}\sigma}(k - \theta_i)\right) \\ Not\ Invest & if \ \ \theta_i \leq \Phi\left(\frac{1}{\sqrt{2}\sigma}(k - \theta_i)\right). \end{cases}$$

Note that the equation

$$\theta_i - \Phi\left(\frac{1}{\sqrt{2}\sigma}(k - \theta_i)\right) = 0$$

has a unique solution in θ_i since $\Phi\left(\frac{1}{\sqrt{2}\sigma}(k-\theta_i)\right)$ is strictly decreasing in θ_i. Let

$$b(k)$$

be the unique value of θ_i solving the previous equation. Moreover $\Phi\left(\frac{1}{\sqrt{2}\sigma}(k-\theta_i)\right)$ is strictly increasing in k. Then, the best response of player i is to follow a cut-off strategy with threshold equal to $b(k)$, increasing in k :

$$s_i^{br}\left(s_{-i}^k(\theta_{-i})\,|\theta_i\right) = \begin{cases} Invest & if \;\; \theta_i > b(k) \\ Not\ Invest\, if & \theta_i \leq b(k). \end{cases}$$

Note that

1. $$k \to -\infty \Rightarrow \Phi\left(\frac{1}{\sqrt{2}\sigma}(k-\theta_i)\right) \to 0 \Rightarrow b(k) \to 0,$$

   i.e. if player 2 always invest, player 1 will invest if the signal θ_i is positive;

2. $$k \to \infty \Rightarrow \Phi\left(\frac{1}{\sqrt{2}\sigma}(k-\theta_i)\right) \to 1 \Rightarrow b(k) \to 1,$$

   i.e. if player 2 never invest, player 1 will invest if the signal θ_i is greater than 1;

3. $$k = \frac{1}{2} \Rightarrow b(k) = \frac{1}{2}$$

   since when $\theta_i = \frac{1}{2}$ then $E(\theta|\theta_i) = \frac{1}{2}$, which implies that according to i's beliefs, player $-i$ will not invest with probability $\frac{1}{2}$:

4. by total implicit differentiation

   $$b'(k) = \frac{1}{1 + \frac{\sqrt{2\pi}}{\phi\left(\frac{1}{\sqrt{2}\sigma}(k-\theta_i)\right)}} \in (0,1)$$

   where ϕ is the density of the standard normal. This confirms that $b(k)$ is strictly increasing in k, which implies that there is a unique 'threshold' equilibrium where each player uses a threshold of $\frac{1}{2}$.

The function $b(\cdot)$ is plotted in the following figure.[3]

[3]Figure 3.1 in Morris-Shin 2003.

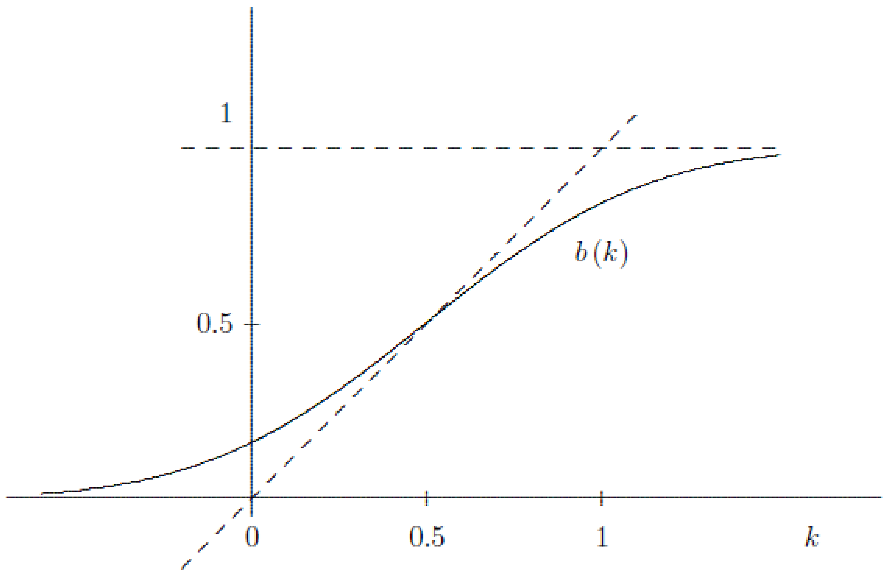


Figure 1: The function $b\left(k\right)$

The unique equilibrium has both players investing if and only if they observe a signal greater than 1/2. Actually, the strategy with threshold $\frac{1}{2}$ is in fact the unique strategy surviving iterated deletion of (interim) strictly dominated strategies, since a strategy s_i survives n rounds of iterated deletion of strict dominated strategies if and only if

$$s_i\left(\theta_i\right) = \begin{cases} Invest \quad if \ \ \theta_i > b^n\left(1\right) \\ Not\ Invest\,if \ \ \theta_i \leq b^n\left(0\right) \end{cases}$$

where

$$b^n\left(k\right) = \overbrace{b\left(b\left(...b\left(k\right)\right)\right)}^{n \ \text{times}}.$$

We argue the second clause by induction (the argument for the first clause is symmetric). The claim is true for $n = 1$, because Invest/Not Invest is a dominant strategy if the expected value of θ is greater than 1/less than 0. Moreover, suppose the claim is true for arbitrary n. If a player knew that his opponent would choose action Not Invest if he had observed a signal less than $b^{n-1}(1)$, his best response would always be to choose action Not Invest if his signal was less than $b(b^{n-1}(1))$. Because $b(\cdot)$ is strictly increasing and has a unique fixed point at 1/2, $b^n(0)$ and $b^n(1)$ both tend to 1/2 as $n \to \infty$.

An alternative argument follows Milgrom and Roberts 1990: if a symmetric game with strategic complementarities has a unique symmetric

Nash equilibrium, then the strategy played in that unique Nash equilibrium is also the unique strategy surviving iterated deletion of strictly dominated strategies.

The key intuition for this argument is that the uniform prior assumption ensures that each player, whatever his signal, attaches probability $\frac{1}{2}$ to his opponent having a higher signal and probability $\frac{1}{2}$ to him having a lower signal. This property remains true no matter how small the noise is, but breaks discontinuously in the limit: when noise is zero, he attaches probability 1 to his opponent having the same signal. The striking feature of this result is that no matter how small ■ is, players' behavior is influenced by the existence of the ex ante possibility that their opponent has a dominant strategy to choose each action. Thus, a "grain of doubt" concerning the opponent's behavior has large consequences. Thus, the probability that either individual invests is

$$\Phi\left(\frac{\frac{1}{2}-\theta}{\sigma}\right)$$

and, conditional on θ, their investment decisions are independent.

3 Two by Two General Global Games

The previous result can be generalized to the class of two-player, two-action games.

Example 3 *Consider a generic 2×2 game:*

1 \ **2**	L	R
U	θ_1, θ_2	θ_3, θ_4
D	θ_5, θ_6	θ_7, θ_8

Game 3

Thus a vector $\theta \in R^8$ describes the payoffs of the game and is drawn from some distribution. For a generic choice of θ, there are three possible configurations of Nash equilibria:

1. a unique Nash equilibrium with both players using strictly mixed strategies;

2. a unique strict Nash equilibrium with both players using pure strategies;

3. two pure strategy strict Nash equilibria and one strictly mixed strategy Nash equilibrium.

In the last case, Harsanyi and Selten 1988 proposed the criterion of risk dominance to select among the multiple Nash equilibria. Suppose that (U, L) and (D, R) are strict Nash equilibria of the above game, which requires

$$\theta_1 > \theta_5, \quad \theta_7 > \theta_3, \quad \theta_2 > \theta_4, \quad \theta_8 > \theta_6.$$

Then (U, L) is a risk dominant equilibrium if

$$(\theta_1 - \theta_5)(\theta_2 > \theta_4) > (\theta_7 - \theta_3)(\theta_8 - \theta_6).$$

Generically, exactly one of the two pure Nash equilibria will be risk dominant.

Now consider the following incomplete information game $G(\sigma)$. Each player i observes a signal

$$\boldsymbol{\theta}_i = \boldsymbol{\theta} + \sigma \boldsymbol{\varepsilon}_i$$

where the $\boldsymbol{\varepsilon}_i$ are eight-dimensional noise terms. Thus we have a class of incomplete information games parameterized by $\sigma > 0$. In the incomplete information game $G(\sigma)$, a strategy for player i is a map of the following type

$$s_i : \mathbb{R}^8 \to A_i, \quad \text{where } A_1 = \{U, D\}, \quad A_2 = \{L, R\}.$$

Then Carlsson and van Damme (1993) study the distribution over action profiles in the game $G(\sigma)$, averaging across signal realizations, for any given strategy profile of players in the game and any actual realization of the payoffs $\boldsymbol{\theta}$:

Theorem 1 *For any sequence of games $G(\sigma^n)$ where $\sigma^n \to 0$ and any sequence of equilibria of those games, the average play converges at almost all payoff realizations to the unique Nash equilibrium (if there is one) and to the risk dominant Nash equilibrium (if there are multiple Nash equilibria).*

Carlsson and van Damme 1993 also generalize the argument from the example described above to show that, if an action is part of a risk dominant equilibrium or a unique strict Nash equilibrium of the complete information game, then - for sufficiently small σ - that action is the unique action surviving iterated deletion of strictly dominated strategies.

The original analysis of Carlsson and van Damme 2003 relaxed the assumption of common knowledge of payoffs in a particular way: they assumed that there was a common prior on payoffs and that each player observes a small conditionally independent signal of payoffs. This is an intuitively small perturbation of the game and this is the perturbation that has been the focus of study in the global games literature. However, when the noise is small one can show that types in the perturbed game are close to common knowledge types in the product topology on the universal type space: that is, for each type t in the perturbed game, there is a common knowledge type t' such that type t and t' almost agree in their beliefs about payoffs, they almost agree about their beliefs about the opponents' beliefs, and so on up to any finite level. Thus the 'discontinuity' in equilibrium outcomes in global games when noise goes to zero is illustrating the same sensitivity to higher order beliefs of the famous example of the electronic mail game by Rubinstein 1989. The natural question to ask is then: how general is the phenomenon that Rubinstein 1989 and Carlsson and van Damme 1993 identified? That is, for which games and actions is it the case that, under common knowledge, the action is **part of an equilibrium** (and thus survives iterated deletion of strictly dominated strategies) but for a type 'close' to common knowledge of that game, that action is the **unique** action surviving iterated deletion of strictly dominated strategies. Weinstein and Yildiz 2007 shows that this is true for every action surviving iterated deletion of strictly dominated strategies in the original game. This observation highlights the following fact:

Remark 2 *the selections that arise in standard global games arise not just because one relaxes common knowledge, but because it is relaxed in a particular way:*

1. *the common prior assumption is maintained and outcomes are analyzed under that common prior,*

2. *the noisy signal technology ensures particular properties of higher-order beliefs, that is, that each player's beliefs about how other players' beliefs differ from his is not too dependent on the level of his beliefs.*

Summary 1 *Carlsson and van Damme 1993 named their perturbed games for the two player, two action case "global games" because all possible payoff profiles were possible. They showed that there was a general way of adding noise to the payoff structure so that, as the noise went to zero,*

