5

Connectivity

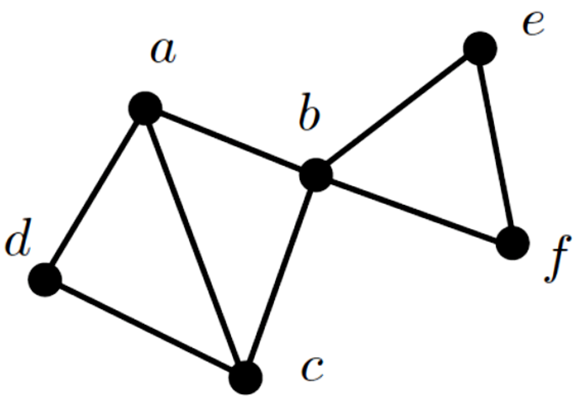
Eulerian and Hamiltonian Graphs, The Travelling Salesperson Problem

Connectivity

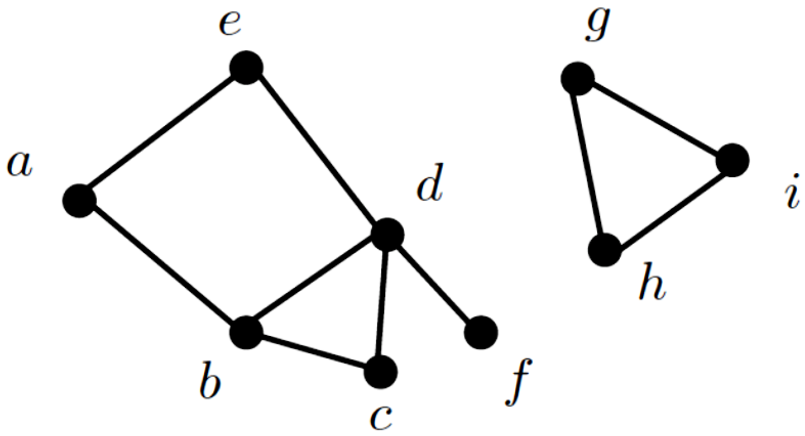
- A graph G is called **connected** if it is non-empty, and any two of its vertices are linked by a path in G.
- If $U \subseteq V$ and U is connected, we also call U itself connected (in G).
- Instead of ‘not connected’ we usually say ‘**disconnected**’.

Connectivity … Cont'd

- A connected graph ($\boldsymbol{a}$) VS a disconnected graph ($\boldsymbol{b}$).



(a)



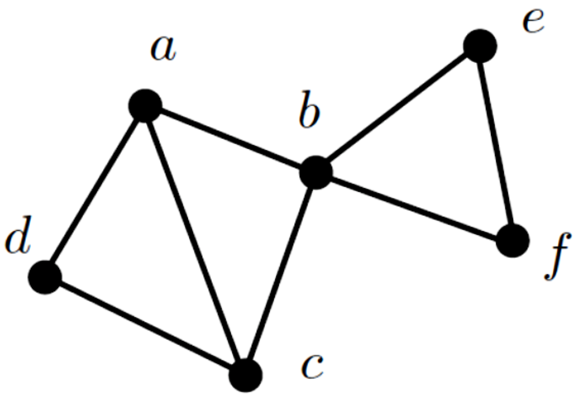
(b)

Connectivity … Cont'd

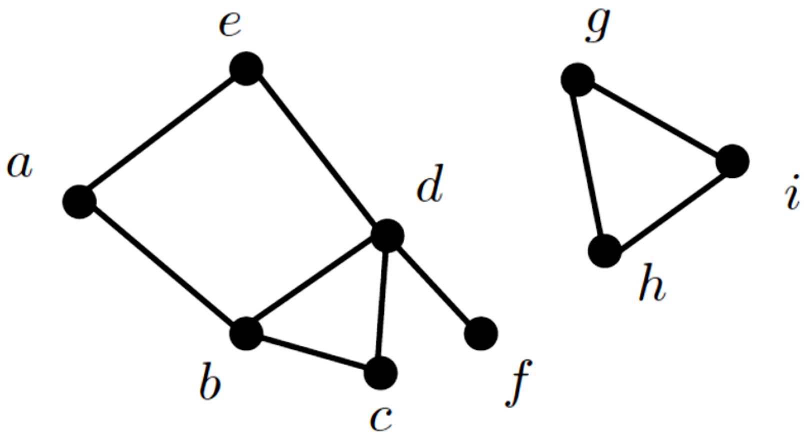
- A **maximal connected subgraph** of G is a subgraph that is connected and is not contained in any other connected subgraph of G.
- A maximal connected subgraph of G is called a **connected component** of G.

