Esercizio 1 con svolgimento

La seguente funzione di utilità definita su X, Y e Z è concava, quasi –concava, altro?

In[∘]:= `U = 20 * Log[X^0.2 * Y^0.2 * Z^0.2]`

Out[∘]= $20\,\text{Log}\left[X^{0.2}\,Y^{0.2}\,Z^{0.2}\right]$

Calcoliamo la matrice hessiana :

In[∘]:= $H = \{\{\partial_{X,X}U, \partial_{X,Y}U, \partial_{X,Z}U\}, \{\partial_{Y,X}U, \partial_{Y,Y}U, \partial_{Y,Z}U\}, \{\partial_{Z,X}U, \partial_{Z,Y}U, \partial_{Z,Z}U\}\}$

Out[∘]= $\left\{\left\{-\frac{4.}{X^{2.}}, 0, 0\right\}, \left\{0, -\frac{4.}{Y^{2.}}, 0\right\}, \left\{0, 0, -\frac{4.}{Z^{2.}}\right\}\right\}$

In[∘]:= `Det[H]`

Out[∘]= $-\frac{64.}{X^{2.}\,Y^{2.}\,Z^{2.}}$

In[∘]:= `MatrixForm[H]`

$$\begin{pmatrix} -\frac{4.`}{X^{2.`}} & 0 & 0 \\ 0 & -\frac{4.`}{Y^{2.`}} & 0 \\ 0 & 0 & -\frac{4.`}{Z^{2.`}} \end{pmatrix}$$

Di che tipo di matrice si tratta ?

A cosa corrisponde il determinate ?

Che uso possiamo fare degli autovalori ?

Guardarne il segno ricordando che ipotizziamo X, Y e Z strettamente positivi ma < ∞

In[∘]:= `Eigenvalues[H]`

Out[∘]= $\left\{-\frac{4.}{Z^{2.}}, -\frac{4.}{Y^{2.}}, -\frac{4.}{X^{2.}}\right\}$

Promemoria

In[∘]:= `J = {{-1, 0, 0}, {0, -4, 0}, {0, 0, -3}}`

Out[∘]= {{-1, 0, 0}, {0, -4, 0}, {0, 0, -3}}

In[∘]:= `MatrixForm[J]`

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

H1 < 0; H2 > 0, H3 < 0

In[∘]:= `NegativeSemidefiniteMatrixQ[J]`

In[∘]:= `True`

P = {{-1, 0, 0}, {0, 4, 0}, {0, 0, -3}}

Out[∘]= {{-1, 0, 0}, {0, 4, 0}, {0, 0, -3}}

In[∘]:= Det[P]

Out[∘]= 12

In[∘]:= MatrixForm[P]

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

In[∘]:= NegativeSemidefiniteMatrixQ[P]

Out[∘]= False

Domande Ordinarie con x, y, z = prezzi e R = somma spendibile

In[∘]:= $\Lambda = U - \lambda * (x * X + y * Y + z * Z - R)$

Out[∘]= $-(-R + x X + y Y + z Z)\,\lambda + 20\,\text{Log}\left[X^{0.2}\,Y^{0.2}\,Z^{0.2}\right]$

In[∘]:= $\nabla_{\{X,Y,Z,\lambda\}}\Lambda$

Out[∘]= $\left\{\frac{4.}{X^{1.}} - x\,\lambda,\ \frac{4.}{Y^{1.}} - y\,\lambda,\ \frac{4.}{Z^{1.}} - z\,\lambda,\ R - x X - y Y - z Z\right\}$

In[∘]:= $\text{Solve}\left[\left\{\frac{4.}{X^{1.}} - x\,\lambda == 0,\ \frac{4.}{Y^{1.}} - y\,\lambda == 0,\ \frac{4.}{Z^{1.}} - z\,\lambda == 0,\ R - x X - y Y - z Z == 0\right\},\ \{X, Y, Z, \lambda\}\right]$

Out[∘]= $\left\{\left\{X \to \frac{0.333333\,R}{x},\ Y \to \frac{0.333333\,R}{y},\ Z \to \frac{0.333333\,R}{z},\ \lambda \to \frac{12.}{R}\right\}\right\}$

Sulla base della risposta alla domanda circa la concavità , il puntio di stazionarietà è max o min ?