1. *there was a unique action surviving iterated deletion of (interim) dominated strategies, a* ***limit uniqueness result***
2. *the action that got played in the limit was independent of the distribution of noise added, a* ***noise independent selection result.***

3.0.1 Comments on Strategic Uncertainty

In global games, the importance of the noisy observation of the underlying state lies in the fact that it generates strategic uncertainty, that is, uncertainty about others' behavior in equilibrium, because of players' uncertainty about other players' payoffs. Thus, understanding global games involves understanding how equilibria depend on players' uncertainty about other players' payoffs. But, clearly, it is not going to be enough to know each player's beliefs about other players' payoffs. We must also take into account each player's beliefs about other players' beliefs about his payoffs, and further such higher-order beliefs. Players' payoffs and higher-order beliefs about payoffs are the true primitives of a game of incomplete information, not the asymmetric information structure. In these introductory examples, we told an asymmetric information story about how there is a true state - the fundamentals - θ drawn from some prior and each player observes a signal of θ generated by some technology. But, our analysis of the resulting game implicitly assumes that there is common knowledge of the prior distribution of θ and of the signaling technologies. It is hard to defend this assumption literally when the original purpose was to get away from the unrealistic assumption that there is common knowledge of the realization of θ. The classic arguments of Harsanyi (1967–1968) and Mertens and Zamir (1985) tell us that we can assume common knowledge of some state space without loss of generality. But such a common knowledge state space makes sense with an incomplete information interpretation (a player's "type" is a description of his higher-order beliefs about payoffs), but not with an asymmetric information interpretation (a player's "type" is a signal drawn according to some ex ante fixed distribution). Thus, the noise structures analyzed in global games are interesting because they represent a tractable way of generating a rich structure of higher-order beliefs. The analysis of global games represents a natural vehicle to illustrate the power of higher-order beliefs at work in applications. But, then, the natural way to understand the "trick" to global games analysis is to go back and understand what is going on in terms of higher-order beliefs.

Even if one is uninterested in the philosophical distinction between incomplete information and asymmetric information, there is a second reason why the higher-order beliefs literature may contribute to our understanding of global games. Even keeping a pure asymmetric information interpretation, we can calculate (from the prior distribution over θ and the signal technologies) the players' higher-order beliefs about payoffs. Statements about higher-order beliefs about payoffs turn out to represent a natural mathematical way of characterizing which properties of the prior distribution and signal technologies matter for the results.

The pedagogical risk of emphasizing higher-order beliefs is that readers may conclude that playing in the uniquely rational way in a global game requires fancy powers of reasoning, some kind of hyperrationality that allows them to reason to an arbitrarily high number of levels. We emphasize that the fact that either the analyst or a player expresses information about the game in terms of higher-order beliefs does not make standard equilibrium concepts any less compelling and does not suggest any particular view about how equilibrium behavior might be arrived at. In particular, recall that there is a very simple heuristic that will generate equilibrium behavior in symmetric binary action games. If there is not common knowledge of the environment you are in, you should hold diffuse beliefs about others' behavior. In particular, if you are on the margin between your two actions, it seems reasonable to take the agnostic view that you are equally likely to hold any rank in the population concerning your evaluation of the desirability of the two actions. Thus, if other people behave like you, you should make your decision on the assumption that the proportion of other players choosing each action is uniformly distributed. This reasoning sound naive, but actually generates a very simple heuristic for behavior that is consistent with the unique rational behavior.

4 Global Games with Many Players and Many Actions

Carlsson and van Damme 1993 results do not extend in general to many player many action games. Thus, in discussing known extensions, we must distinguish which of their results extend. We start with an example.

4.1 The Investment Game with Many Players

Example 4 *Consider a game with the following characteristics:*

1. *a continuum of players* $i \in [0, 1]$
2. *who have to decide whether to invest or not,* $L \in \{Invest, \quad Not \quad Invest\}$

3. *with payoff*

$$U_i(L,l) = \begin{cases} \theta + l - 1 & L = Invest \\ 0 & L = Not \;\; Invest \end{cases}$$

where l is the proportion of other players choosing to invest.

4. *The* ***information structure*** *is standard:*

   (a) *each player i observes a private signal*

   $$\theta_i = \theta + \sigma\varepsilon_i$$

   (b) *where ε_i are identically and independently normally distributed with mean 0 and standard deviation 1*

   $$\varepsilon_i \sim N(0,1)$$

   (c) *and*

   $$\theta \sim U(\mathbb{R}).$$

This game has the following solution.

Result 2 *The unique strategy surviving iterated deletion of strictly dominated strategies is*

$$s_i(\theta_i) = \begin{cases} Invest & if \;\; \theta_i > \frac{1}{2} \\ Not\ Invest & if \;\; \theta_i \leq \frac{1}{2} \end{cases}$$

as for the two player game.

Proof. We will briefly sketch why this is the case. Consider a player i who has observed signal θ_i and thinks that all his opponents are following the threshold strategy with cutoff point k. Since $E(\theta|\theta_i) = \theta_i$, i will assign probability $\Phi\left(\frac{k-\theta_i}{\sqrt{2}\sigma}\right)$ to any given opponent observing a signal less than k, which is also $i's$ expectation of the proportion of players who observe a signal less than k, because the realization of the signals are independent conditional on θ. Thus, $i's$ expected payoff to investing will be

$$\theta_i - \Phi\left(\frac{k-\theta_i}{\sqrt{2}\sigma}\right),$$

as in the previous 2×2 example, and all the previous arguments go through. ■

This argument shows the importance of keeping track of the layers of beliefs across players, and as such may seem rather daunting from the point of view of an individual player. However, the equilibrium outcome is also consistent with a "Laplacian" procedure that places far less demands on the capacity of the players, and that seems to be far removed from equilibrium of any kind.

Algorithm 1 *The **"Laplacian" procedure** has the following three steps:*

1. *Estimate θ from the signal θ_i*

2. *Postulate that l is distributed uniformly on the unit interval* $[0, 1]$

3. *Take the optimal action.*

Proof. Since $E(\theta|\theta_i) = \theta_i$, the expected payoff to investing if l is uniformly distributed is $\theta_i - \frac{1}{2}$, whereas the expected payoff to not investing is zero. Thus, a player following this procedure will choose to invest if and only if $\theta_i > \frac{1}{2}$, which is identical to the unique equilibrium strategy previously outlined. ■

The belief that l is distributed uniformly on the unit interval $[0, 1]$ is Laplacian in the sense that it represents a "diffuse" or sort of logic of "insufficient reason" view on the actions of other players in the game. The previous results shows that an apparently naive and simplistic strategy coincides with the equilibrium strategy. This is not an accident. There are good reasons why the Laplacian action is the correct one in this game, and why it turns out to be an approximately optimal action in many binary action global games. The key to understanding this feature is to consider the following question asked by a player in this game:

Question 1 *"My signal has realization θ_i. What is the probability that proportion less than z of my opponents have a signal higher than mine?"*

Answer 1 *The answer to this question is crucial if everyone is using the cutoff strategy, since the proportion of players who invest is equal to the proportion whose signal is above $E(\theta|\theta_i) = \theta_i$. If the true state is θ, the proportion of players who receive a signal higher than θ_i is given by $1 - \Phi\left(\frac{\theta_i - \theta}{\sigma}\right)$, which is less than z if the state θ is such that*

$$1 - \Phi\left(\frac{\theta_i - \theta}{\sigma}\right) \leq z \Rightarrow \theta \leq \theta_i - \sigma\Phi^{-1}(1 - z),$$

that has probability, conditional on θ_i

$$\Phi\left(\frac{\theta_i - \sigma\Phi^{-1}(1 - z) - \theta_i}{\sigma}\right) = 1 - z.$$

Therefore the density of z is uniform over the unit interval. If θ_i is to serve as the cutoff point of an equilibrium cutoff strategy, a player must be indifferent between choosing to invest and not to invest given that the proportion who invest is uniformly distributed on $[0,1]$. Moreover, even away from the switching point, the optimal action motivated by this belief coincides with the equilibrium action, even though the (Laplacian) belief may not be correct. Away from the cutoff point, the density of the random variable representing the proportion of players who invest will not be uniform. However, as long as the payoff advantage to investing is increasing in θ, the Laplacian action coincides with the equilibrium action. Thus, the naive procedure outlined in the algorithm gives the correct prediction as to what the equilibrium action will be.

In most of this note, we will focus on games with a continuum of players, however as suggested by examples 2 and 4, the qualitative analysis is often very similar irrespective of the number of players.

Result 3 *The analysis of the continuum player game* ***with linear payoffs*** *applies equally well to any finite number of players (where each player observes a signal with an independent normal noise term): independent of the number of players, the cutoff threshold in the unique equilibrium is* $\frac{1}{2}$.

However, let we stress an interesting difference.

Remark 3 *A distinctive implication of the infinite player case is that the outcome is a deterministic function of the realized state. In particular, once we know the realization of* θ, *we can calculate exactly the proportion of players who will invest.*

Result 4 *In the investment game with a continuum of players, in the unique equilibrium with cutoff threshold* $\frac{1}{2}$, *the proportion of players who will invest is exactly*

$$\widehat{\xi}(\theta) = 1 - \Phi\left(\frac{\frac{1}{2} - \theta}{\sigma}\right)$$

which also holds when the finite number of players increase to ∞.

Proof. Consider the investment game with N players. Then, the probability that at least proportion λ out of the N players invest when the realized state is θ is

$$\xi(\theta; \lambda, N) = \sum_{n \geq \lambda N} \binom{N}{n} \left[\Phi\left(\frac{\frac{1}{2} - \theta}{\sigma}\right)\right]^{N-n} \left[1 - \Phi\left(\frac{\frac{1}{2} - \theta}{\sigma}\right)\right]^{n}.$$

Note that

$$\lim_{N\to\infty} \xi(\theta;\lambda,N) = \begin{cases} 1\ if & \lambda < \widehat{\xi}(\theta) \\ 0\, if & \lambda > \widehat{\xi}(\theta)\,. \end{cases}$$

■

5 A General Approach to Symmetric Binary Action Global Games with a Continuum of Players

5.1 Introduction

Before going into the details, let we make some general considerations referring to the literature.

5.1.1 Global Games with Limit Dominance Property or Supermodular Payoffs

Frankel, Morris and Pauzner 2003 consider games with strategic complementarities. Rather than allowing for all possible payoff profiles, they restrict attention to a one-dimensional set of possible payoff functions, or states, which are ordered so that higher states lead to higher actions. The idea of global games is captured by a **'limit dominance' property**: for sufficiently low values of θ, each player has a dominant strategy to choose his lowest action, and for sufficiently high values of θ, each player has a dominant strategy to choose his highest action. Under these restrictions, they are able to present a complete analysis of the case with many players, asymmetric payoffs and many actions. In particular, a **limit uniqueness result** holds: if each player observes the state with noise, and the size of noise goes to zero, then in the limit there is a unique strategy profile surviving iterated deletion of strictly dominated strategies. Note that while Carlsson and van Damme 1993 required no strategic complementarity, when there are multiple equilibria in a two-player, two-action game - the interesting case for Carlsson and van Damme's analysis - there are automatically strategic complementarities. Within this class of games where limit uniqueness holds, Frankel, Morris and Pauzner 2003 also provide sufficient conditions for **'noise independent selection**', i.e. which action gets played in the limit as noise goes to zero does not depend on the shape of the noise. They show that a generalization of the potential maximizing action profile is sufficient for noise independent selection. This sufficient condition encompasses the risk dominant selection in two player binary action games; the selection of the 'Laplacian' action (a best response to a uniform distribution over others' actions) in many player, binary action games. Frankel, Morris and Pauzner 2003 also provide an example of a two-player, four-action, symmetric payoff game where noise independent selection fails: there is a unique limit as the noise goes to zero, but the nature of the limit depends on the exact distribution of the noise, while Carlsson 1989 gave a three- player, two-action example in which noise independent selection failed. Also Corsetti et al. 2004 describe a global games model of currency crises, where there is a continuum of small traders and a single large trader, where the equilibrium selected as noise goes to zero depends on the relative informativeness of the large and small traders' signals, i.e. an application where noise-independent selection fails.