Connectivity … Cont'd

- The graph in Fig. ($\boldsymbol{a}$) has one connected component whereas the graph in Fig. ($\boldsymbol{b}$) has two connected components.



(a)



(b)

Connectivity … Cont'd

- The **connectivity** $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal results in a disconnected graph or a single vertex graph K_1.

- A graph G is ***k*-connected** if $\kappa(G) \geq k$.

Connectivity … Cont'd

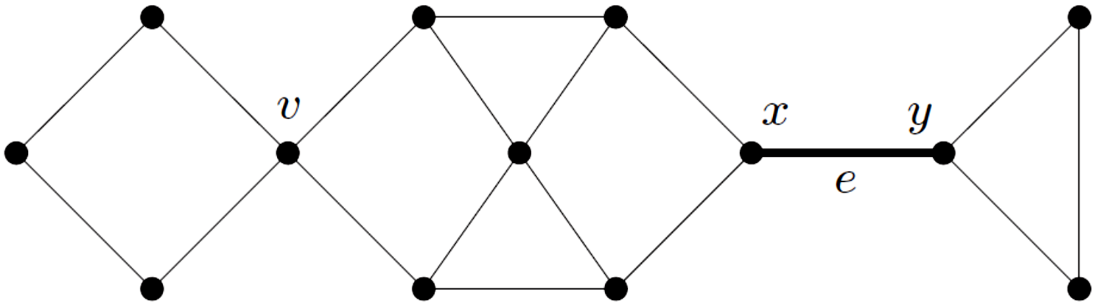
- A **separating set** (or a vertex cut) of a connected graph G is a set $S \subseteq V$ such that $G - S$ has more than one component.
- If a vertex cut contains exactly one vertex, then we call the vertex cut a **cut vertex** (or articulation point).
- If a vertex cut in a 2-connected graph contains exactly two vertices, then we call the two vertices a **separation-pair**.

Connectivity … Cont'd

- The **edge connectivity** $\kappa'(G)$ of a connected graph G is the minimum number of edges whose removal results in a disconnected graph.
- A graph is ***k*-edge-connected** if $\kappa'(G) \geq k$.
- A **disconnecting set of edges** in a connected graph G is a set $F \subseteq E$ such that $G - F$ has more than one component.
- If a disconnecting set contains exactly one edge, it is called a **bridge**.

Connectivity ... Some Examples

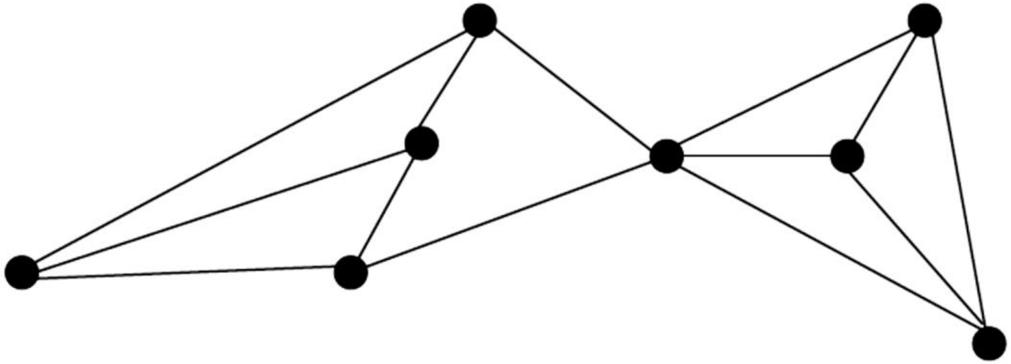
- A graph with cut vertices v, x, y and bridge $e = xy$



Connectivity … Cont'd

- **REMIND**: $\delta(G)$ indicates the the **minimum degree** of a connected simple graph.
- For **complete graphs** of $n \geq 1$ vertices, $\kappa(G) = \kappa'(G) = \delta(G) = n - 1$.
- **Lemma**. Let G be a connected simple graph. Then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

