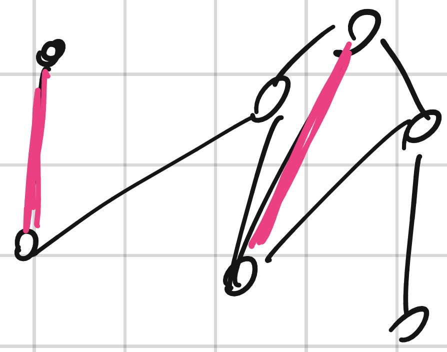
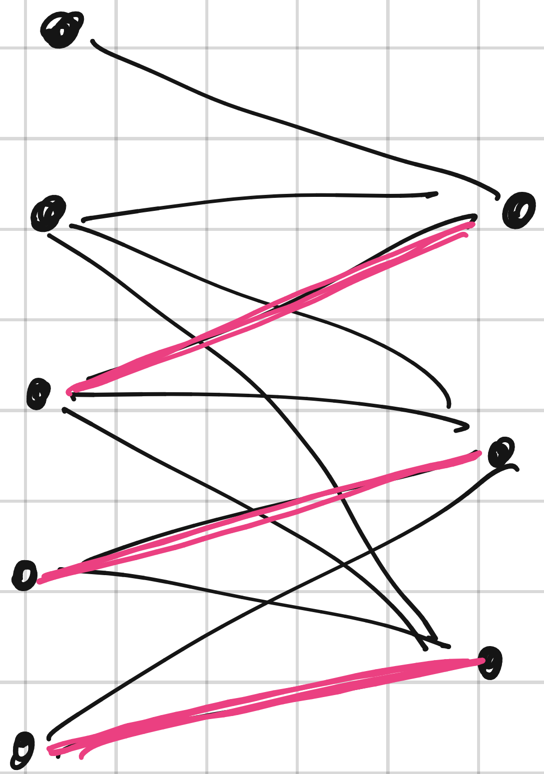
Matching $M \subseteq E$ such that each vertex $v \in V$ is incident on at most one edge of M.



Problem: Find a largest (maximum) matching in G

Bipartite graph

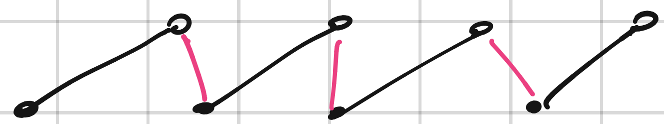


Let $X \subseteq L$ where (L, R) is the bipartition of G

A matching M is X-saturating if all vertices of X are incident on some edge of M

M-augmenting path

Let M be a matching of G, a path $\langle v_0, \dots, v_k \rangle$ is M-augmenting if $(v_i, v_{i+1}) \in M$ if i is odd and k is odd and v_0, v_k are not saturated by M



$\Downarrow$

Augment M by swapping matching and non-matching edges

Hall's Theorem

Let G be a bipartite graph with bipartition (L, R), and let $X \subseteq L$.

Then there exists an X-saturating matching M iff $|N(W)| \geq |W|$ for each $W \subseteq X$.

Proof (constructive)

We maintain:

- M matching
- X_k set of marked vertices of L at iteration k
- Y_k " " " " " R " " "

$X_0 = \{x_0\}$ where x_0 is any vertex not saturated by M

$Y_0 = \{\}$

$M = \{\}$

1. if $|N(X_i)| < |X_i| \Rightarrow$ Hall violator

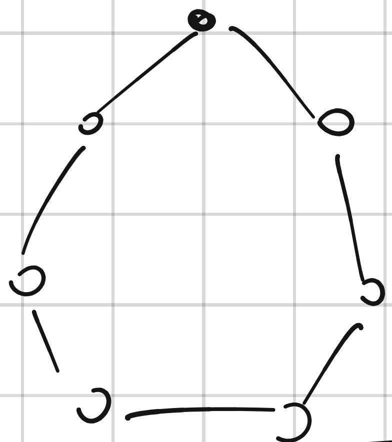
2. $y_{i+1} :=$ any vertex in $N(X_i) \setminus Y_i$