Alcune Proprietà

$$x * \frac{0.33333333333333333\grave{}\ R}{x} + y * \frac{0.33333333333333333\grave{}\ R}{y} + z * \frac{0.33333333333333333\grave{}\ R}{z}$$

1.\` R (Walras)

In[∘]:= $X = \frac{0.33333333333333333\grave{}\ R}{x}$

Out[∘]= $\frac{0.333333\,R}{x}$

In[∘]:= $Y = \frac{0.33333333333333333\grave{}\ R}{y}$

Out[∘]= $\frac{0.333333\,R}{y}$

In[∘]:= $Z = \frac{0.33333333333333333\grave{}\ R}{z}$

Out[∘]= $\frac{0.333333\,R}{z}$

In[∘]:= $(\partial_x X) * \left(\frac{x}{X}\right) + (\partial_y X) * \left(\frac{y}{X}\right) + (\partial_z X) * \left(\frac{z}{X}\right) + (\partial_R X) * \left(\frac{R}{X}\right)$

Out[∘]= 0.

0 (Sfrutta omogeneità grado 0 della domanda nei prezzi e in R)

In[∘]:=

$(\partial_R X) * (x) + (\partial_R Y) * (y) + (\partial_R Z) * (z)$

In[∘]:= $\mathcal{L} = 20 * \left(\text{Log}\left[\left(\frac{0.3333333333333333 * R}{x}\right)^{0.2}\right] \text{Log}\left[\left(\frac{0.3333333333333333 * R}{y}\right)^{0.2}\right] + \text{Log}\left[\left(\frac{0.3333333333333333 * R}{z}\right)^{0.2}\right]\right)$

Out[∘]= $20 \left(\text{Log}\left[0.802742 \left(\frac{R}{x}\right)^{0.2}\right] \text{Log}\left[0.802742 \left(\frac{R}{y}\right)^{0.2}\right] + \text{Log}\left[0.802742 \left(\frac{R}{z}\right)^{0.2}\right]\right)$

In[∘]:= $\partial_R \mathcal{L}$

Out[∘]= $20 \left(\frac{0.2}{\left(\frac{R}{z}\right)^{1.} z} + \frac{0.2 \text{Log}\left[0.802742 \left(\frac{R}{x}\right)^{0.2}\right]}{\left(\frac{R}{y}\right)^{1.} y} + \frac{0.2 \text{Log}\left[0.802742 \left(\frac{R}{y}\right)^{0.2}\right]}{\left(\frac{R}{x}\right)^{1.} x}\right)$

In[∘]:= Simplify[α]

Out[∘]= $\frac{2.24222 + 0.8 \text{Log}\left[\frac{R}{x}\right] + 0.8 \text{Log}\left[\frac{R}{y}\right]}{R}$

In[∘]:= FullySimplify[%]

Out[∘]= $\text{FullySimplify}\left[\frac{12. + 2.24222 R + 0.8 R \text{Log}\left[\frac{R}{x}\right] + 0.8 R \text{Log}\left[\frac{R}{y}\right]}{R^2}\right]$

In[∘]:= Simplify[%55]

Out[∘]= $\text{FullySimplify}\left[\frac{12. R + 12. x X + 12. y Y + 12. z Z}{R^2}\right]$

In[∘]:= $\Omega = 20 * \text{Log}\left[\left(\left(\frac{0.333333333333333 * R}{x}\right) \text{^}0.2\right) * \left(\left(\frac{0.333333333333333 * R}{y}\right) \text{^}0.2\right) * \left(\left(\frac{0.333333333333333 * R}{z}\right) \text{^}0.2\right)\right]$

In[∘]:= $20 \text{Log}\left[0.5172818579717834` \left(\frac{R}{x}\right)^{0.2`} \left(\frac{R}{y}\right)^{0.2`} \left(\frac{R}{z}\right)^{0.2`}\right]$

$r = \partial_R \Omega$

Out[∘]= $20 \text{Log}\left[0.517282 \left(\frac{R}{x}\right)^{0.2} \left(\frac{R}{y}\right)^{0.2} \left(\frac{R}{z}\right)^{0.2}\right]$