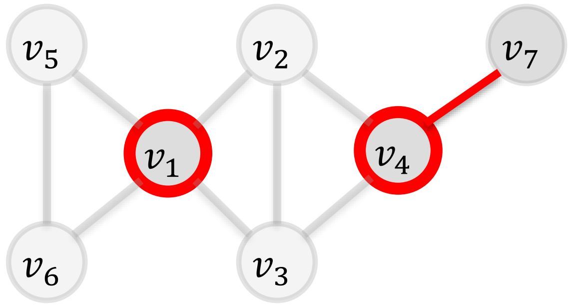
Connectivity … Cont'd



- A graph G with $\kappa(G) = 1$, $\kappa'(G) = 2$, and $\delta(G) = 3$.

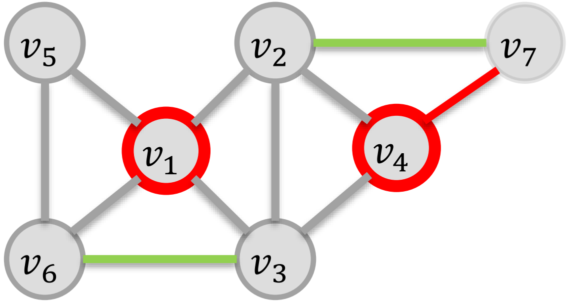
Biconnectivity

- When $\kappa(G) = k'(G) = 2$
  - No single edge or vertex removal disconnects the graph
  - No network failure points compromise the network itself
- The graph of the example is NOT biconnected:
  - A bridge: $\{v_4, v_7\}$
  - Two articulation points: v_1, v_4



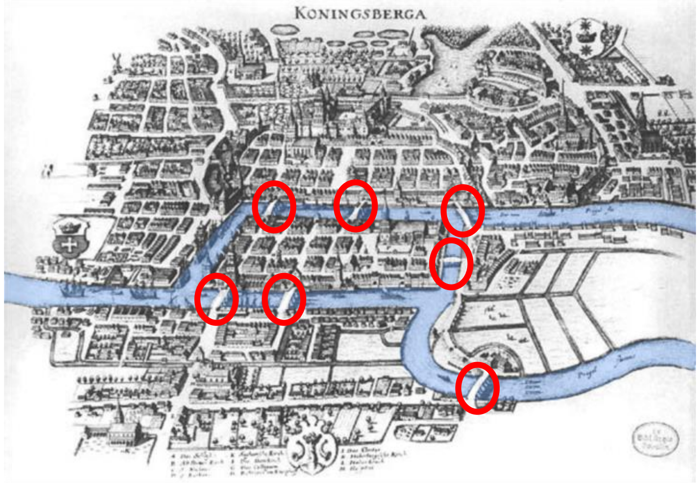
Obtaining Biconnectivity

- Example:
  - $\{v_3, v_6\}$
    - Resolves the articulation point v_1
  - $\{v_2, v_7\}$
    - Resolves the articulation point v_4
  - Alternative to the bridge $\{v_4, v_7\}$



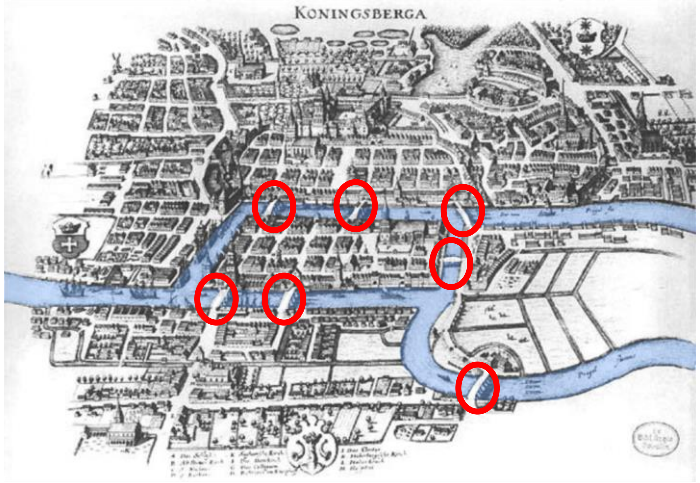
Euler and the Bridges of Königsberg (1736)