3. if y_{i+1} is saturated in $M \Rightarrow Y_{i+1} := Y_i \cup \{y_{i+1}\}$, $X_{i+1} := X \cup \{x\}$
   where x is the vertex matched with y_{i+1}

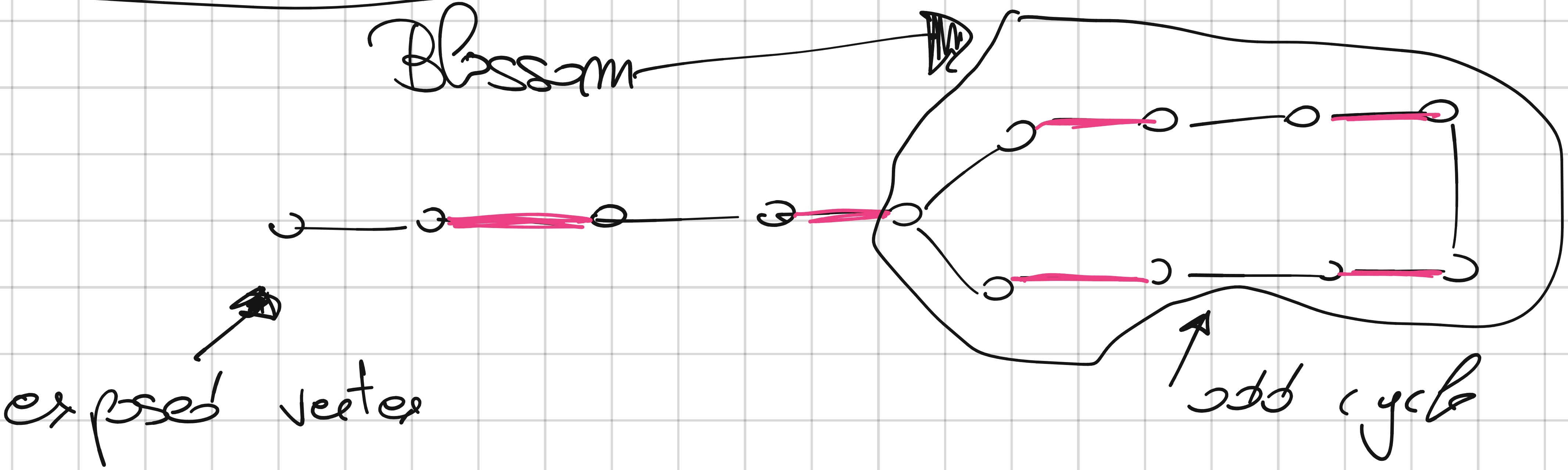
   else
   there is an M-augmenting path ending in y_{i+1}

