Exercises of Dynamic Optimization

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The exercises with "*" are difficult !

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1 Optimal control with variational method

Find the optimal control function and the optimal state function of the following problems:

1.1 The "simplest problem"

In this first section we consider optimal control problems where appear only a initial condition on the trajectory.

a) $$\begin{cases} \min \int_1^3 [x + 2t(1 - e^t)u]\, \mathrm{d}t \\ \dot{x} = 2x + 4ut \\ x(1) = 0 \\ 0 \le u \le 2 \end{cases}$$

b) $$\begin{cases} \min \int_0^2 (u^2 - xe^t)\, \mathrm{d}t \\ \dot{x} = -x + u \\ x(0) = 1 \end{cases}$$

c) $$\begin{cases} \max \int_0^{1/3} (-u^2 - 2x^2)\, \mathrm{d}t \\ \dot{x} = 2u + x \\ x(0) = 1 \end{cases}$$

d) $$\begin{cases} \min \int_0^1 (x^2 + 2x - 2u + u^2)\, \mathrm{d}t \\ \dot{x} = u \\ x(0) = 0 \end{cases}$$

e) $$\begin{cases} \max \int_0^1 (x - u^2)\, \mathrm{d}t \\ \dot{x} = u \\ x(0) = 0 \end{cases}$$

f) $$\begin{cases} \max \int_1^2 -2xe^t\, \mathrm{d}t \\ \dot{x} = \dfrac{e^t}{u} + x \\ x(1) = 0 \\ 1 \le u \le 2 \end{cases}$$

g) $$\begin{cases} \max \int_0^2 (2x - 4u)\, \mathrm{d}t \\ \dot{x} = x + u \\ x(0) = 5 \\ 0 \le u \le 2 \end{cases}$$

h) $$\begin{cases} \min \int_1^2 (u^2 + x^2)\, \mathrm{d}t \\ \dot{x} = x + u \\ x(1) = 2 \\ u \ge 0 \end{cases}$$

i)
$$\begin{cases} \min \int_1^2 (3x + 2u)\, \mathrm{d}t \\ \dot{x} = e^{-u} + t^3 \\ x(1) = e^{-2} \\ 2 \le u \le 3 \end{cases}$$

l)
$$\begin{cases} \max \int_0^4 (u - x + t)\, \mathrm{d}t \\ \dot{x} = \frac{t}{u} + x \\ x(0) = 1 \\ 1 \le u \le 2 \end{cases}$$

m)
$$\begin{cases} \max \int_{-1}^1 (-2tx + 3t^3 u)\, \mathrm{d}t \\ \dot{x} = tu \\ x(-1) = 1 \\ 0 \le u \le 2 \end{cases}$$

n)
$$\begin{cases} \max \int_0^3 (x - 2u)\, \mathrm{d}t \\ \dot{x} = e^{-u} - x \\ x(0) = 0 \end{cases}$$

o)
$$\begin{cases} \max \int_{-3}^{-1} (-x + u^2) t\, \mathrm{d}t \\ \dot{x} = x + 3u \\ x(-3) = 2 \\ -2 \le u \le 0 \end{cases}$$

p)
$$\begin{cases} \min \int_0^{\sqrt{2}} (x^2 - x\dot{x} + 2\dot{x}^2)\, \mathrm{d}t \\ x(0) = 1 \end{cases}$$

1.2 More general problems

a)
$$\begin{cases} \max \int_0^2 (2x - u^2)\, \mathrm{d}t \\ \dot{x} = 1 - u \\ x(0) = 1 \\ x(2) = 0 \end{cases}$$

b)
$$\begin{cases} \max \int_0^{11} x\, \mathrm{d}t \\ \dot{x} = u \\ x(0) = 0 \\ x(11) = 1 \\ -1 \le u \le 1 \end{cases}$$

c) $$\begin{cases} \min \displaystyle\int_{-1}^{1} (2u - 3x)\,\mathrm{d}t \\ \dot{x} = t - u - 2x \\ x(1) = -\frac{5}{4} \\ 0 \le u \le 3 \end{cases}$$

d) $$\begin{cases} \max \displaystyle\int_{0}^{4} 3x\,\mathrm{d}t \\ \dot{x} = x + u \\ x(0) = 0 \\ x(4) = \frac{3}{2}e^4 \\ 0 \le u \le 2 \end{cases}$$