5.2 Uniform Prior and Private Values

In this subsection we deal with the case where there is a uniform prior on the initial state, and each player's signal is a sufficient statistic for how much they care about the state, called by Morris-Shin 2006 the private values case. Under this assumptions, the analysis is especially clean, it is possible to prove a uniqueness result and to characterize the unique equilibrium independent of both the structure and size of the noise in players' signals. In the subsequent subsection, we review how the analysis can be extended to deal with general priors and payoffs that depend on the realized state.

Example 5 *Consider the following class of games, characterized as follows:*

1. *There is a continuum of players,* $i \in [0,1]$;
2. *Each player has to choose an action* $a \in \{0,1\}$;
3. *All players have the same payoff function*

$$u : \{0,1\} \times [0,1] \times \mathbb{R} \to \mathbb{R}$$

where $u(a,l,\theta_i)$ *is* i*'s player's payoff if she chooses action* a*, a proportion* l *of the other players choose action* 1*, and her "private signal" is* θ_i*.*

Remark 4 *Since* i*'s payoff is independent of which of the opponents choose action* 1*, to analyze best responses, it is enough to know the payoff gain from choosing one action rather than the other. Thus, the utility function is parameterized by a function*

$$\pi : [0,1] \times \mathbb{R} \to \mathbb{R} \text{ such that } \pi(l,\theta_i) \equiv u(1,l,\theta_i) - u(0,l,\theta_i).$$

Definition 1 *An action is the Laplacian action if it is a best response to a uniform prior over the opponents' choice of action. Thus,*

1. $a = 1$ *is the Laplacian action at* θ_i *if*

$$\int_0^1 \pi(l, \theta_i) dl = \int_0^1 u(1, l, \theta_i) - \int_0^1 u(0, l, \theta_i) > 0$$

2. $a = 0$ *is the Laplacian action at* θ_i *if*

$$\int_0^1 \pi(l, \theta_i) dl = \int_0^1 u(1, l, \theta_i) - \int_0^1 u(0, l, \theta_i) < 0.$$

Remark 5 *Generically, a game with a continuum of players, symmetric payoff, and two-action will have exactly one Laplacian action.*

Assumption 1 *On* ***players' information****, the assumptions are the following:*

I.1 *a state* $\theta \in \mathbb{R}$ *is drawn according to the improper uniform density on the real line*

$$\theta \sim U(\mathbb{R});$$

I.2 *player* i *observes a private signal*

$$\theta_i = \theta + \sigma \varepsilon_i$$

with $\sigma > 0$;

I.3 ε_i *is a noise distributed on* $\mathbb{R}$ *according to a continuous density* $f(\cdot)$, *possibly non symmetric and with mean different from* 0.

Result 5 *Even if the prior is improper, the conditional density function of* $\theta | \theta_i$ *is well defined and is*

$$\left(\frac{1}{\sigma}\right) f\left(\frac{\theta_i - \theta}{\sigma}\right).$$ [4]

Assumption 2 *The* ***players' payoffs*** *satisfy the following properties*

P.1 ***Action Monotonicity:*** $\pi(l, \theta)$ *is nondecreasing in* l, *i.e. the players actions are strategic complements;*

P.2 ***State Monotonicity:*** $\pi(l, \theta)$ *is nondecreasing in* θ, *i.e. a player's optimal action is increasing in the unknown state;*

[4] See Hartigan 1983.

P.3 ***Strict Laplacian State Monotonicity****: there exists a unique* θ^* *solving* $\int_{l=0}^{1} \pi(l, \theta^*)dl = 0$, *i.e. there is at most one crossing for a player with Laplacian beliefs;*

P.4 ***Limit Dominance****: there exist* $\underline{\theta} \in \mathbb{R}$ *and* $\overline{\theta} \in \mathbb{R}$, *such that*

(a) $\pi(l, \theta_i) < 0$ *for all* $l \in [0,1]$ *and for all* $\theta_i \leq \underline{\theta}$, *i.e. action* $a = 0$ *is a dominant strategy for sufficiently low signals;*

(b) $\pi(l, \theta_i) > 0$ *for all* $l \in [0,1]$ *and for all* $\theta_i \geq \overline{\theta}$, *i.e. action* $a = 1$ *is a dominant strategy for sufficiently high signals;*

P.5 ***Continuity****:* $\int_{l=0}^{1} g(l)\, \pi(l, \theta_i)dl$ *is continuous with respect to* θ_i *and density* g *with respect to the weak topology, therefore the payoff function might be discontinuous at one value of* l.

Definition 2 *Let define the game satisfying these assumptions as* $G^*(\sigma)$.

Then it is possible to prove the following result.

Proposition 1 *In game* $G^*(\sigma)$, *there is essentially a unique iterated strictly undominated strategy profile* $(s_i^*)_{i \in [0,1]}$ *such that*

$$\forall i \in [0,1] \qquad s_i^*(\theta_i) = \begin{cases} 0\, if & \theta_i < \theta^* \\ 1\, if & \theta_i > \theta^* \end{cases} \quad \textit{where} \;\; \theta^* \; \textit{satisfies} \; \int_0^1 \pi(l, \theta^*)\, dl = 0.$$

Proof. The key idea of the proof is that, with a uniform prior on θ, observing θ_i gives no information to a player on her ranking within the population of signals. Thus, she will have a uniform belief over the proportion of players who will observe higher signals. Formally, write $\pi_\sigma^*(\theta_i, k)$ for the expected payoff gain by choosing action 1 for a player who has observed a signal θ_i and knows that all other players will choose action 0 if they observe signals less than k:

$$\pi_\sigma^*(\theta_i, k) = \int_{-\infty}^{\infty} \left(\frac{1}{\sigma}\right) f\left(\frac{\theta_i - \theta}{\sigma}\right) \pi\left(1 - F\left(\frac{k - \theta}{\sigma}\right), \theta_i\right) d\theta.$$

Note that $\pi_\sigma^*(\theta_i, k)$ is continuous in θ_i and k, increasing in θ_i, and decreasing in k, $\pi_\sigma^*(\theta_i, k) < 0$ if $\theta_i < \underline{\theta}$ and $\pi_\sigma^*(\theta_i, k) > 0$ if $\theta_i > \overline{\theta}$. Working by induction, we argue that a strategy survives n rounds of iterated deletion of strictly interim dominated strategies if and only if

$$\forall i \in [0,1] \qquad s_i^*(\theta_i) = \begin{cases} 0\, if & \theta_i < \underline{\xi}_n \\ 1\, if & \theta_i > \overline{\xi}_n \end{cases}$$

where $\underline{\xi}_0 = -\infty$, $\overline{\xi}_0 = \infty$, and $\underline{\xi}_n$, $\overline{\xi}_n$ are defined inductively by

$$\underline{\xi}_{n+1} = \min\left\{\theta_i : \pi^*_\sigma\left(\theta_i, \underline{\xi}_n\right) = 0\right\}$$

$$\overline{\xi}_{n+1} = \max\left\{\theta_i : \pi^*_\sigma\left(\theta_i, \overline{\xi}_n\right) = 0\right\}.$$

Suppose the claim was true for n. By strategic complementarities, if action1 were ever to be a best response to a strategy surviving n rounds, it must be a best response to the cutoff strategy with cutoff $\underline{\xi}_n$ and $\underline{\xi}_{n+1}$ is defined to be the lowest signal where this occurs. Similarly, if action 0 were ever to be a best response to a strategy surviving n rounds, it must be a best response to the cutoff strategy with cutoff $\overline{\xi}_n$ and $\overline{\xi}_{n+1}$ is defined to be the highest signal where this occurs. Note that $\underline{\xi}_n$ and $\overline{\xi}_n$ are, respectively, increasing and decreasing sequences, because $\underline{\xi}_0 = -\infty < \underline{\theta} < \underline{\xi}_1$, $\overline{\xi}_0 = \infty > \overline{\theta} > \overline{\xi}_1$ and $\pi^*_\sigma(\theta_i, k)$ is increasing in θ_i and decreasing in k. Thus,

$$\lim_{n\to\infty} \underline{\xi}_n = \underline{\xi} \text{ and } \lim_{n\to\infty} \overline{\xi}_n = \overline{\xi}.$$

Because of the continuity of π^*_σ,

$$\pi^*_\sigma\left(\underline{\xi}, \underline{\xi}\right) = 0 = \pi^*_\sigma\left(\overline{\xi}, \overline{\xi}\right).$$

Then, we should now show that θ^* is the unique solution to the equation $\pi^*_\sigma(\theta_i, \theta_i) = 0$. To prove it, let denote by $\Psi^*_\sigma(l; \theta_i, k)$ the probability that a player assigns to proportion less than l of the other players observing a signal greater than k, if he has observed signal θ_i. Note that if the true state is θ, the proportion of players observing a signal greater than k is $1 - F\left(\frac{k-\theta}{\sigma}\right)$. This proportion is less than l if $\theta \leq k - \sigma F^{-1}(1-l)$. Therefore

$$\Psi^*_\sigma(l; \theta_i, k) = \int_{-\infty}^{k-\sigma F^{-1}(1-l)} \frac{1}{\sigma} f\left(\frac{\theta_i - \theta}{\sigma}\right) d\theta =$$

$$= \int_{\frac{\theta_i - k}{\sigma} + F^{-1}(1-l)}^{\infty} f(z)\, dz = 1 - F\left(\frac{\theta_i - k}{\sigma} + F^{-1}(1-l)\right)$$

where $z = \frac{\theta_i - \theta}{\sigma}$. Note that if $\theta_i = k$, then $\Psi^*_\sigma(l; \theta_i, k) = l$, i.e. $\Psi^*_\sigma(\cdot; \theta_i, k)$ is the identity function, so it coincides with the cumulative distribution function of the uniform density. Thus,

$$\pi^*_\sigma(\theta_i, \theta_i) = \int_0^1 \pi^*_\sigma(l, \theta_i)\, dl.$$

Then by Strict Laplacian State Monotonicity there exists a unique θ^* solving $\pi^*_\sigma(\theta_i, \theta_i) = \int_{l=0}^1 \pi(l, \theta^*)\, dl = 0$. ■

5.2.1 General Prior and Common Values

In this subsection we deal with the case with general priors and payoffs that depend on the realized state.