Matching on general graphs

Bipartite = no odd cycle



odd cycle

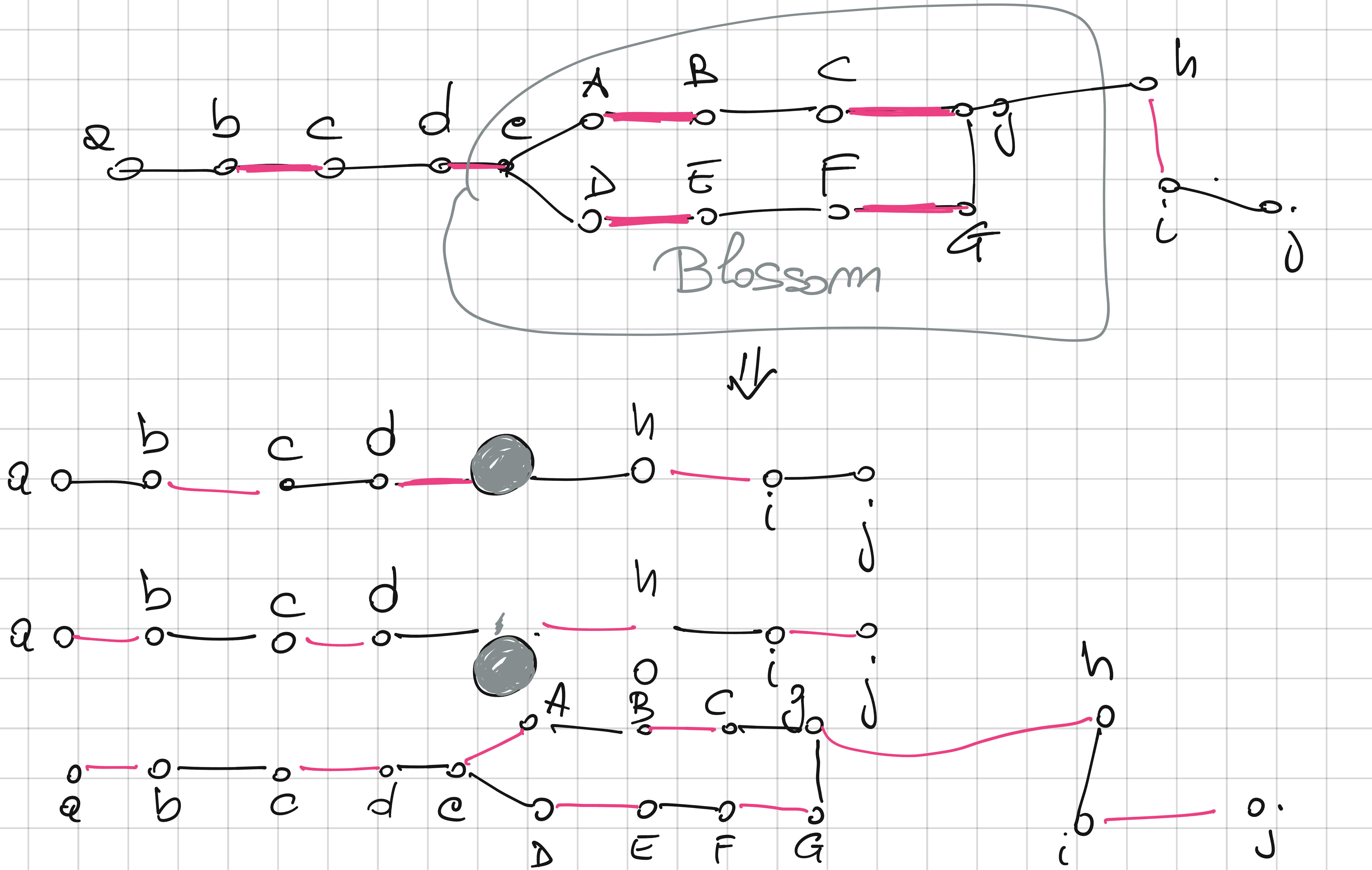


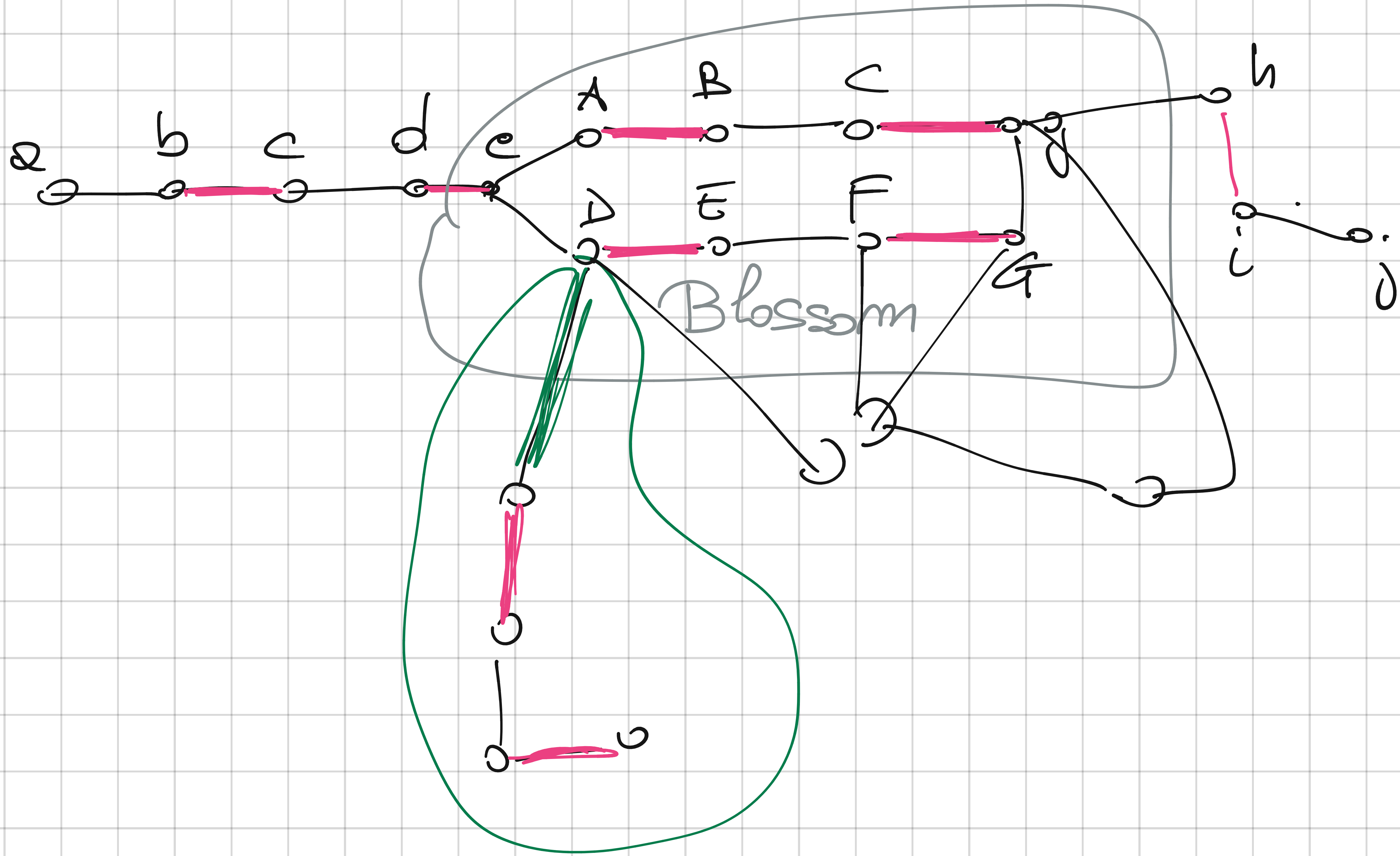
Algorithm

1. Find an augmenting path in G

    OR

2. Find a blossom B
3. Contract B into a single vertex (new graph G_1)
4. Find an augmenting path in G_1
5. Expand B, adapting to the new augmenting path