e) $$\begin{cases} \min \displaystyle\int_{1}^{4} t^2 \left(\frac{1}{u} - x\right)\,\mathrm{d}t \\ \dot{x} = -x - tu \\ x(4) = 2 \\ 1 \le u \le 3 \end{cases}$$

f) $$\begin{cases} \min \displaystyle\int_{0}^{e} (u - x)\,\mathrm{d}t \\ \dot{x} = e^{-u} + t^2 \\ x(e) = 0 \\ 1 \le u \le 3 \end{cases}$$

g) $$\begin{cases} \min \displaystyle\int_{0}^{2} (u + 2tx)\,\mathrm{d}t \\ \dot{x} = tx + u \\ x(2) = 0 \\ 1 \le u \le 3 \end{cases}$$

h) $$\begin{cases} \min \displaystyle\int_{1}^{e} (t\dot{x}^2 + 2x)\,\mathrm{d}t \\ x(1) = 1 \\ x(e) = 0 \end{cases}$$

i)* $$\begin{cases} \min\limits_{u} \displaystyle\int_{0}^{2} (u^2 + 4x)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = 0 \\ x(2) = 2 \\ u \ge 0 \end{cases}$$

l) $$\begin{cases} \min\limits_{u} \displaystyle\int_{0}^{2} (x - u)\;\mathrm{d}t + x(2) \\ \dot{x} = 1 + u^2 \\ x(0) = 1 \end{cases}$$

m) $$\begin{cases} \min\limits_{u} \displaystyle\int_{0}^{1} u^2\;\mathrm{d}t + (x(1))^2 \\ \dot{x} = x + u \\ x(0) = 1 \end{cases}$$

n)
$$\begin{cases} \min\limits_u \int_0^1 (2-5t)u \ \mathrm{d}t \\ \dot{x} = 2x + 4te^{2t}u \\ x(0) = 0 \\ x(1) = e^2 \\ |u| \le 1 \end{cases}$$

o)
$$\begin{cases} \min\limits_u \int_0^1 u^2 \ \mathrm{d}t \\ \dot{x} = -2x + u \\ x(0) = 1 \\ x(1) = 0 \end{cases}$$

1.3 Using Arrow's sufficient condition

a)
$$\begin{cases} \max \int_0^4 (1-u)x \, \mathrm{d}t \\ \dot{x} = ux \\ x(0) = 2 \\ 0 \le u \le 1 \end{cases}$$

b)
$$\begin{cases} \max\limits_u \int_0^5 x_2 \, \mathrm{d}t \\ \dot{x}_1 = 2ux_1 \\ \dot{x}_2 = 2(1-u)x_1 \\ x_1(0) = 1 \\ x_2(0) = 3 \\ 0 \le u \le 1 \end{cases}$$

c)
$$\begin{cases} \max \int_{-1}^1 (tx - u^2) \, \mathrm{d}t \\ \dot{x} = x + u^2 \\ x(-1) = -\frac{2}{e} - 1 \\ 0 \le u \le 1 \end{cases}$$

1.4 Singular control

a)
$$\begin{cases} \min \int_{-1}^1 (x - 1 + t^2)^2 \, \mathrm{d}t \\ \dot{x} = u \\ |u| \le 1 \end{cases}$$

b)
$$\begin{cases} \min \int_{-1}^1 (x - e^t)^2 \, \mathrm{d}t \\ \dot{x} = u \\ |u| \le 1 \end{cases}$$

1.5 Abnormal controls

In the next exercises, find the optimal control and prove that it is abnormal.

a) $$\begin{cases} \max \int_0^1 \left(t - \frac{1}{2}\right) u \, \mathrm{d}t \\ \dot{x}_1 = u \\ \dot{x}_2 = (x_1 - tu)^2 \\ x_1(0) = 0 \\ x_2(0) = x_2(1) = 0 \end{cases}$$

b) $$\begin{cases} \max \int_0^1 (u_1 - 2u_2) \, \mathrm{d}t \\ \dot{x} = (u_1 - u_2)^2 \\ x(0) = x(1) = 0 \\ |u_1| \leq 1 \\ |u_2| \leq 1 \end{cases}$$

c) $$\begin{cases} \max \int_0^1 u \, \mathrm{d}t \\ \dot{x} = (u - u^2)^2 \\ x(0) = 0 \\ x(1) = 0 \\ 0 \leq u \leq 2 \end{cases}$$