Example 6 *Consider the following class of games, characterized as follows:*

1. *There is a continuum of players,* $i \in [0,1]$ *;*
2. *Each player has to choose an action* $a \in \{0,1\}$*;*
3. *All players have the same payoff function*

$$u : \{0,1\} \times [0,1] \times \mathbb{R} \to \mathbb{R}$$

where $u(a,l,\theta)$ *is* i'*s player's payoff if she chooses action* a*, a proportion* l *of the other players choose action* 1*, and the realized state is* θ*.*

Remark 6 *As before,* i'*s payoff is independent of which of his opponents choose action* 1*, thus to analyze best responses, it is enough to know the payoff gain from choosing one action rather than the other. Thus, the utility function is parameterized by a function*

$$\pi : [0,1] \times \mathbb{R} \to \mathbb{R} \text{ such that } \pi(l,\theta) \equiv u(1,l,\theta) - u(0,l,\theta).$$

Assumption 3 *On* ***players' information****, the assumptions are the following:*

*I.1** *a state* $\theta \in \mathbb{R}$ *is drawn according to a continuously differentiable strictly positive density on the real line* $\mathbb{R}$, $p(\mathbb{R})$*:*

$$\theta \sim p(\mathbb{R});$$

I.2 player i *observes a private signal*

$$\theta_i = \theta + \sigma\varepsilon_i$$

with $\sigma > 0$;

I.3 ε_i *is a noise distributed on* $\mathbb{R}$ *according to a continuous density* $f(\cdot)$*, possibly non symmetric and with mean different from* 0*.*

I.4 ***Finite Expectations of Signals****:* $\int_{-\infty}^{\infty} z f(z)\, dz$ *is well defined.*

Remark 7 *Assumption I.4 simply requires that the distribution of noise is integrable, while assumption I.1* is clearly a generalization of previous assumption I.1.*

We must impose two extra technical assumptions on players' payoff functions.

Assumption 4 ***Players' payoffs*** *satisfy the following conditions:*

*P.4** ***Uniform Limit Dominance****: there exist* $\underline{\theta} \in \mathbb{R}$, *a* $\overline{\theta} \in \mathbb{R}$, *and a strictly positive* $\epsilon \in \mathbb{R}_{++}$ *such that*

(a) $\pi(l, \theta) < -\epsilon$ *for all* $l \in [0,1]$ *and for all* $\theta \leq \underline{\theta}$;

(b) $\pi(l, \theta) > \epsilon$ *for all* $l \in [0,1]$ *and for all* $\theta \geq \overline{\theta}$.

Remark 8 *Assumption P.4* strengthens assumption P.4 of Limit Dominance by requiring that the payoff gain to choosing action 0 is uniformly negative for sufficiently low values of* θ, *and the payoff gain to choosing action 1 is uniformly positive for sufficiently high values of* θ.

Definition 3 *Let define the game satisfying assumptions P.1, P.2, P.3, P.4*, P.5 and I.6 as* $G(\sigma)$.

Then it is possible to prove the following result.

Proposition 2 *Let* θ^* *be defined solving* $\int_0^1 \pi(l, \theta^*)\, dl = 0$. *For any* $\delta > 0$, *there exists* $\overline{\sigma} > 0$ *such that for all* $\sigma \geq \overline{\sigma}$, *if strategy* s_i *survives iterated deletion of strictly dominated strategies in the game* $G(\sigma)$, *then*

$$\forall i \in [0,1] \qquad s_i(\theta_i) = \begin{cases} 0 \; if & \theta_i < \theta^* - \delta \\ 1 \; if & \theta_i > \theta^* + \delta. \end{cases}$$

Proof. See Morris and Shin 2006. ■

5.2.2 Comments and Possible Generalizations

Assumptions P.1 and P.2 represent very strong monotonicity assumptions: P.1 requires that each player's utility function is supermodular in the action profile, whereas P.2 requires that each player's utility function is supermodular in his own action and the state. Vives (1990) showed that the supermodularity property P.2 of complete information game payoffs is inherited by the incomplete information game. Thus, the existence of a largest and smallest strategy profile surviving iterated deletion of dominated strategies when payoffs are supermodular, noted by Milgrom and Roberts (1990), can be applied also to the incomplete information game. The state monotonicity assumption P.2 implies, in addition, that the largest and smallest equilibria consist of cutoff strategies. Once we know that we are interested in cutoff strategies, the very weak assumption P.3 is sufficient to ensure the equivalence of the largest and smallest equilibria and thus the uniqueness of equilibrium.

Question 2 *Can one dispense with the full force of the supermodular payoffs assumption P.1?*

Answer 2 *Unfortunately, as long as P.1 is not satisfied at the cutoff point θ^*, i.e., $\pi(l, \theta^*)$ is decreasing in l over some range, then one can find a problematic noise distribution $f(\cdot)$ such that the symmetric cutoff strategy profile with cutoff point θ^* is not an equilibrium, and thus there is no cutoff strategy equilibrium. To obtain positive results, one must either impose additional restrictions on the noise distribution or relax P.1 only away from the cutoff point.*

Global Games without Supermodular Payoffs but satisfying a Single Crossing Property More limited results are available on global games without supermodular payoffs. In many applications - such as bank runs - there are some strategic complementarities but payoffs are not supermodular everywhere: conditional on enough people running on the bank to cause collapse, a player is better off if she run but few people run and share in the liquidation of the bank's assets. An important paper of Goldstein and Pauzner 2005 has shown equilibrium uniqueness for 'bank run payoffs' - satisfying a single crossing property - with uniform prior and uniform noise. This analysis has been followed in a number of applications. They establish that there is a unique equilibrium in threshold strategies and there are no non-threshold equilibria. However, their analysis does not address the question of which strategies survive iterated deletion of strictly dominated strategies.

Morris and Shin 2003 discuss how the existence of a unique threshold equilibrium can be established more generally under a single crossing property on payoffs and a monotone likelihood ratio property on signals (not required for global games analysis with supermodular payoffs); however, these arguments do not rule out the existence of non - monotonic equilibria. Results of van Zandt and Vives 2007 can be used more generally to establish the existence of a unique monotone equilibrium under weaker conditions than supermodularity.

Generalizations and Noise Distribution In this paragraph we follow Athey (2001) and (2002), where she provides a general description of how monotone comparative static results can be preserved in stochastic optimization problems, when supermodular payoff conditions are weakened to single crossing properties, but signals are assumed to be sufficiently well behaved (i.e., satisfy a monotone likelihood ratio property). We follow Morris and Shin (2006) adapting her results to the global games setting, using two new assumptions.

Assumption 5 *P.1* **Action Single Crossing**: For each $\theta \in \mathbb{R}$, there exists $l^* \in [-\infty.\infty]$ such that $\pi(l,\theta) < 0$ if $l < l^*$ and $\pi(l,\theta) < 0$ if $l > l^*$.*

Assumption 6 *I.5 **Monotone Likelihood Ratio Property**: If $\overline{\theta_i} > \underline{\theta_i}$, then $\frac{f\left(\overline{\theta_i}-\theta\right)}{f\left(\underline{\theta_i}-\theta\right)}$ is increasing in θ.*

Definition 4 *Denote by $\widetilde{G}(\sigma)$ the incomplete information game with a uniform prior satisfying P.1*, P.2, P.3, P.4, P.5, and I.5.*

Then it is possible to prove the following result.

Proposition 3 *Let θ^* be defined solving $\int_0^1 \pi(l,\theta^*)\,dl = 0$. Then, the game $\widetilde{G}(\sigma)$, has a unique (symmetric) cutoff strategy equilibrium,such that*

$$\forall i \in [0,1] \qquad s_i(\theta_i) = \begin{cases} 0\, if & \theta_i < \theta^* \\ 1\, if & \theta_i > \theta^*. \end{cases}$$

Proof. See Morris and Shin 2006. ∎

Remark 9 *Notice that this result does not show the nonexistence of other, non-monotonic, equilibria. Additional arguments are required to rule out non-monotonic equilibria.*

Generalizations and Payoff Properties It is also possible to generalize assumption P.1 away from $\theta^{*.}$

Definition 5 *Let $g(\cdot)$ and $h(\cdot)$ be densities on the interval $[0,1]$; then g stochastically dominates h, $g \succeq h$ if $\int_0^l g(z)\,dz \leq \int_0^l h(z)\,dz$ for all $l \in [0,1]$.*

Let $\overline{g}(\cdot)$ be the uniform density on $[0,1]$, so that $\overline{g}(l) = 1$ for all $l \in [0,1]$. Then, consider the following generalization of P.1, P.2 and P.3.

Assumption 7 *P.7 There exists a θ^* which solves $\int_0^1 \pi(l,\theta^*)\,dl = 0$ such that*

(a) $\int_0^1 g(l)\,\pi(l,\theta)\,dl \geq 0$ *for all* $\theta \geq \theta^*$ *and* $g \succeq \overline{g}$, *with strict inequality if* $\theta > \theta^*$;

(b) $\int_0^1 g(l)\,\pi(l,\theta)\,dl \leq 0$ *for all* $\theta \geq \theta^*$ *and* $g \preceq \overline{g}$, *with strict inequality if* $\theta < \theta^*$.

Remark 10 *Note that P.1, P.2 and P.3 imply P.7, but P.7 allows some failure of P.1 away from* θ^*.

Remark 11 *P.7 implies that* $\pi(l,\theta^*)$ *is nondecreasing in* l.

Definition 6 *Let define the game satisfying these assumptions as* $\overline{G}(\sigma)$.

Then it is possible to prove the following results.

Proposition 4 *In game* $\overline{G}(\sigma)$, *there is essentially a unique iterated strictly undominated strategy profile* $(s_i^*)_{i\in[0,1]}$ *such that*

$$\forall i \in [0,1] \qquad s_i^*(\theta_i) = \begin{cases} 0\, if & \theta_i < \theta^* \\ 1\, if & \theta_i > \theta^* \end{cases} \quad \textit{where } \theta^* \textit{ satisfies } \int_0^1 \pi(l,\theta^*)\,dl = 0.$$

Proof. See Morris-Shin 2006. ■

Proposition 5 *Let* θ^* *be defined solving* $\int_0^1 \pi(l,\theta^*)\,dl = 0.$ *. For any* $\delta > 0$, *there exists* $\sigma > 0$ *such that for all* $\sigma \geq \overline{\sigma}$, *if strategy* s_i *survives iterated deletion of strictly dominated strategies in the game* $\overline{G}(\sigma)$, *then*

$$\forall i \in [0,1] \qquad s_i(\theta_i) = \begin{cases} 0\, if & \theta_i < \theta^* - \delta \\ 1\, if & \theta_i > \theta^* + \delta. \end{cases}$$

Proof. See Morris and Shin 2006. ■

6 Inefficiency of Equilibrium Outcomes in Global Games

Previous results deliver strong negative conclusions about the efficiency of equilibrium outcomes in global games.