a
b
c
d
e
A
B
C
D
E
F
G
g
h
i
j
Blossom


a
b
c
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F
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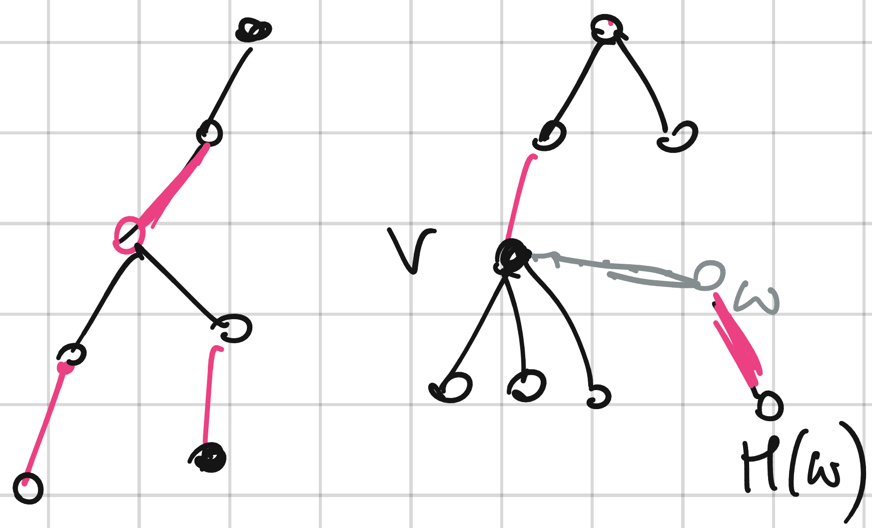
j
Blossom

Forest F, whose roots are not saturated and paths are alternating



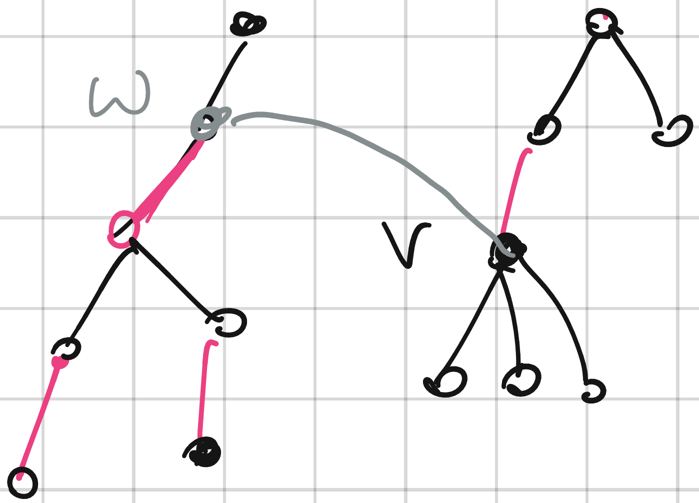
1) Find a vertex v with $d(v, \text{root})$ even

2) Find an unmarked edge (v,w)