1.6 Infinite horizon problems

a) $$\begin{cases} \min \int_0^\infty e^{-2t}(x^2 + u) \, \mathrm{d}t \\ \dot{x} = u \\ x(0) = -1 \\ \lim_{t \to \infty} x(t) = -1 \end{cases}$$

b) $$\begin{cases} \min \int_0^\infty e^{-t}(2x^2 + 3x + u + u^2) \, \mathrm{d}t \\ \dot{x} = u \\ x(0) = 1 \\ \lim_{t \to \infty} x(t) = -1 \end{cases}$$

c) $$\begin{cases} \min \int_0^\infty e^{2t}(\dot{x}^2 + 3x^2) \, \mathrm{d}t \\ x(0) = 2 \end{cases}$$

d) $$\begin{cases} \min \int_1^\infty (t^4 \dot{x}^2 + 4t^2 x^2) \, \mathrm{d}t \\ x(1) = 1 \end{cases}$$

e)* $$\begin{cases} \max \int_0^\infty e^{-3t} \ln u \, \mathrm{d}t \\ \dot{x} = 2x - u \\ x(0) = 4 \\ u \geq 0 \\ \lim_{t \to \infty} x(t) = 0 \end{cases}$$

f)*
$$\begin{cases} \min \displaystyle\int_0^\infty e^{-2t}(u^2 + 3x^2)\,\mathrm{d}t \\ \dot{x} = u \\ |u| \le 1 \\ x(0) = 2 \\ \displaystyle\lim_{t\to\infty} x(t) = 0 \end{cases}$$

g)
$$\begin{cases} \min \displaystyle\int_0^\infty e^{-2t}(u^2 + 3x^2)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = 2 \\ \displaystyle\lim_{t\to\infty} x(t) = 0 \end{cases}$$

h)*
$$\begin{cases} \max \displaystyle\int_0^\infty e^{-t/2}(x - u)\,\mathrm{d}t \\ \dot{x} = ue^{-t} \\ x(0) = 1 \\ 0 \le u \le 1 \end{cases}$$

1.7 Time optimal problems

a)
$$\begin{cases} \min\limits_u T \\ \ddot{x} = u \\ x(0) = \dot{x}(0) = -1 \\ x(T) = \dot{x}(T) = 0 \\ |u| \le 1 \end{cases}$$

Suggestion: use an existence result in order to prove that the extremal control is optimal.

b)
$$\begin{cases} \min\limits_u T \\ \dot{x} = x + u \\ x(0) = 5 \\ x(T) = 11 \\ |u| \le 1 \end{cases}$$

Suggestion: use the Gronwall's inequality in order to prove that the extremal control is optimal.

c)
$$\begin{cases} \min\limits_u T \\ \dot{x} = x + \frac{3}{u} \\ x(0) = 1 \\ x(T) = 2 \\ u \ge 3 \end{cases}$$

Suggestion: use the Gronwall's inequality in order to prove that the extremal control is optimal.

d)
$$\begin{cases} \min\limits_u T \\ \dot{x} = 2x + \frac{1}{u} \\ x(0) = \frac{5}{6} \\ x(T) = 2 \\ 3 \le u \le 5 \end{cases}$$

1.8 Constraints problems

a)
$$\begin{cases} \max \displaystyle\int_0^1 (v - x)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = \dfrac{1}{8} \\ u \in [0,1] \\ v^2 \leq x \end{cases}$$

b)
$$\begin{cases} \max \displaystyle\int_0^1 x\,\mathrm{d}t \\ \dot{x} = x + u \\ x(0) = 0 \\ |u| \leq 1 \\ 2 - x - u \geq 0 \end{cases}$$

c)
$$\begin{cases} \max \displaystyle\int_0^3 (4 - t)u\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = 0 \\ x(3) = 3 \\ t + 1 - x \geq 0 \\ u \in [0,2] \end{cases}$$

2 Optimal control with dynamic programming

Find the value function, the optimal control function and the optimal state function of the following problems.