Result 6 *In general, in global games the equilibrium outcomes are not efficient.*

Proof. In the limit, in equilibrium all players will be choosing action 1 when the state is θ if

$$\int_0^1 \pi(l,\theta)\,dl > 0.$$

On the other hand, efficiency requires to choose action 1 at state θ if

$$u(1,1,\theta) > u(0,0,\theta)$$

and these conditions will not coincide in general. For example, in the investment example, we had

$$u(1,l,\theta) = \theta + l - 1 \text{ and } u(0,l,\theta) = 0$$

and thus

$$\pi(l,\theta) = u(1,l,\theta) - u(0,l,\theta) = \theta + l - 1$$

that implies

$$\int_0^1 \pi(l,\theta)\, dl = \int_0^1 (\theta + l - 1) = \theta - \frac{1}{2}.$$

So in the limiting equilibrium

$$\int_0^1 \pi(l,\theta)\, dl > 0 \Leftrightarrow \theta > \frac{1}{2},$$

i.e. both players will be investing if the state θ is at least $\frac{1}{2}$, although it is efficient for them to be investing if the state is at least 0. ■

The analysis of the unique noncooperative equilibrium serves as a benchmark describing what will happen in the absence of other considerations. In practice, repeated play or other institutions will often allow players to do better.

Consider briefly what happens in the game if players were allowed to make cheap talk statements about the signals that they have observed in the investment example. The arguments here follow Baliga and Morris (2000).

For this exercise, it is most natural to consider a finite player case, thus let consider the two-player investment example. The investment example as formulated has a nongeneric feature, which is that if a player plans not to invest, he is exactly indifferent about which action his opponent will take. To make the problem more interesting, let us perturb the payoffs to remove this tie.

Example 7 *Let consider a parametric class of* 2×2 *games,* $G(\theta, \delta)$, *of the following type:*

		2 Invest	Not Invest
1	Invest	$\theta+\delta,\ \theta+\delta$	$\theta-1,\ \delta$
	Not invest	$\delta,\ \theta-1$	$0,\ 0$

Game $G(\theta)$

Thus, each player receives a small payoff δ (which may be positive or negative) if the other player invests, independent of his own action. This change does not influence each player's best responses, and the analysis of this game in the absence of cheap talk is unchanged by the payoff change. But, observe that if $\delta \leq 0$, there is an equilibrium of the game with cheap talk, where each player truthfully announces his signal, and invests if the (common) expectation of θ conditional on both announcements is greater than $-\delta$, and this gives the efficient outcome. On the other hand, if $\delta > 0$, then each player would like to convince the other to invest even if he does not plan to do so. In this case, there cannot be a truth-telling equilibrium where the efficient equilibrium is achieved, although there may be equilibria with some partially revealing cheap talk that improves on the no cheap talk outcome.

7 Public and Private Signals

To understand the effects of public signals for global games, consider the Investment Game with a continuum of players previously analyzed with private information only.

Example 8 *Consider a game with the following characteristics:*

1. *a continuum of players* $i \in [0, 1]$
2. *who have to decide whether to invest or not,* $L \in \{Invest, \quad Not \quad Invest\}$
3. *with payoff*

$$U_i(L, l) = \begin{cases} \theta + l - 1 & L = Invest \\ 0 & L = Not \quad Invest \end{cases}$$

   where l *is the proportion of other players choosing to invest.*

4. *The* ***information structure*** *is the following:*

   (a) *each player* i *observes a private signal*

   $$\theta_i = \theta + \sigma \varepsilon_i$$

   (b) *where* ε_i *are identically and independently normally distributed with mean 0 and standard deviation 1*

   $$\varepsilon_i \sim N(0, 1)$$

(c)

$$\theta \sim N(y, \tau)$$

where

(d) *y is a public signal.*

From standard statistics, we get the following result.

Result 7

$$E(\theta|\theta_i) = \frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2}$$

Consider a cutoff strategy

$$s(E(\theta|\theta_i)) = \begin{cases} \text{\textit{Invest}} & \text{if } E(\theta|\theta_i) > \kappa \\ \text{\textit{Not Invest}} & \text{if } E(\theta|\theta_i) \leq \kappa. \end{cases}$$

Definition 7 *Let define*

$$\widetilde{\gamma}(\sigma, \tau) = \frac{\sigma^2}{\tau^4}\left(\frac{\sigma^2 + \tau^2}{\sigma^2 + 2\tau^2}\right).$$

Then, Morris and Shin (2006) prove the following result.

Proposition 6 *The game has a symmetric switching strategy equilibrium with cutoff κ if κ solves the equation*

$$\kappa = \Phi\left(\sqrt{\widetilde{\gamma}}(\kappa - y)\right);$$

then

1. *if $\widetilde{\gamma}(\sigma, \tau) \leq 2\pi$, there is a unique value of κ solving the previous equation and the strategy with cutoff κ is the essentially unique strategy surviving iterated deletion of strictly dominated strategies;*

2. *if $\widetilde{\gamma}(\sigma, \tau) > 2\pi$,, then (for some values of y) there are multiple values of κ solving the previous equation and multiple symmetric cutoff strategy equilibria.*

Proof. See Morris and Shin (2006) for a full proof, here we just sketch the intuition. From standard statistics, we know that

$$\theta|\theta_i \sim N\left(\frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2}, \sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}}\right).$$

Define

$$\overline{\theta} := E(\theta|\theta_i) = \frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2}.$$

Moreover, any other player's signal, θ_j', is distributed as follows

$$\theta_j' \sim N\left(\overline{\theta}, \sqrt{\frac{2\sigma^2\tau^2 + \sigma^4}{\sigma^2 + \tau^2}}\right)$$

since

$$\theta_j' = \theta + \sigma\varepsilon.$$

Suppose that all other players follow a cutoff strategy such that

$$s\left(E\left(\theta|\theta_j'\right)\right) = \begin{cases} Invest & \text{if } E\left(\theta|\theta_j'\right) > \kappa \\ Not \;\; Invest & \text{if } E\left(\theta|\theta_j'\right) \leq \kappa \end{cases} \Rightarrow$$

$$\Rightarrow s\left(\theta_j'\right) = \begin{cases} Invest & \text{if } \frac{\sigma^2 y + \tau^2 \theta_j'}{\sigma^2 + \tau^2} > \kappa \\ Not \;\; Invest & \text{if } \frac{\sigma^2 y + \tau^2 \theta_j'}{\sigma^2 + \tau^2} \leq \kappa. \end{cases}$$

Thus, player i assigns probability

$$1 - \Phi\left(\frac{\kappa - \overline{\theta} + \frac{\sigma^2}{\tau^2}(\kappa - y)}{\sqrt{\frac{2\sigma^2\tau^2 + \sigma^4}{\sigma^2 + \tau^2}}}\right)$$

to j investing. Then, $i's$ expectation of the proportion of opponents investing must be equal to the probability he assigns to any one investing, so that $i's$ expected payoff is

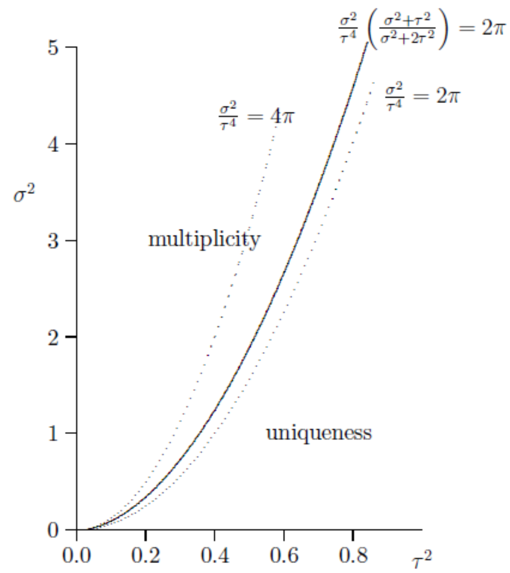
$$v\left(\overline{\theta}, \kappa\right) = \begin{cases} E(\theta + l - 1) & \text{if Invest} \\ 0 & \text{if Not Invest} \end{cases} = \begin{cases} \overline{\theta} - \Phi\left(\frac{\kappa - \overline{\theta} + \frac{\sigma^2}{\tau^2}(\kappa - y)}{\sqrt{\frac{2\sigma^2\tau^2 + \sigma^4}{\sigma^2 + \tau^2}}}\right) & \text{if Invest} \\ 0 & \text{if Not Invest.} \end{cases}$$

Note that $v\left(\overline{\theta}, \kappa\right)$ is increasing in $\overline{\theta}$, therefore there exists a symmetric equilibrium with cutoff at κ if

$$v^*(\kappa) := v(\kappa, \kappa) = 0$$

i.e. if

$$v^*(\kappa) = \kappa - \Phi\left(\frac{\frac{\sigma^2}{\tau^2}(\kappa - y)}{\sqrt{\frac{2\sigma^2\tau^2 + \sigma^4}{\sigma^2 + \tau^2}}}\right) = \kappa - \Phi\left(\sqrt{\gamma}(\kappa - y)\right) = 0.$$



Note that the solution to this equation depends on the values of γ, which is large if $\sigma \gg \tau$, i.e. if public information is more informative than private information, small if $\tau \gg \sigma$, i.e. if public information is less informative than private information. In particular

$$\frac{dv^*(\kappa)}{d\kappa} = 1 - \sqrt{\gamma}\phi\left(\sqrt{\gamma}(\kappa - y)\right).$$

Since ϕ, being the density of the standard normal distribution, has the maximum for $\theta_i = 0$, at value $\frac{1}{\sqrt{2\pi}}$, then

$$\gamma \leq 2\pi \Rightarrow \frac{dv^*(\kappa)}{d\kappa} = 1 - \sqrt{\gamma}\phi\left(\sqrt{\gamma}(\kappa - y)\right) \geq 0$$

so that the equation

$$v^*(\kappa) = 0$$

has a unique solution for $\kappa = y$. on the other hand

$$\gamma > 2\pi \Rightarrow \left.\frac{dv^*(\kappa)}{d\kappa}\right|_{\kappa=y} < 0$$

so that the equation

$$v^*(\kappa) = 0$$

has two other solutions. ■

The following picture plots the two regions in the space (τ^2, σ^2) :

Suppose $\widetilde{\gamma}(\sigma,\tau) \leq 2\pi$, then

$$\kappa = \Phi\left(\sqrt{\widetilde{\gamma}}(\kappa - y)\right)$$

has a unique solution and can be solved for y :

$$y = \kappa - \frac{1}{\sqrt{\widetilde{\gamma}}}\Phi^{-1}(\kappa),$$

which is decreasing in κ since

$$\frac{dy}{d\kappa} = 1 - \frac{1}{\sqrt{\widetilde{\gamma}}}\frac{1}{\phi\left(\Phi^{-1}(\kappa)\right)} \leq 0 \Leftrightarrow \frac{1}{\phi\left(\Phi^{-1}(\kappa)\right)} \geq \sqrt{\widetilde{\gamma}}$$

which is always satisfied when $\widetilde{\gamma} \leq 2\pi$. In particular, the smaller $\widetilde{\gamma}(\sigma,\tau)$, the steeper the function, which is intuitive since $\widetilde{\gamma}$ is directly related to the informativeness of the public versus the private signal. Since player i would invest if $\overline{\theta} \geq \kappa$, then we can use the previous map substituting $\overline{\theta}$ for κ, getting players' investment if and only if

$$y \geq \overline{\theta} - \frac{1}{\sqrt{\widetilde{\gamma}}}\Phi^{-1}\left(\overline{\theta}\right),$$

which has the following interesting implications.