Case 1: $w \notin F \Rightarrow \exists\, (w, M(w)) \in M$

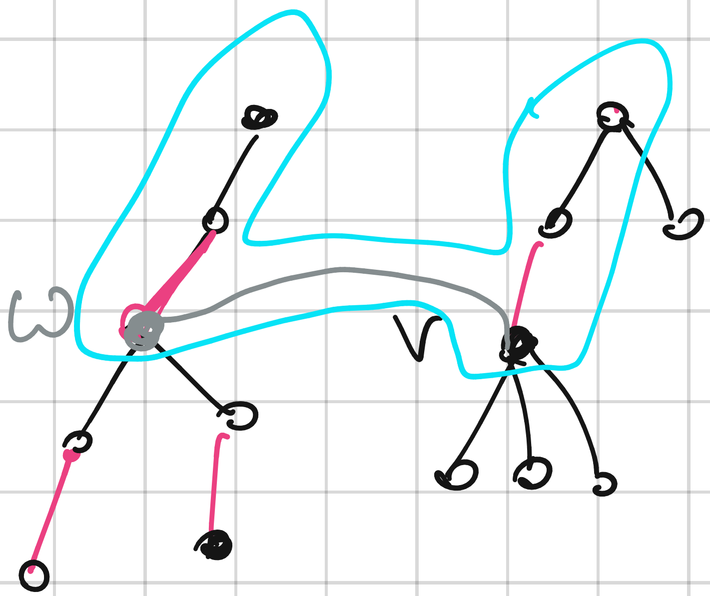
Add (v,w) $(w, M(w))$ to F

Case 2: $w \in F$, $d(w, root)$ is odd



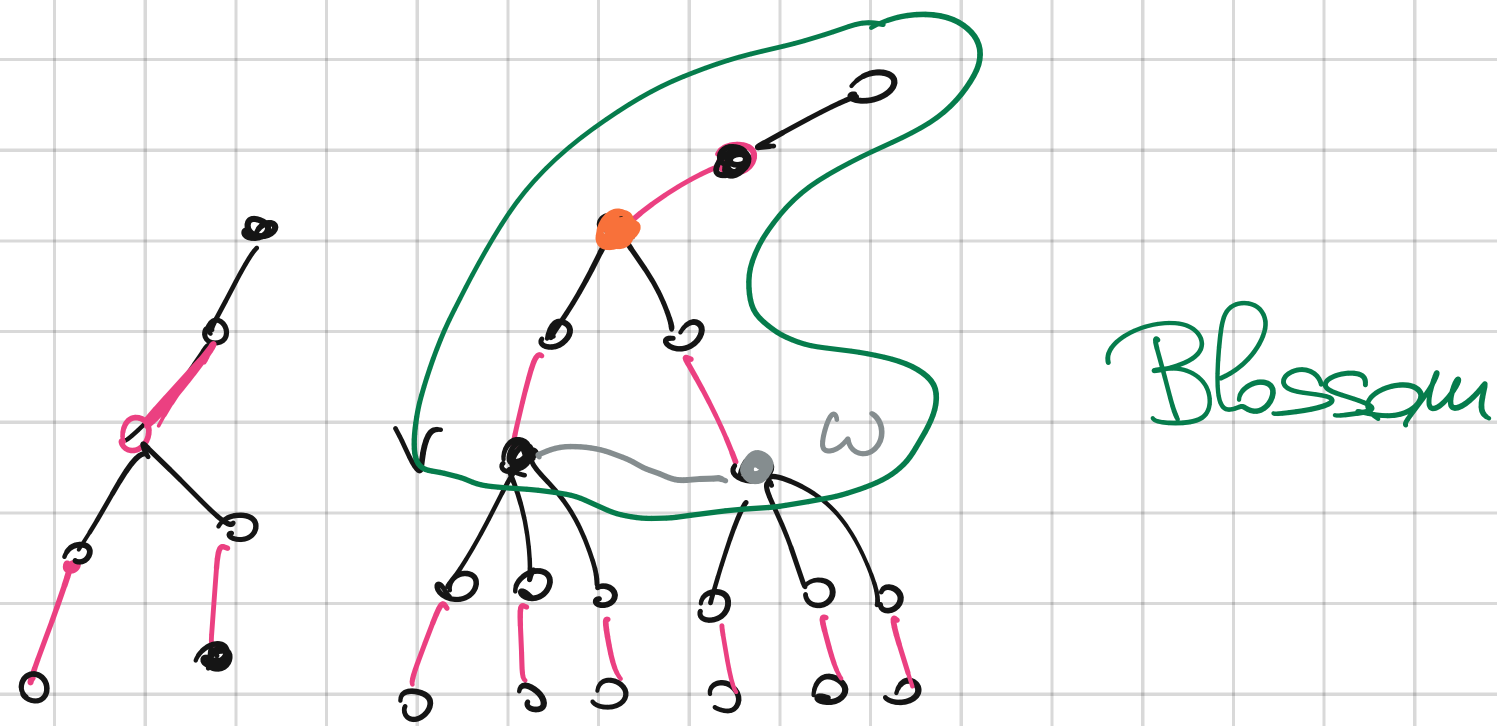
No augmenting path

Case 3: $w \in F$, $d(w, \text{root})$ is even, v and w are in two different trees of F