Out[∘]= $\frac{38.6636 \left(\frac{0.103456 \left(\frac{R}{y}\right)^{0.2} \left(\frac{R}{z}\right)^{0.2}}{\left(\frac{R}{x}\right)^{0.8} x} + \frac{0.103456 \left(\frac{R}{x}\right)^{0.2} \left(\frac{R}{z}\right)^{0.2}}{\left(\frac{R}{y}\right)^{0.8} y} + \frac{0.103456 \left(\frac{R}{x}\right)^{0.2} \left(\frac{R}{y}\right)^{0.2}}{\left(\frac{R}{z}\right)^{0.8} z}\right)}{\left(\frac{R}{x}\right)^{0.2} \left(\frac{R}{y}\right)^{0.2} \left(\frac{R}{z}\right)^{0.2}}$

In[•]:= FullSimplify[r]

Out[•]= $\frac{12.}{R}$

Domande Compensate

Min spesa con vincolo U = T = livello utilità dato

In[•]:= L = (x * X + y * Y + z * Z) - μ * (U - T)

Out[•]= $x X + y Y + z Z - \mu \left(-T + 20 \text{Log}\left[X^{0.2} Y^{0.2} Z^{0.2}\right]\right)$

In[•]:= $\nabla_{\{X, Y, Z, \mu\}} L$

Out[•]= $\left\{x - \frac{4.\ \mu}{X^{1.}},\ y - \frac{4.\ \mu}{Y^{1.}},\ z - \frac{4.\ \mu}{Z^{1.}},\ T - 20 \text{Log}\left[X^{0.2} Y^{0.2} Z^{0.2}\right]\right\}$

In[•]:= $\text{Solve}\left[\left\{x - \frac{4.\ \mu}{X^{1.}} == 0,\ y - \frac{4.\ \mu}{Y^{1.}} == 0,\ z - \frac{4.\ \mu}{Z^{1.}} == 0,\ T - 20 \text{Log}\left[X^{0.2} Y^{0.2} Z^{0.2}\right] == 0\right\}, \{X, Y, Z, \mu\}\right]$

Out[•]=

$$\left\{\left\{X \to -\frac{1}{x^{2/3}} (0.5 - 0.866025\ i)\ e^{0.0833333\ T} y^{1/3} z^{1/3},\right.\right.$$

$$Y \to -\frac{1}{y^{2/3}} (0.5 - 0.866025\ i)\ e^{0.0833333\ T} x^{1/3} z^{1/3},$$

$$Z \to -\frac{1}{z^{2/3}} (0.5 - 0.866025\ i)\ e^{0.0833333\ T} x^{1/3} y^{1/3},$$

$$\left.\mu \to (-0.125 + 0.216506\ i)\ e^{0.0833333\ T} x^{1/3} y^{1/3} z^{1/3}\right\},$$

$$\left\{X \to -\frac{1}{x^{2/3}} (0.5 + 0.866025\ i)\ e^{0.0833333\ T} y^{1/3} z^{1/3},\right.$$

$$Y \to -\frac{1}{y^{2/3}} (0.5 + 0.866025\ i)\ e^{0.0833333\ T} x^{1/3} z^{1/3},$$

$$Z \to -\frac{1}{z^{2/3}} (0.5 + 0.866025\ i)\ e^{0.0833333\ T} x^{1/3} y^{1/3},$$

$$\left.\mu \to (-0.125 - 0.216506\ i)\ e^{0.0833333\ T} x^{1/3} y^{1/3} z^{1/3}\right\},$$

$$\left\{X \to \frac{1.\ e^{0.0833333\ T} y^{1/3} z^{1/3}}{x^{2/3}},\ Y \to \frac{1.\ e^{0.0833333\ T} x^{1/3} z^{1/3}}{y^{2/3}},\right.$$

$$\left.\left.Z \to \frac{1.\ e^{0.0833333\ T} x^{1/3} y^{1/3}}{z^{2/3}},\ \mu \to 0.25\ e^{0.0833333\ T} x^{1/3} y^{1/3} z^{1/3}\right\}\right\}$$

Solo soluzioni reali

In[•]:= Last[%44]