Corollary 1 *Suppose* $\widetilde{\gamma}(\sigma,\tau) \leq 2\pi$, *then*

1. *if* $E(\theta|\theta_i) < 0$, *then in equilibrium for any* y *it is optimal not to invest;*

2. *if* $E(\theta|\theta_i) > 1$, *then in equilibrium for any* y *it is optimal to invest;*

3. *if* $E(\theta|\theta_i) \in [0,1]$, *then in equilibrium the higher* y, *the more likely it is optimal to invest. Thus, the players will always invest for sufficiently high* y, *and not invest for sufficiently low* y. *This implies in particular that changing* y *has a larger impact on a player's action than changing his private signal (controlling for the informativeness of the signals), the* ***"publicity" effect****.*

7.1 The Role of Public and Private Information

To explore the strategic impact of public information, we examine how much a player's private signal must adjust to compensate for a given change in the public signal. Consider the cutoff equation

$$\kappa = \Phi\left(\sqrt{\widetilde{\gamma}}(\kappa - y)\right)$$

when

$$\kappa = \overline{\theta} = \frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2},$$

i.e.

$$\frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2} = \Phi\left(\sqrt{\widetilde{\gamma}}\left(\frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2} - y\right)\right).$$

Considering this equation as an implicit function $\theta_i(y)$, we have

$$\frac{d\theta_i(y)}{dy} = -\frac{\frac{\sigma^2}{\tau^2} + \sqrt{\widetilde{\gamma}}\phi(\cdot)}{1 - \sqrt{\widetilde{\gamma}}\phi(\cdot)}$$

which measures how much the private signal would have to change to compensate for a change in the public signal leaving the player indifferent between investing or not investing. On the other hand, if we ignore strategic effect of a change in y, the private signal has a different substitution ratio that can be derived maintaining constant

$$E(\theta|\theta_i) = \overline{\theta} = \frac{\sigma^2 y + \tau^2 \theta_i}{\sigma^2 + \tau^2}$$

so that

$$\frac{d\theta_i}{dy} = -\frac{\sigma^2}{\tau^2}.$$

The ratio between these two substitution ratio defines the **publicity multiplier**

$$\zeta = \frac{1 + \frac{\tau^2}{\sigma^2}\sqrt{\widetilde{\gamma}}\phi(\cdot)}{1 - \sqrt{\widetilde{\gamma}}\phi(\cdot)}$$

which is increasing in $\widetilde{\gamma}$, which is intuitive since $\widetilde{\gamma}$ is directly related to the informativeness of the public versus the private signal. However, remember that $\widetilde{\gamma}$ is bounded above by 2π, otherwise we go into the regions of multiple equilibria.

There is plentiful anecdotal evidence that in settings where coordination is important, public signals play a role in coordinating outcomes that exceed the information content of those announcements. For example, financial markets apparently "overreact" to announcements from the Federal Reserve Board and public announcements in general. If market participants are concerned about the reaction of other participants to the news, the "overreaction" may be rational and determined by the type of equilibrium logic of our example. Further evidence for this is briefings on market conditions by key players in financial markets using conference calls with hundreds of participants. Such public briefings have a larger impact on the market than bilateral briefings with the same information, because they automatically convey to participants not only information about market conditions, but also valuable information about the beliefs of the other participants.

8 Dynamic Global Games

8.1 Recurring Random Matching

This approach has been developed in Burdzy, Frankel, and Pauzner (2001) and Frankel and Pauzner (1999), then applied in Frankel and Pauzner (2000) and Levin (2009).

A continuum of players are periodically randomly matched in a two-player, two-action game, for simplicity suppose it is the game of example 2.

Example 9 *Consider a parametric class of games, $G(\theta)$, of the following type:*

		2	
		Invest	Not Invest
1	Invest	θ, θ	$\theta - 1$, 0
	Not invest	0, $\theta - 1$	0, 0

Game $G(\theta)$

In particular suppose that

Hypothesis 2 *1. the publicly observed common payoff parameter evolves through time according to some random process;* ■

2. each player can only occasionally alter his behavior: revision opportunities arrive according to a Poisson process and arrive slowly relative to changes in the game's payoffs.

Result 8 *Under certain conditions on the noise process (roughly equivalent to the sufficiently uniform prior conditions in global games), there is a unique equilibrium where each player invests when $\theta \geq \frac{1}{2}$and not when $\theta \leq \frac{1}{2}$.*

8.2 Overlapping Generation Global Games

Levin (2001) describes another approach:

1. At discrete time t, player t chooses an action;
2. t payoff may depend on the actions of players choosing before or after t, but also depends on a payoff parameter θ,
3. θ is publicly observed and evolves according to a random walk.

Result 9 *If players act as if they cannot influence or do not care about the action of the decision maker in the next period, then under weak monotonicity conditions (a player's best response is increasing in others' actions and the payoff parameter) and limit dominance conditions (the highest (lowest) action is a dominant strategy for sufficiently high (low) values of* θ*), there is a unique equilibrium.*

The no influence assumption makes sense if there are in fact a continuum of players at each date or if actions are observed only with a sufficiently long lag.

8.3 Recurring Incomplete Information

Consider the following example, with infinitely many periods

$$t \in \mathbb{N}.$$

Example 10 *Consider a game with the following characteristics:*

1. *a continuum of players* $i \in [0, 1]$
2. *who have to decide whether to invest or not,* $L \in \{Invest, \quad Not \quad Invest\}$
3. *with payoff*

$$U_i(L, l) = \begin{cases} \theta + l - 1 & L = Invest \\ 0 & L = Not \quad Invest \end{cases}$$

where l *is the proportion of other players choosing to invest.*

4. *The **information structure** is the following:*

   (a) θ_t *follow a random walk,such that*

$$\theta_t = \theta_{t-1} + \eta_t$$

   where

(b) η_t is independently and normally distributed

$$\eta_t \sim N(0, \tau)$$

(c) in each period t, the players observes

i. a public signal θ_{t-1}, and

ii. a private signal

$$\theta_{it} = \theta_t + \varepsilon_{it}$$

where

iii. ε_{it} is independently and normally distributed

$$\varepsilon_{it} \sim N(0, \sigma).$$

This dynamic game represents a crude way of embedding the static global games analysis in a dynamic setting. In particular, each period's play of this dynamic game can be analyzed independently and is exactly equivalent to the public signals model examined previously, where

- θ_{t-1} is the public signal
- θ_{it} is player i 's private signal.

8.4 Herding Models

In the herding models of Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992), players sequentially make some discrete choice. Players do not care about each other's actions directly, but they have private information, and so each player may partially learn the information of players who choose before him. But, if a number of early-moving players happen to observe signals favoring one action, late-moving players may start ignoring their own private information, leading to inefficient herding because of the negative informational externality.

Herding models share with global game models the feature that outcomes are highly sensitive to fine details of the information structure. However, it is important to note that the mechanisms are quite different. The global games analysis is driven by strategic complementarities and the highly correlated signals generated by the noisy observations technology. However, sensitivity to the information structure arises in a purely static setting. The herding stories have no payoff complementarities and simple information structures, but rely on sequential choice.

Dasgupta (2001) analyzes a simple model where it is possible to see both kinds of effects at work. A finite set of players decide sequentially (in an exogenous order) whether to invest or not. Investment conditions are either bad (when each player has a dominant strategy to not invest) or good (in which case it pays to invest if all other players invest). Each player observes a signal from a continuum, with high signals implying a higher probability that investment conditions are good. All equilibria in this model are cutoff equilibria: each player invests only if all previous players invested and his private signal exceeds some cutoff. Such equilibria encompass herding effects: previous players' decisions to invest convey positive information to later players and make it more likely that they will invest. They also encompass higher-order belief effects: an increase in a player's signal makes it more likely that he will invest both because he thinks it more likely that investment conditions are good and because he thinks it more likely that later players will observe high signals and choose to invest.

8.5 Public Signals and Dynamic Games

Complete information models are often used in applied economic analysis for tractability, assuming that the assumption of common knowledge of game payoffs capture the essence of the social, political or economic problem. Presumably there is not in fact common knowledge of payoffs, and if asymmetries of information are not the focus of the economic analysis, this assumption may seem harmless. However, complete information games often have multiple equilibria, and policy analysis - and comparative statics more generally - are hard to carry out in multiple equilibrium models.

The global games analysis has highlighted how natural relaxations of the common knowledge assumptions often lead to intuitive selections of a unique equilibrium. This suggests these ideas might be useful in applications. Fukao 1994 and Morris and Shin 1995 were two early papers that pursued this agenda. The latter paper - published as Morris and Shin 1998 - was an application to currency crises, where the existing literature builds on a dichotomy between 'fundamentals-driven' models and multiple equilibrium or 'sunspot' equilibria views of currency crises. This dichotomy does not make sense in a global games model of currency crises: currency attacks are 'self- fulfilling' in the sense that speculators are attacking only because they expect others to do so, but their expectations of others' behavior may nonetheless be pinned down by higher order beliefs (see Heinemann, 2000, for an important correction of the equilibrium characterization in Morris and Shin, 1998).

Morris and Shin 2000 laid out the methodological case for using global games as a framework for economic applications. Morris and Shin 2003 surveys many early applications to currency crises, bank runs, the design of international institutions and asset pricing, and there have been many more since. The following subsection will highlight two important methodological issues - public signals and dynamics - that have played an important role in the developing applied literature, before going to further applications of global games theory.

9 Applications

Let us turn to some applications that make specific assumptions about the distribution of payoffs and signals. But, if one is interested only in analyzing the limiting behavior as noise about θ becomes small, the results of the previous section imply that we can identify the limiting behavior independently of the prior beliefs and the shape of the noise.

9.1 Pricing Debt

This application refers to Morris and Shin (2004). Consider the following simple model.

1. There are two periods: in period 1, a continuum of investors hold collateralized debt that will pay
   - 1 in period 2 if it is rolled over and if an underlying investment project is successful;
   - 0 in period 2 if it is rolled over and the project is not successful;
   - $\kappa \in (0, 1)$, the value of the collateral, if an investor does not roll over his debt.
2. The success of the project depends on
   - the proportion l of investors who do not roll over and
   - the state of the economy, θ, which is distributed according to a continuum density $p(\cdot)$

   Specifically, the project is successful if the proportion of investors not rolling over is less than θ/z.
3. Write $a = 1$ for the action "roll over" and $a = 0$ for the action "do not roll over", then the payoffs can be written as follows:

$$u(a, l, \theta) = \begin{cases} 1 \text{ if } & a = 1 \text{ and } \frac{\theta}{z} \geq 1 - l \\ 0 \text{ if } & a = 1 \text{ and } \frac{\theta}{z} < 1 - l \\ \kappa & \text{if } \ a = 0 \end{cases}$$

   or alternatively

		Percentage of investors rolling over	
		$1-l\leq\frac{\theta}{z}$	$1-l>\frac{\theta}{z}$
$i\in[0,1]$	$a=1$	1	0
	$a=0$	κ	κ

Payoff structure

Hence,

$$\pi(l,\theta)=u(1,l,\theta)-u(0,l,\theta)=\begin{cases}1-\kappa\,\text{if} & \frac{\theta}{z}\geq 1-l\\ -\kappa\ \text{if} & \frac{\theta}{z}<1-l\end{cases}$$

so that

$$\int_0^1\pi(l,\theta)\,dl=\begin{cases}-\kappa & \text{if }\ \theta\leq 0\\ \frac{\theta}{z}-\kappa & \text{if }\ 0\leq\theta\leq z\\ 1-\kappa & \text{if }\ \theta\geq z.\end{cases}$$

Remark 12 *The game representing the model satisfies assumptions P.1* and P.2, and therefore Proposition 3 holds.*

Thus, we can state the following result.