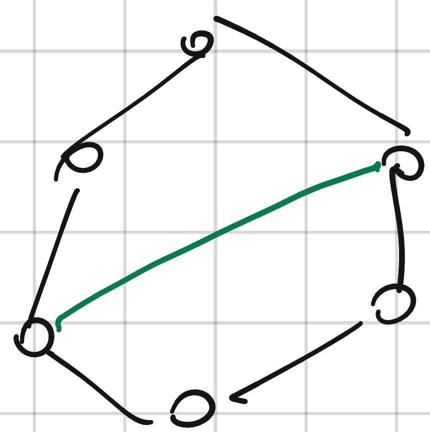
Augmenting path

Case 4: $w \in F'$, $d(v, root)$ is even, v and w are in the same tree of F'

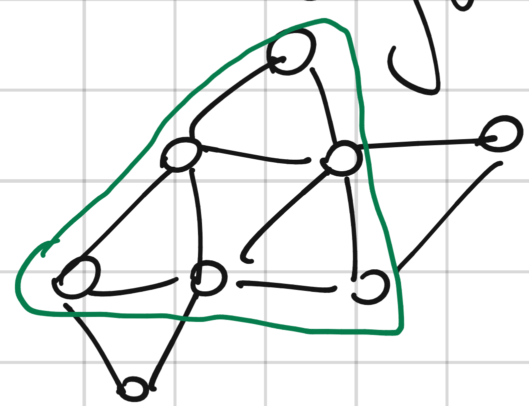


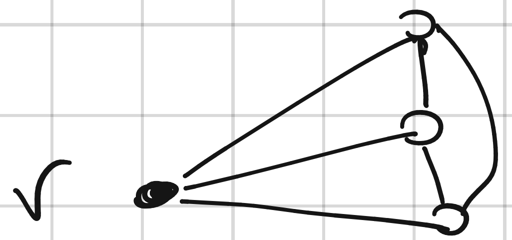
Chordal graphs

Let G be a graph. G is chordal if for any cycle C with at least 4 vertices, C has at least a chord (an edge connecting two nonconsecutive vertices)



Triangulated graphs





v is simplicial

$\Updownarrow$

$N(v)$ is a clique of G

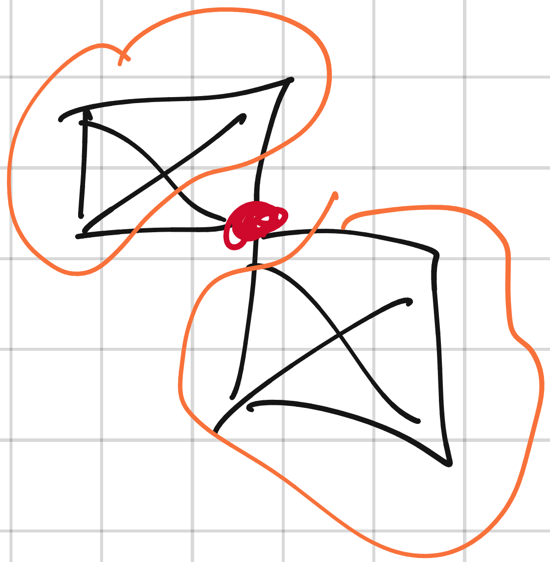
Let $s = (v_1, \dots, v_n)$ be an ordering of the vertices of V

s is a perfect elimination scheme if

v_i is a simplicial vertex in $G[\{v_1, \dots, v_i\}]$

1) G is chordal iff G has a perfect elimination scheme

Graph Decomposition

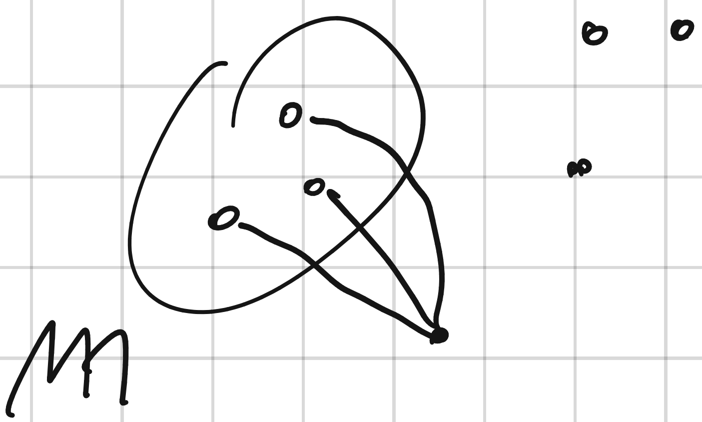


Modular decomposition

$M \subseteq V$ is a module if $\forall v \notin M$,

either v is adjacent to all $w \in M$, or

v is not " " " "