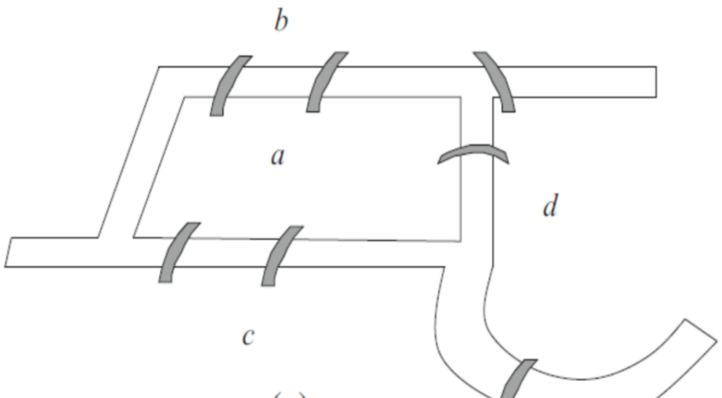
- Graph theory was introduced with the Euler famous paper in which he solved, **using graphs**, the so called «Bridges of Königsberg» problem (a city at the time Prussian, now Russified in Kaliningrad)



Description of the Problem

- From any part of the city, is it possible to take a walk-in order to cross all the bridges once and only once?

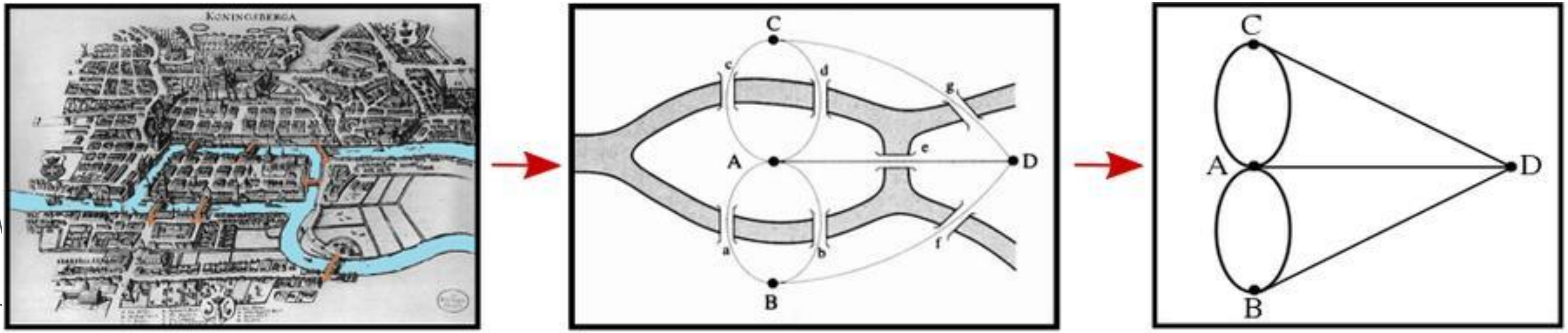




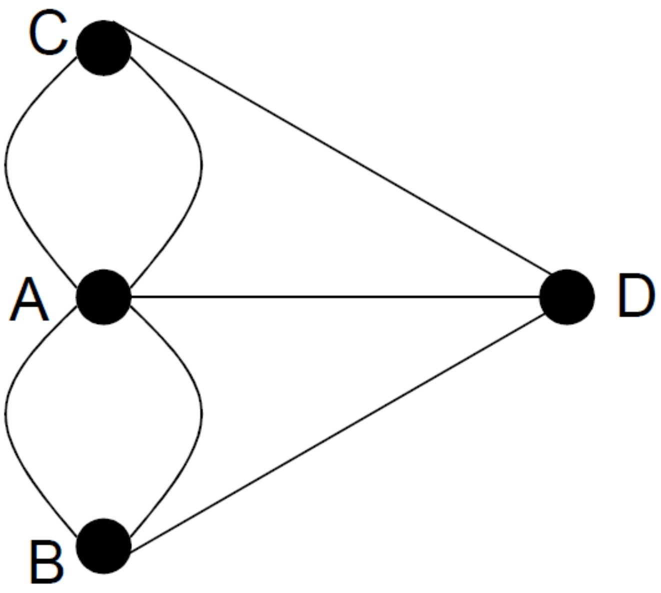
(a)