Out[•]=

$$\left\{X \to \frac{1.\ e^{0.0833333\ T} y^{1/3} z^{1/3}}{x^{2/3}},\ Y \to \frac{1.\ e^{0.0833333\ T} x^{1/3} z^{1/3}}{y^{2/3}},\right.$$

$$\left.Z \to \frac{1.\ e^{0.0833333\ T} x^{1/3} y^{1/3}}{z^{2/3}},\ \mu \to 0.25\ e^{0.0833333\ T} x^{1/3} y^{1/3} z^{1/3}\right\}$$

Funzione di spesa : domande compensate nel vincolo di spesa

In[•]:= $S = x * \frac{1.` e^{0.08333333333333333` T} y^{1/3} z^{1/3}}{x^{2/3}} + y * \frac{1.` e^{0.08333333333333333` T} x^{1/3} z^{1/3}}{y^{2/3}} + z * \frac{1.` e^{0.08333333333333336` T} x^{1/3} y^{1/3}}{z^{2/3}}$

Out[•]= $2.\ e^{0.0833333\ T} x^{1/3} y^{1/3} z^{1/3} + 1.\ e^{0.0833333\ T} x^{1/3} y^{1/3} z^{1/3}$

Lemma Shephard (solo per domanda X)

In[•]:= $\partial_x S$

Out[•]= $\frac{0.666667\ e^{0.0833333\ T} y^{1/3} z^{1/3}}{x^{2/3}} + \frac{0.333333\ e^{0.0833333\ T} y^{1/3} z^{1/3}}{x^{2/3}}$

In[•]:= Simplify$\left[\frac{0.666667\ e^{0.0833333\ T} y^{1/3} z^{1/3}}{x^{2/3}} + \frac{0.333333\ e^{0.0833333\ T} y^{1/3} z^{1/3}}{x^{2/3}}\right]$

Out[•]= $\frac{1}{x^{2/3}} \left(0.666667\ e^{0.0833333\ T} + 0.333333\ e^{0.0833333\ T}\right) y^{1/3} z^{1/3}$

A cosa corrisponde ?

Utilità indiretta : MAX Utilità (domande ordinarie nella funzione di utilità)

In[•]:= $V = 20 * \text{Log}\left[\left(\frac{1}{x} 0.3333333333333333` R\right)\text{^}0.2 * \left(\frac{1}{y} 0.3333333333333333` R\right)\text{^}0.2 * \left(\frac{1}{z} 0.3333333333333333` R\right)\text{^}0.2\right]$

Out[•]= $20\ \text{Log}\left[0.517282 \left(\frac{R}{x}\right)^{0.2} \left(\frac{R}{y}\right)^{0.2} \left(\frac{R}{z}\right)^{0.2}\right]$

Applichiamo Roy (solo X)

In[•]:= $\frac{-\partial_x V}{\partial_R V}$

Out[•]= $\left(0.103456\ R \left(\frac{R}{y}\right)^{0.2} \left(\frac{R}{z}\right)^{0.2}\right) \Big/ \left(\left(\frac{R}{x}\right)^{0.8} x^2 \left(\frac{0.103456 \left(\frac{R}{y}\right)^{0.2} \left(\frac{R}{z}\right)^{0.2}}{\left(\frac{R}{x}\right)^{0.8} x} + \frac{0.103456 \left(\frac{R}{x}\right)^{0.2} \left(\frac{R}{z}\right)^{0.2}}{\left(\frac{R}{y}\right)^{0.8} y} + \frac{0.103456 \left(\frac{R}{x}\right)^{0.2} \left(\frac{R}{y}\right)^{0.2}}{\left(\frac{R}{z}\right)^{0.8} z}\right)\right)$

In[•]:= Simplify[%]

Out[•]= $\frac{0.333333\ R}{x}$

Applichiamo Teorema di Hotelling – Wold (domanda inversa)

In[•]:= $u = 20 * 0.2 * (\text{Log}[X] + \text{Log}[Y] + \text{Log}[Z])$

Out[•]= $4.\ (\text{Log}[X] + \text{Log}[Y] + \text{Log}[Z])$

In[•]:= $q = \frac{\partial_{\text{Log}[X]} u}{X * (\partial_{\text{Log}[X]} u + \partial_{\text{Log}[Y]} u + \partial_{\text{Log}[Z]} u)}$

Out[•]= $\frac{0.3333333333333333`}{X}$

dove q è il prezzo di X normalizzato con R,
ovvero x / R. Okkio : si deriva rispetto a Log[X] non X (o Y o Z). In
che quota % viene speso R per l ' acquisto di X? Rifare per Y e Z.