trivial modules

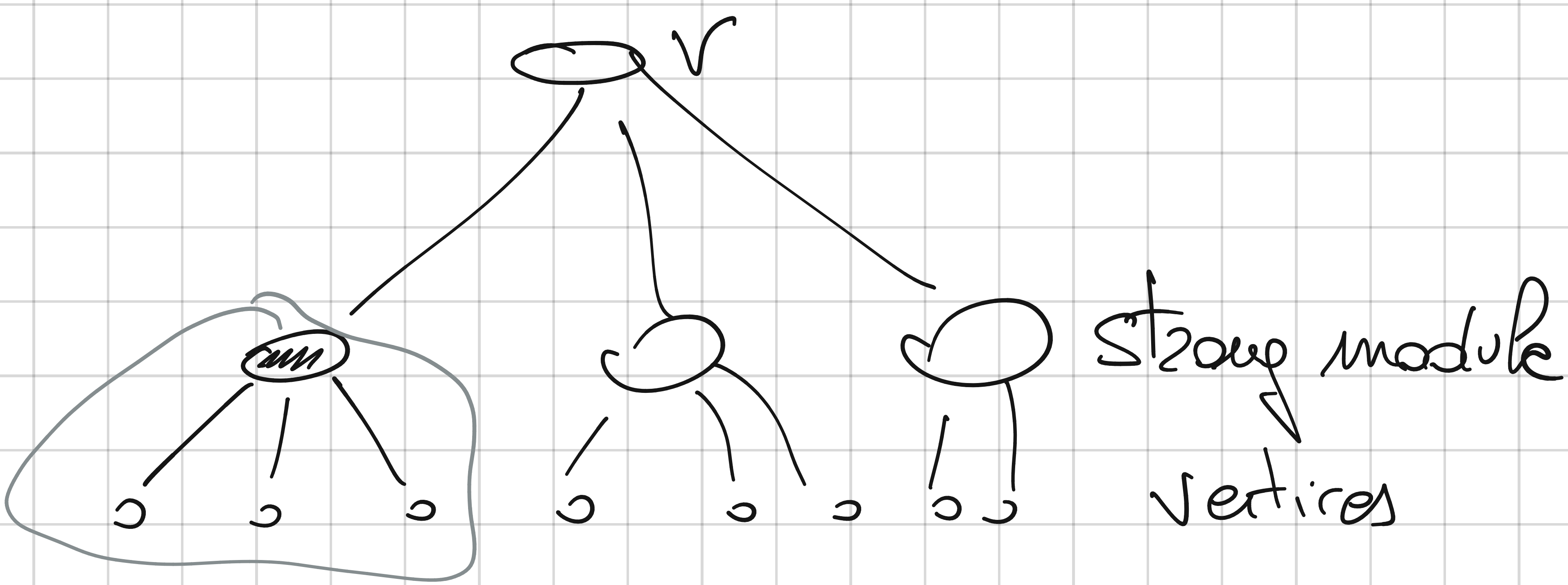
$\emptyset$, V, singletons

M_1, M_2 modules

$M_1 \cap M_2 \neq \emptyset$ and $M_1 \nsubseteq M_2$ and $M_2 \subseteq M_1$

$$\Downarrow$$

$M_1 \setminus M_2$, $M_2 \setminus M_1$, $M_1 \cap M_2$ are all modules



A module M is strong if any other module X is such that $X \subseteq M$, or $M \subseteq X$, or $X \cap M = \emptyset$

Quotient graph

Ear decomposition