Modeling the Problem

- If the land areas are associated with **points** (nodes or vertices) and the bridges are associated with **line sections** (arcs or edges) the Königsberg bridge problem is modeled by the graph:

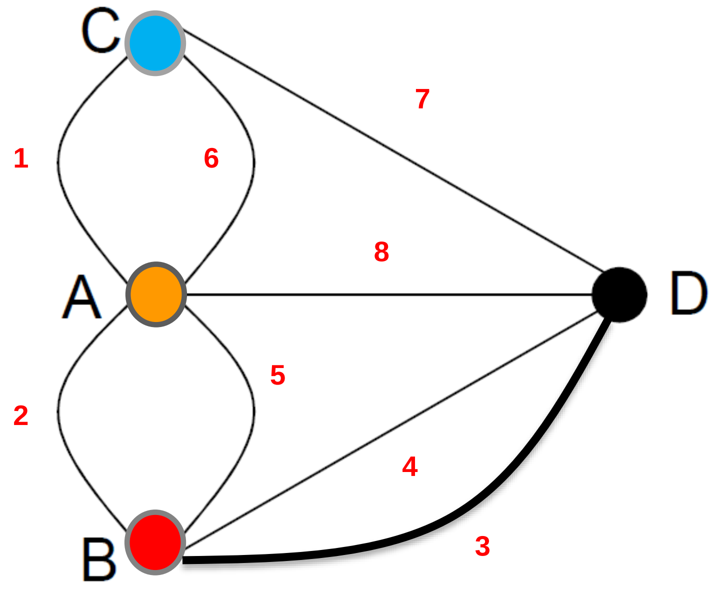


Problem Solution



- Euler used this graph to establish that it is **impossible** to find the required path, with the bridges thus distributed.

Problem Solution … Cont'd

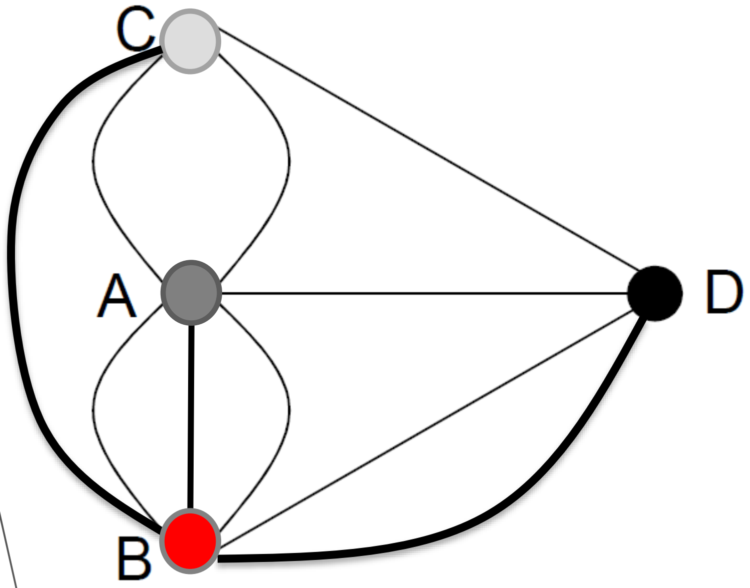


Addition of the eighth "bridge"

- Instead, it is possible:
  - If the **number of incident arcs in each node is even** (there are 0 nodes with odd number of incident arcs).
  - Or if **only two nodes have an odd number of incident edges**.

- The **blue node** is the starting point, the **orange node** is the arrival point.

Modified problem



- **Exercise**: find the path that pass only once for each bridge and back to the point of departure.
- To solve the problem formulated this way it is necessary to add two more "bridges".