Result 10 *Let $\theta^*=z\kappa$, then the game has a unique (symmetric) cutoff strategy equilibrium, such that*

$$\forall i\in[0,1]\qquad s_i(\theta_i)=\begin{cases}0\ if & \theta_i\leq\theta^*\\ 1\ if & \theta_i>\theta^*.\end{cases}$$

Remark 13 *In other words, if private information about θ among the investors is sufficiently accurate, the project will collapse exactly if*

$$\theta\leq z\kappa.$$

Question 3 *We can now ask how debt would be priced ex ante in this model, i.e. before anyone observed private signals about θ.*

Recalling that $p(\cdot)$ is the density of the prior on θ, and writing $P(\cdot)$ for the corresponding cdf, the value of the collateralized debt will be

$$V(\kappa)\equiv\kappa P(z\kappa)+1-P(z\kappa)=1-(1-\kappa)P(z\kappa)$$

that implies

$$\frac{dV(\kappa)}{d\kappa}=P(z\kappa)-z(1-\kappa)p(z\kappa).$$

Thus, increasing the value of collateral has two effects:

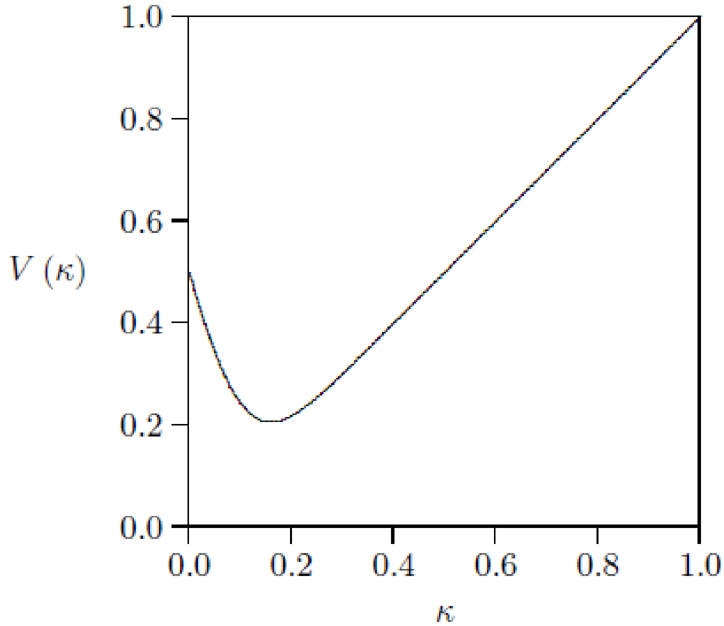


Figure 2: The function $V(\kappa)$.

1. it increases the value of debt in the event of default (the direct effect);
2. it increases the range of θ at which default occurs (the strategic effect).

For small κ, the strategic effect outweighs the direct effect, whereas for large κ, the direct effect outweighs the strategic effect. The following figure plots $V(\kappa)$ for the case where $z = 10$ and $p(\cdot)$ is the standard normal density.

9.2 Currency Crises

This application refers to Morris and Shin (1998). Consider the following simple model.

1. There is a continuum of speculators must decide whether to attack a fixed–exchange rate regime by selling the currency short.
2. Each speculator may short only a unit amount.

3. There is a fixed transaction cost t of attacking, that can be interpreted as an actual transaction cost or as the interest rate differential between currencies.

4. The current value of the currency is e^*;

5. the monetary authority may defend or not the currency

   (a) if the monetary authority does not defend the currency, the currency will float to the shadow rate $\zeta(\theta)$, where θ is the state of fundamentals, so that $\zeta(\theta)$ is increasing in θ. Assume $\zeta(\theta) \leq e^* - t$ for all θ;

   (b) if the monetary authority does defend the currency, its value remains at e^*;

   (c) The monetary authority defends the currency if the cost of doing so is not too large, where the costs of defending the currency are increasing in the proportion of speculators who attack and decreasing in the state of fundamentals:

   (d) Hence, there will be a critical proportion of speculators, $b(\theta)$, increasing in θ, who must attack in order for a devaluation to occur.

6. Write $a = 1$ for the action "not attack" and $a = 0$ for the action "attack", then the payoffs can be written as follows:

$$u(a, l, \theta) = \begin{cases} 0 & \text{if } \ a = 1 \\ e^* - \zeta(\theta) - t & \text{if } \ a = 0 \text{ and } 1 - l \geq b(\theta) \\ -t & \text{if } \ a = 0 \text{ and } 1 - l < b(\theta) \end{cases}$$

   or alternatively

		Percentage of attacking speculators	
		$1 - l < b(\theta)$	$1 - l \geq b(\theta)$
$i \in [0, 1]$	$a = 1$	0	0
	$a = 0$	$-t$	$e^* - \zeta(\theta) - t$

Payoff structure

Hence,

$$\pi(l, \theta) = u(1, l, \theta) - u(0, l, \theta) = \begin{cases} \zeta(\theta) + t - e^* & \text{if } \ l \leq 1 - b(\theta) \\ t & \text{if } \ l \leq 1 - b(\theta). \end{cases}$$

Result 11 *Suppose θ is common knowledge, then*

1. *if $\theta < b^{-1}(0)$, then there is unique equilibrium in dominant strategies, $a^* = 0$ for all $i \in [0,1]$;*

2. *if $b^{-1}(0) \leq \theta \leq b^{-1}(1)$, then there two equilibria such that $a^* = 0$ for all $i \in [0,1]$ and $a^{**} = 1$ for all $i \in [0,1]$;*

3. *if $\theta > b^{-1}(1)$, then there is unique equilibrium in dominant strategies, $a^* = 1$ for all $i \in [0,1]$.*

On the other hand, if θ is observed with noise, we can apply the previous results, because the previous assumptions are satisfied.

Remark 14 *Note that the game representing the model satisfies assumptions P.1 and P.2, and therefore Proposition 2 holds.*

In particular

$$\int_0^1 \pi(l,\theta)\,dl = [1 - b(\theta)]\,[\zeta(\theta) + t - e^*] + b(\theta)\,t$$

which implies

$$\int_0^1 \pi(l,\theta^*)\,dl = 0 \Leftrightarrow [1 - b(\theta^*)]\,[\zeta(\theta^*) - e^*] = t.$$

Thus, we can state the following result.

Result 12 *The game representing our model with private information on θ has a unique (symmetric) cutoff strategy equilibrium, such that*

$$\forall i \in [0,1] \qquad s_i(\theta_i) = \begin{cases} 0\; if & \theta_i \leq \theta^* \\ 1\; if & \theta_i > \theta^*. \end{cases}$$

9.3 Bank Runs

Consider the following simple model by Goldstein and Pauzner (2005), who add noise to the classic bank runs model of Diamond and Dybvig (1983).

1. There are two periods, 1 and 2;

2. There is a continuum of depositors $i \in [0,1]$ (with total deposits normalized to 1);

3. Each depositor must decide whether to withdraw their money from a bank at period 1, denoted $a = 0$, or at period 2, denoted by $a = 1$;

4. The withdrawn resources are entirely used for consumption that gives utility $U(\cdot)$;

5. A proportion λ of depositors will have consumption needs only in period 1 and will thus have a dominant strategy to withdraw, thus we are concerned with the game among the proportion $1 - \lambda$ of depositors. Let denote by l the proportion of late consumers who do not withdraw in period 1;

6. The monetary payoffs are:

    - $r > 1$ if the depositors withdraw their money in period 1 and there are enough resources;
    - $\frac{1-\lambda r}{(1-l)(1-\lambda)}$ if there are not enough resources to fund all those who try to withdraw, i.e. the remaining cash $1-\lambda r$ is divided equally among early withdrawers. This happens when

    $$\lambda r + (1-l)(1-\lambda) r \geq 1 \Leftrightarrow l \leq \frac{r-1}{(1-\lambda) r};$$

    - $\max\{0, 1 - \lambda r + (1-l)(1-\lambda) r\} R(\theta) \geq 0$ in period 2 for those who chose to wait until period 2 to withdraw their money, i.e. any remaining money after period 1 withdraws, $\max\{0, 1 - \lambda r + (1-l)(1-\lambda) r\}$, earns a total return $R(\theta) > 0$ in period 2, which is increasing in θ, and it is divided equally among those who chose to wait until period 2 to withdraw their money, $l(1-\lambda)$.

Then the consumers monetary payoffs can be written as follows:

		Percentage of late consumers	
		$l \leq \frac{r-1}{(1-\lambda)r}$	$l \geq \frac{r-1}{(1-\lambda)r}$
$i \in [0,1]$	$a = 0$	$\frac{1-\lambda r}{(1-l)(1-\lambda)}$	r
	$a = 1$	0	$\left[\frac{1-\lambda r+(1-l)(1-\lambda)r}{l(1-\lambda)}\right] R(\theta)$

Monetary payoff structure

Thus, the utilities of late consumers are

$$u(a,l,\theta)=\begin{cases} U\left(\frac{1}{1-l(1-\lambda)}\right) & \text{if } \ a=0 \text{ and } l\leq\frac{r-1}{(1-\lambda)r} \\ U(r) & \text{if } \ a=0 \text{ and } l\geq\frac{r-1}{(1-\lambda)r} \\ U(0) & \text{if } \ a=1 \text{ and } l\leq\frac{r-1}{(1-\lambda)r} \\ U\left(\left[\frac{1-\lambda r+(1-l)(1-\lambda)r}{l(1-\lambda)}\right]R(\theta)\right) & \text{if } \ a=0 \text{ and } l\geq\frac{r-1}{(1-\lambda)r} \end{cases}$$

Hence,

$$\pi(l,\theta)=u(1,l,\theta)-u(0,l,\theta)=\begin{cases} U(0)-U\left(\frac{1}{1-l(1-\lambda)}\right) & \text{if } \ l\leq\frac{r-1}{(1-\lambda)r} \\ U\left(\left[\frac{1-\lambda r+(1-l)(1-\lambda)r}{l(1-\lambda)}\right]R(\theta)\right)-U(r) & \text{if } \ l\geq\frac{r-1}{(1-\lambda)r} \end{cases}$$

Result 13 *Suppose θ is common knowledge, then for late consumers*

1. *if θ is small so that also $R(\theta)$ is small, then there is unique equilibrium in dominant strategies, $a^*=0$;*

2. *if θ is intermediate so that also $R(\theta)$ is intermediate, then there are two equilibria, $a^*=0$ for all $i\in[0,1]$ and $a^{**}=1$;*

3. *if θ is large so that also $R(\theta)$ is large, then there is unique equilibrium in dominant strategies, $a^*=1$.*

On the other hand, if θ is observed with noise, we can apply the previous results, because the previous assumptions are satisfied.