2.1 The "simplest problem"

In this first section we consider optimal control problems where appear only a initial condition on the trajectory.

a) $$\begin{cases} \min \displaystyle\int_1^2 2xe^t \, \mathrm{d}t \\ \dot{x} = \dfrac{e^t}{u} + x \\ x(1) = -e/4 \\ 1 \le u \le 2 \end{cases}$$

In order to solve B–H–J equation, we suggest to find the solution in the family of functions $\mathcal{F} = \{V(t,x) = Axe^t + Bxe^{-t} + Ct + De^{2t} + E, \ A,B,C,D,E \in \mathbb{R}\}$.

b) $$\begin{cases} \max \displaystyle\int_{-1}^1 tx \, \mathrm{d}t \\ \dot{x} = u \\ x(-1) = 2 \\ 0 \le u \le 1 \end{cases}$$

In order to solve B–H–J equation, we suggest to find the solution in the family of functions $\mathcal{F} = \{V(t,x) = A + Bt + Cx + Dt^3 + Ext^2, \ A,B,C,D,E \in \mathbb{R}\}$.

c) $$\begin{cases} \min \displaystyle\int_{-1}^1 (tx + u^2) \, \mathrm{d}t \\ \dot{x} = x + 2u \\ x(-1) = 0 \end{cases}$$

In order to solve B–H–J equation, we suggest to find the solution in the family of functions $\mathcal{F} = \{V(t,x) = Ax + Btx + Ct^3 + Dt^2 + Et + F, \ A,B,C,D,E,F \in \mathbb{R}\}$.

d) $$\begin{cases} \max \displaystyle\int_0^1 (tx - u^2) \, \mathrm{d}t \\ \dot{x} = 1 - 4u \\ x(0) = 0 \end{cases}$$

In order to solve B–H–J equation, we suggest to find the solution in the family of functions $\mathcal{F} = \{V(t,x) = At^5 + Bt^4 + Ct^3 + Dt^2x + Et + Fx + G, \ A,B,C,D,E,F,G \in \mathbb{R}\}$.

e) $$\begin{cases} \max \displaystyle\int_0^2 (2x - 4u) \, \mathrm{d}t \\ \dot{x} = x + u \\ x(0) = 5 \\ 0 \le u \le 2 \end{cases}$$

In order to solve the PDE $Ax + xF_x + F_t = 0$ (with A constant), we suggest to find the solution in the family of functions $\mathcal{F} = \{F(t,x) = ax + bxe^{-t} + c, \ a,b,c \in \mathbb{R}\}$; for the PDE $Ax + xF_x + BF_x + F_t + C = 0$ (with A, B and C constants), we suggest the family $\mathcal{F} = \{F(t,x) = ax + bt + ce^{-t} + dxe^{-t} + f, \ a,b,c,d,f \in \mathbb{R}\}$.

f) $$\begin{cases} \max \displaystyle\int_0^4 (u - x + t)\,\mathrm{d}t \\ \dot{x} = \dfrac{t}{u} + x \\ x(0) = 1 \\ 1 \le u \le 2 \end{cases}$$

In order to solve the PDE $F_t - x + t + xF_x + AtF_x + B = 0$ (with A and B constants), we suggest to find the solution in the family of functions $\mathcal{F} = \{F(t,x) = a + bx + ct + dt^2 + (fx + g + ht)e^{-t},\ a,b,c,d,f,g,h \in \mathbb{R}\}$.

g) $$\begin{cases} \max \displaystyle\int_0^3 (1 - u)x\,\mathrm{d}t \\ \dot{x} = ux \\ x(0) = 1 \\ 0 \le u \le 1 \end{cases}$$

In order to solve the PDE $AxF_x + BF_t = 0$ (with A and B constants), we suggest to find the solution in the family of functions $\mathcal{F} = \{F(t,x) = axe^{-t},\ \text{with } a \text{ constant}\}$.

h) $$\begin{cases} \max \displaystyle\int_0^1 (x - u^2)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = 2 \end{cases}$$

In order to solve the PDE $x + A(F_x)^2 + BF_t = 0$ (with A and B constants), we suggest to find the solution in the family of functions $\mathcal{F} = \{F(t,x) = at^3 + bt^2 + ct + dx + fxt + g,\ a,b,c,d,f,g \in \mathbb{R}\}$.

i) $$\begin{cases} \min \displaystyle\int_0^2 (x^2 + u^2)\,\mathrm{d}t \\ \dot{x} = x + u \\ x(0) = 2 \\ u \ge 0 \end{cases}$$

In order to solve the PDE $xF_x + Ax^2 + F_t = 0$ (with A constant), we suggest to find the solution in the family of functions $\mathcal{F} = \{F(t,x) = x^2G(t),\ \text{with } G = G(t) \text{ function}\}$.

l) $$\begin{cases} \max \displaystyle\int_0^2 (2tx - u^2)\,\mathrm{d}t \\ \dot{x} = 1 - u^2 \\ x(0) = 0 \\ 0 \le u \le 1 \end{cases}$$