Euler Tour (or Trail)

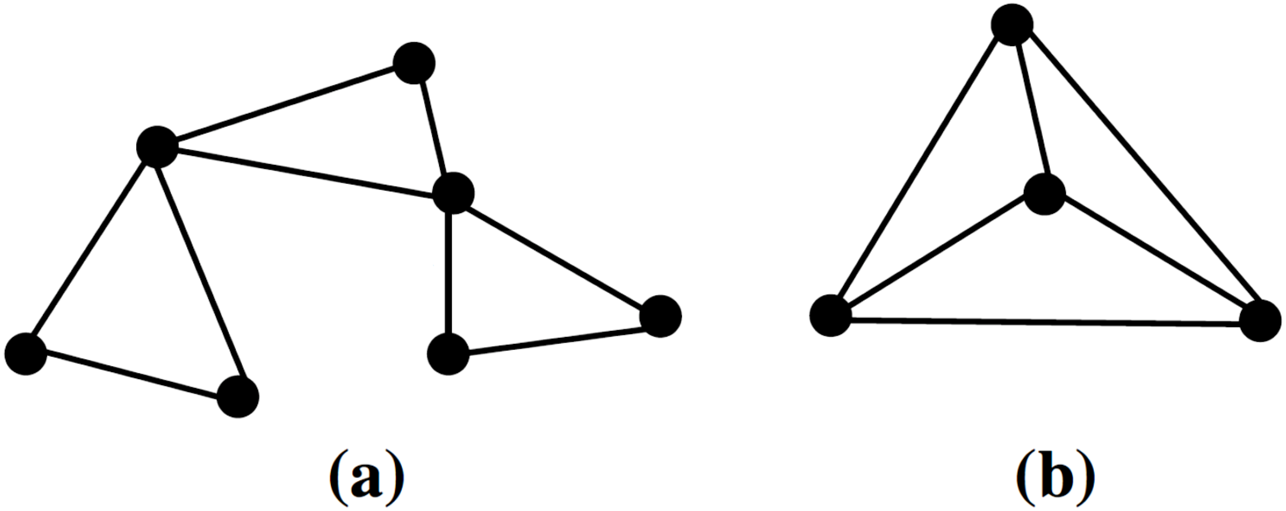
- Let us call a walk in a graph a **Euler tour** (or trail) if it traverses every edge of the graph exactly once.
- Repetition of vertices is possible.

Eulerian Graph

- A connected graph is Eulerian (**Eulerian graph**) if it admits an Euler tour (trail).
- **Theorem**. A connected graph G is Eulerian if and only if every vertex of G has even degree.

Eulerian Graph … Cont'd

- An Eulerian graph ($\boldsymbol{a}$) and a graph which is not Eulerian ($\boldsymbol{b}$).



Eulerian Circuit

- A circuit in a connected graph is an **Eulerian circuit** if it contains every edge of the graph.
- A connected graph with an Eulerian circuit is an Eulerian graph.

Hamiltonian Graph

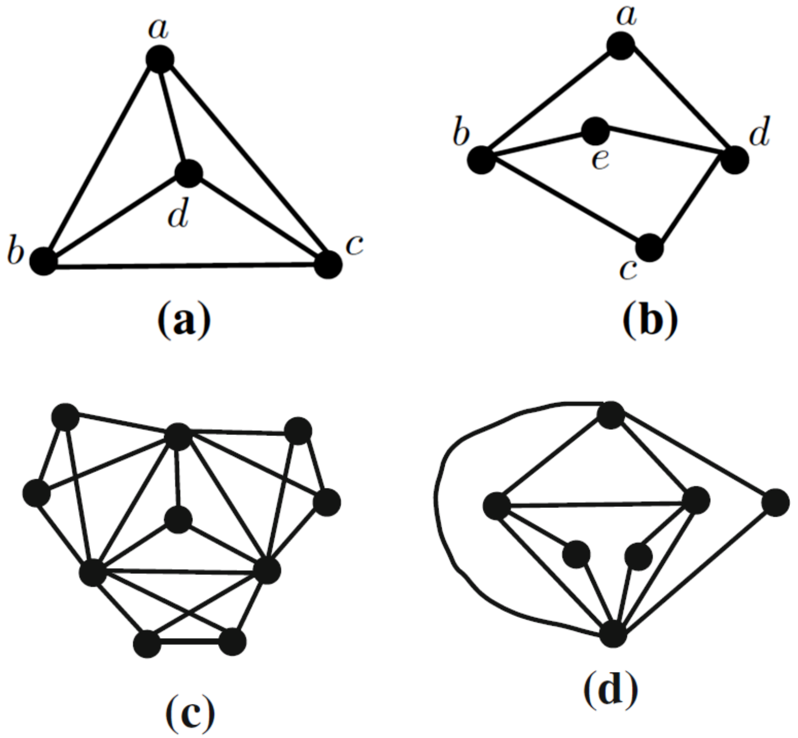
- An Eulerian circuit visits each edge exactly once, but may visit some vertices more than once.
- What happens if we consider a round trip through a given graph G such that **every vertex is visited exactly once**?