In particular θ^* is defined by the following equation

$$\int_0^1 \pi(l,\theta^*)\,dl=0\Leftrightarrow$$

$$\Leftrightarrow\int_0^{\frac{r-1}{(1-\lambda)r}}\left[U(0)-U\left(\frac{1}{1-l(1-\lambda)}\right)\right]dl+$$

$$+\int_{\frac{r-1}{(1-\lambda)r}}^1\left[U\left(\left[\frac{1-\lambda r+(1-l)(1-\lambda)r}{l(1-\lambda)}\right]R(\theta^*)\right)-U(r)\right]dl=0.$$

Thus, we can state the following result.

Result 14 *The game representing our model with private information on θ has a unique (symmetric) cutoff strategy equilibrium, such that*

$$s_i(\theta_i)=\begin{cases} 0\; if & \theta_i\leq\theta^* \\ 1\; if & \theta_i>\theta^*. \end{cases}$$

9.4 Introduction to Regime Change Games

To introduce this class of global games, it is useful to consider an example that has become a workhorse of the applied literature, dubbed the 'regime change' game in Angeletos, Hellwig and Pavan 2007. The example comes from a 1999 working paper on 'Coordination Risk and the Price of Debt' presented as a plenary talk at the 1999 European meetings of the Econometric Society, eventually published as Morris and Shin 2004.

Consider the following basic model:

1. There is a continuum of players;
2. Each player must decide whether to invest or not invest.
3. The cost of investing is $c \in (0,1)$. The payoff to investing is 1 if the proportion investing, denoted by n, is at least $1-\theta$, 0 otherwise. Thus the payoff matrix is

	$n < 1-\theta$	$n \geq 1-\theta$
I	$-c$	$1-c$
NI	0	0

Result 15 *If there is common knowledge of θ and $\theta \in (0,1)$, there are multiple Nash equilibria of this continuum player complete information game: 'all invest' and 'all not invest'.*

But now suppose that there is private asymmetric information such that

1. $$\theta \sim N\left(y, \tau^2\right)$$

   and

2. each player in the continuum population observes the mean y, which is thus a **public signal of** θ.
3. In addition, each player i observes a private signal

$$\theta_i \sim N\left(\theta, \sigma^2\right).$$

Morris and Shin 2004 show that

Result 16 *the resulting game of incomplete information has a unique equilibrium if and only if* $\sigma \leq \sqrt{2\pi\tau}$*, that is, if private signals are sufficiently accurate relative to the accuracy of public signals.*

This result is intuitive: we know that if there is common knowledge of θ, there are multiple equilibria. A very small value of τ means that the public signal is very accurate and there is 'almost' common knowledge.

This result makes it possible to conduct comparative statics within a unique equilibrium not only in the uniform prior, no 'public' information, limit but also with non-trivial public information. A distinctive comparative static results that arises is that the unique equilibrium is very sensitive to the public signal y, even conditioning on the true state θ (see Morris and Shin, 2003; 2004; Angeletos and Werning, 2006). This is because, **for each player, the public signal y becomes a more accurate prediction of others' behavior than his private signal, even if they are of equal precision**. But the sensitivity of the uniqueness result to public signals also raises a robustness question.

Public information is endogenously generated in economic settings, and thus a question that arises in many dynamic applications of global games in general and the regime change game in particular is when endogenous information generates enough public information to get back multiplicity (Tarashev, 2003; Dasgupta, 2007; Angeletos, Hellwig and Pavan, 2006; 2007; Angeletos and Werning, 2006; Hellwig, Mukherji and Tsyvinski, 2006). This literature has highlighted the importance of endogenous information revelation and the variety of channels through which such revelation may lead to multiplicity or enhance uniqueness. In addition, these and other dynamic applications of global games raise many other important methodological issues, such as the interaction between the global game uniqueness logic and 'herding', i.e. informational externalities in dynamic settings without payoff complementarities, and 'signalling', biasing choices from static best responses in order to influence opponents' beliefs in the future.

10 Conclusion

Global games rest on the premise that the information received by economic agents is informative, but not so informative so as to achieve common knowledge of the underlying fundamentals. Indeed, as the information concerning the fundamentals become more and more accurate, the actions elicited in equilibrium resemble behavior when the uncertainty concerning the actions of other agents becomes more and more diffuse. This points to the potential pitfalls if we rely too much on our intuitions that are based on complete information games that allow perfectly coordinated switching of beliefs and actions. Decentralized decision making cannot be relied on to rule out inefficient outcomes, so that there may be room for policies that mitigate the inefficiencies. The analysis of economic problems using the methods from global games is in its infancy, but the method seems promising.

Global games also present a "user-friendly" face of games with incomplete information in the tradition of Harsanyi. The potentially daunting task of forming an infinite hierarchy of beliefs over the actions of all players in the game can be given a representation in terms of beliefs (and the behavior that they elicit) that are simple to the point of being naive. Global games go some way to bridging the gap between those who believe that rigorous game theory has a role in economics (as we do) and those who insist on tractable and usable tools for applied economic analysis.

11 References

References

[1] Angeletos, G.-M., Hellwig, C. and Pavan, A. 2006. Signalling in a global game: coordination and policy traps. Journal of Political Economy 114, 452—84.

[2] Angeletos, G.-M., Hellwig, C. and Pavan, A. 2007. Dynamic global games of regime change: learning, multiplicity and timing of attacks. Econometrica 75, 711-56.

[3] Angeletos, G.-M. and Werning, I. 2006. Crises and prices: information aggregation, multiplicity and volatility. American Economic Review 96, 1720-36.

[4] Athey, S. (2001), "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," Econometrica, 69, 861–889.

[5] Athey, S. (2002), "Monotone Comparative Statics under Uncertainty," Quarterly Journal of Economics, 117(1), 187–223.

[6] Baliga, S. and S. Morris (2000), "Coordination, Spillovers and Cheap Talk," Journal of Economic Theory.

[7] Banerjee, A. (1992), "A Simple Model of Herd Behavior," Quarterly Journal of Economics, 107, 797–818

[8] Bikhchandani, S., D. Hirshleifer, and I.Welch (1992), "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades," Journal of Political Economy, 100, 992–1026.

[9] Burdzy, K., D. Frankel, and A. Pauzner (2001), "Fast Equilibrium Selection by Rational Players Living in a Changing World," Econometrica, 69, 163–189.

[10] Carlsson, H. 1989. Global games and the risk dominance criterion. Unpublished manuscript, Department of Economics, University of Lund.

[11] Carlsson, H. and van Damme, E. 1993a. Global games and equilibrium selection. Econometrica 61, 989-1018.

[12] Carlsson, H. and E. van Damme (1993b), Equilibrium Selection in Stag Hunt Games, in Frontiers of Game Theory, (ed. by K. Binmore, A. Kirman, and A. Tani), Cambridge, MA: MIT Press.

[13] Corsetti, G., Dasgupta, A., Shin, H.S. and Morris, S. 2004. Does one Soros make a difference? The role of a large trader in currency crises. Review of Economlc Studies 71, 87-114.

[14] Dasgupta, A. (2001), "Social Learning and Payoff Complementarities," https://personal.lse.ac.uk/dasgupt2/sl.pdf.

[15] Dasgupta, A. 2007. Coordination and delay in global games. Journal of Economic Theor y 134, 195-225.

[16] Diamond, D. and P. Dybvig (1983), "Bank Runs, Deposit Insurance, and Liquidity," Journal of Political Economy, 91, 401–419.

[17] Frankel, D., Morris, S. and Pauzner, A. 2003. Equilibrium selection in global games with strategic complementarities. Journal of Economic Theory 108, 1-44.

[18] Frankel, D., and A. Pauzner (2002), "Expectations and the Timing of Neighborhood Change," Journal of Urban Economics 51(2):295-314.

[19] Frankel, D. and A. Pauzner (2000), "Resolving Indeterminacy in Dynamic Settings: The Role of Shocks," Quarterly Journal of Economics, 115, 285–304.

[20] Fukao, K. 1994. Coordination failures under incomplete information and global games. Discussion Paper No. A299, Institute of Economic Research, Hitotsubashi University.

[21] Goldstein, I. and Pauzner, A. 2005. Demand-deposit contracts and the probability of bank runs. Journal of Finance 60, 1293-327.

[22] Guimaeres, B. and Morris, S. 2004. Risk and wealth in a model of self-fulfilling currency attacks. Discussion Paper No. 1433R, Cowles Foundation.

[23] Hartigan, J. (1983), Bayes Theory. New York: Springer-Verlag

[24] Harsanyi, J. and Selten, R. 1988. A General Theory of Equilibrium Selection in Games. Cambridge, MA: MIT Press.

[25] Heinemann, F. 2000. Unique equilibrium in a model of self- fulfilling cui rency attacks: comment. American Economic Review 90, 316—8.

[26] Hellwig, C., Mukherji, A. and Tsyvinski, A. 2006. Self- fulfilling currency crises: the role of interest rates. American Economic Review 96, 1769-87.

[27] Kajii, A. and Morris, S. 1997. The robustness of equilibria to incomplete information. Econometrica 65, 1283—309.

[28] Levin, J. (2001), "A Note on Global Equilibrium Selection in Overlapping Generations Games," https://web.stanford.edu/~jdlevin/Papers/OLG.pdf.

[29] Levin, J. (2009), "The Dynamics of Collective Reputation," The B.E. Journal of Theoretical Economics, Volume 9, Issue 1.

[30] Milgrom, P. and J. Roberts (1990), "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," Econometrica, 58, 1255–1277.

[31] Morris, S. 2008. Global Games. In The New Palgrave Dictionary of Economics. Second edition, ed. S. N. Durlauf and L. E. Blume, vol. 3. London: Palgrave Macmillan.

[32] Morris, S. and Shin, H.S. 1995. Informational events that trigger

currency attacks. Working Paper No. 95-24, Federal Reserve Bank of Philadelphia.

[33] Morris, S. and Shin, H.S. 1998. Unique equilibrium in a model of self-fulfilling currency attacks. American Economic Review 85, 557—97.

[34] Morris, S. and Shin, H.S. 2000. Rethinking multiple equilibria in macroeconomics. In NBER Macroeconomic Annual 2000, ed. B.S. Bernanke and K. Rogoff. Cambridge, MA: MIT Press.

[35] Morris, S. and Shin, H.S. 2003. Global games: theory and applications. In Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Societ y), ed. M. Dewatripont, L. Hansen and S. Turnovsky. Cambridge: Cambridge University Press.

[36] Morris, S. and Shin, H.S. 2004. Coordination risk and the price of debt. European Economic Review 48, 133-153.

[37] Morris, S. and Ui, T. 2005. Generalized potentials and robust sets of equilibria. Journal of Economic Theor y 124, 45-78.

[38] Rubinstein, A. 1989. The electronic mail game: strategic behavior under almost common knowledge. American Economic Review 79, 385—91.

[39] Tarashev, N. 2003. Speculative attacks and informational role of the interest rate. Journal of the European Economic Association.

[40] van Zandt, T. and Vives, X. 2007. Monotone equilibria in Bayesian games of strategic complementarities. Journal of Economic Theor y 134, 339-60.

[41] Weinstein, J. and Yildiz, M. 2007. A structure theorem for rationalizability with application to robust predictions of refinements. Econometrica 75, 365-400.