Esercizio 2 con svolgimento

(simboli cambiati ma riconoscibili)

In[∘]:= $W = \sqrt{Q * M}$

Out[∘]= $\sqrt{M\,Q}$

In[∘]:= $G = \{\{\partial_{Q,Q}W, \partial_{Q,M}W\}, \{\partial_{M,Q}W, \partial_{M,M}W\}\}$

Out[∘]= $\left\{\left\{-\frac{M^2}{4\,(M\,Q)^{3/2}}, -\frac{M\,Q}{4\,(M\,Q)^{3/2}}+\frac{1}{2\,\sqrt{M\,Q}}\right\}, \left\{-\frac{M\,Q}{4\,(M\,Q)^{3/2}}+\frac{1}{2\,\sqrt{M\,Q}}, -\frac{Q^2}{4\,(M\,Q)^{3/2}}\right\}\right\}$

In[∘]:= MatrixForm[G]

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{M^2}{4\,(M\,Q)^{3/2}} & -\frac{M\,Q}{4\,(M\,Q)^{3/2}}+\frac{1}{2\,\sqrt{M\,Q}} \\ -\frac{M\,Q}{4\,(M\,Q)^{3/2}}+\frac{1}{2\,\sqrt{M\,Q}} & -\frac{Q^2}{4\,(M\,Q)^{3/2}} \end{pmatrix}$$

In[∘]:= Eigenvalues[G]

Out[∘]= $\left\{0, \frac{-M^2-Q^2}{4\,(M\,Q)^{3/2}}\right\}$

Ci aspettiamo det (G) nullo :

In[∘]:= Det[G]

0

Che diciamo della funzione W?

Domande ordinarie:

In[∘]:= $\Psi = W - \rho * (q * Q + m * M - B)$

Out[∘]= $\sqrt{M\,Q} - (-B + m\,M + q\,Q)\,\rho$

In[∘]:= Solve[{$\partial_Q\Psi == 0, \partial_M\Psi == 0, \partial_\rho\Psi == 0$}, {M, Q, ρ}]

Out[∘]= $\left\{\left\{M \to \frac{B}{2\,m}, Q \to \frac{B}{2\,q}, \rho \to \frac{\sqrt{\frac{B^2}{m\,q}}}{2\,B}\right\}\right\}$

In[∘]:= $M = \frac{B}{2\,m}$

Out[∘]= $\frac{B}{2\,m}$

In[∘]:= $Q = \frac{B}{2\,q}$

Out[∘]= $\frac{B}{2\,q}$

In[∘]:= $q * \left(\frac{B}{2\,q}\right) + m * \left(\frac{B}{2\,m}\right)$

Out[∘]= B

(Walras)

In[∘]:= $(\partial_B Q) * (q) + (\partial_B M) * (m)$

Out[∘]= 1

In[∘]:= $(\partial_q Q) * \left(\frac{q}{Q}\right) + (\partial_m Q) * \left(\frac{m}{Q}\right) + (\partial_B Q) * \left(\frac{B}{Q}\right)$

Out[∘]= 0

In[∘]:= $\Phi = \sqrt{\frac{B}{2\,m} * \frac{B}{2\,q}}$

Out[∘]= $\frac{1}{2}\sqrt{\frac{B^2}{m\,q}}$

Utilità indiretta, da cui con Roy

In[∘]:= $\frac{-\partial_m \Phi}{\partial_B \Phi}$

Out[∘]= $\frac{B}{2\,m}$

Hotelling – Wold (no log)

In[∘]:= $\frac{\partial_Q W}{(Q * \partial_Q W) + (M * \partial_M W)}$

Out[∘]= $\frac{1}{2\,Q}$

In[∘]:= Plot$\left[\frac{1}{2\,Q}, \{Q, 0, 8\}\right]$

Out[∘]=