Hamiltonian Graph ... Cont'd

- Let G be a graph. A path in G that includes every vertex of G is called a **Hamiltonian path** of G.
- A cycle in G that includes every vertex in G is called a **Hamiltonian cycle** (or circuit) of G.
- If G contains a Hamiltonian cycle, then G is called a **Hamiltonian graph**.

Hamiltonian Graph … Cont'd

- $(\boldsymbol{a})$ A Hamiltonian graph
- $(\boldsymbol{b})$–$(\boldsymbol{d})$ non-Hamiltonian graphs.



The Travelling Salesperson Problem*

*Previously, Salesman Problem

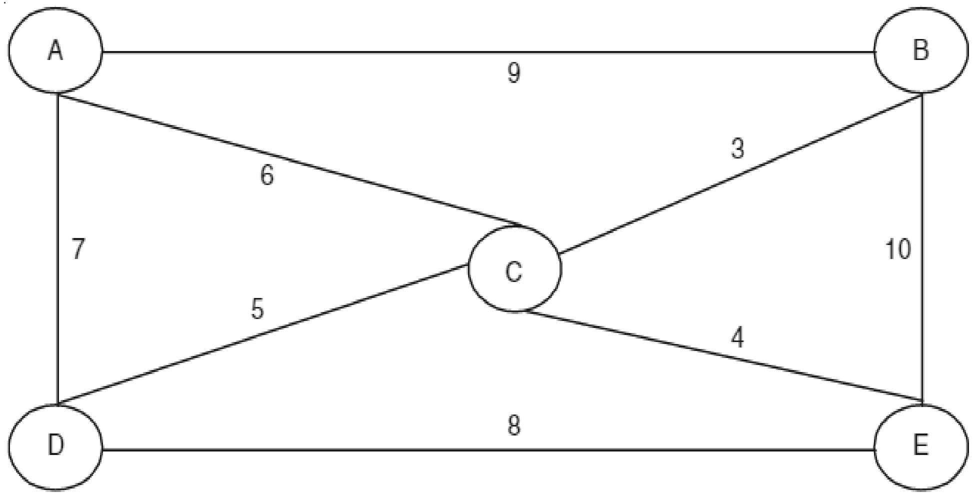
- In the ordinary form of the **Travelling Salesperson Problem**, a map of cities is given to the salesperson, and s/he has to visit all the cities only once to complete a tour such that the length of the tour is the shortest among all possible tours for this map.

- The data consist of weights assigned to the edges of a finite complete graph, and **the objective is to find a Hamiltonian cycle**, a cycle passing through all the vertices of the graph while having the minimum total weight.

- In the Travelling Salesperson Problem context, Hamiltonian cycles are commonly called tours.

The Travelling Salesperson Problem (Example)

- For example, given the map shownin the figure, the lowest cost route would be the one written (A, B, C, E, D, A), with the cost 31.



The Travelling Salesperson Problem (Issues)

- The Travelling Salesperson Problem is an **NP-hard problem** and is often used for testing the optimization algorithms.

- Traditional solution strategies for such problem (and, in general, for NP-hard problems) are:
  - **Find a specific case of the problem** (subproblem) for which either an exact solution or a better heuristic is possible.
  - **Design heuristic algorithms**, i.e., algorithms that produce solutions that are probably good, but impossible to prove to be optimal.
  - Design algorithms to find the exact solution, reasonably fast only for problems with a relatively **low number of cities**.

6

Possible Assignements

Some Possible Assignements

- Discuss the linear time solution for **longest path detection** in Directed Acyclic Graphs.
- Discuss the **PageRank algorithm** (which is based on Random Walks).
- Discuss a specific solution to the **Travelling Salesperson Problem** (*Next Lesson*).
- *You can either present and discuss one of the above-mentioned problems, and/or present an implementation of the algorithm.*