In order to solve the BHJ equation, we suggest to find the solution in the family of functions $\mathcal{F} = \{F(t,x) = At^3 + Bxt^2 + Ct + Dx + E,\ \text{with } A,B,C,D,E \text{ constants}\}$.

m)* $$\begin{cases} \min \displaystyle\int_0^2 (x^2 + u^2)\,\mathrm{d}t \\ \dot{x} = x + u \\ x(0) = -2 \\ u \ge 0 \end{cases}$$

In order to solve the PDE $xF_x + Ax^2 + BF_x^2 + F_t = 0$ (with A and B constants), we suggest to find the solution in the family of functions $\mathcal{F} = \{F(t,x) = x^2G(t),\ \text{with } G = G(t) \text{ function}\}$.

2.2 More general problems

a) $$\begin{cases} \min\limits_u \int_0^1 u^2 \, dt + (x(1))^2 \\ \dot{x} = x + u \\ x(0) = 1 \end{cases}$$

In order to solve BHJ equation, we suggest to find the solution in the family of functions

$\mathcal{F} = \{V(t, x) = h(t)x^2, \ h \in C^1(\mathbb{R})\}.$

b) $$\begin{cases} \min\limits_u \int_0^2 (x - u) \, dt + x(2) \\ \dot{x} = 1 + u^2 \\ x(0) = 1 \end{cases}$$

In order to solve BHJ equation, we suggest to find the solution in the family of functions

$\mathcal{F} = \{V(t, x) = A + Bt + Ct^2 + D\ln(3 - t) + E(3 - t)x, \text{ with } A, B, C, D, E \text{ constants}\}.$

c)* $$\begin{cases} \min\limits_u \int_0^2 (u^2 + 4x) \, dt \\ \dot{x} = u \\ x(0) = A \qquad\qquad |A| < 2 \text{ fixed} \\ x(2) = 2 \\ u \geq 0 \end{cases}$$

In order to solve the BHJ equation we suggest to consider the family of functions

$\mathcal{F} = \{V(t, x) = a(t - 2)^3 + b(x + 2)(t - 2) + c\frac{(x-2)^2}{t-2}, \text{ with } a, b, c \text{ non zero constants } \}.$

d)* $$\begin{cases} \max \int_{-1}^0 -\frac{(|u| + 2)^2}{4} \, dt + |x(0)| \\ \dot{x} = u \\ |u| \leq 2 \\ x(-1) = 1 \end{cases}$$

i. Prove that $V(t, x) = |x| + t$ is a viscosity solution of BHJ system associated to the problem;

ii. Find the optimal control.

e) $$\begin{cases} \max\limits_u \left(-\frac{1}{2} x_1(1)^2 + x_2(1) \right) \\ \dot{x_1} = x_1 + \sqrt{2}u \\ \dot{x_2} = -u^2 \\ x_1(0) = 1 \\ x_2(0) = 0 \end{cases}$$

In order to solve the BHJ equation we suggest to consider the family of functions

$\mathcal{F} = \{V(t, x_1, x_2) = ax_1^2 + bx_2, \text{ with } a = a(t), \ b = \textit{constant} \ \}.$

f) $$\begin{cases} \min\limits_u \int_0^1 u^2 \, dt \\ \dot{x} = u \\ x(0) = 0 \\ x(1) = 1 \end{cases}$$

Find the value function $V = V(t, x)$ and the optimal control.

In order to solve the BHJ equation we suggest to consider the family of functions

$\mathcal{F} = \{V(t, x) = a + bx + cx^2, \text{ with } a = a(t), \ b = b(t), \ c = c(t)\}.$

2.3 Infinite horizon problems

Find the current value function, the optimal control and the optimal state function of the following problems:

a) $$\begin{cases} \min \displaystyle\int_0^\infty e^{-2t}(u^2 + 3x^2)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = 1 \end{cases}$$

In order to solve B–H–J equation for the current value function, we suggest to find the solution in the family of functions $\mathcal{F} = \{V^c(x) = Ax^2,\ A \in \mathbb{R}\}$.

b) $$\begin{cases} \max \displaystyle\int_0^\infty e^{-2t} \ln u\,\mathrm{d}t \\ \dot{x} = x - u \\ x(0) = 1 \\ u \geq 0 \end{cases}$$

In order to solve B–H–J equation for the current value function, we suggest to find the solution in the family of functions $\mathcal{F} = \{V^c(x) :\ (V^c)'(x) = Ax^k,\ A, k \in \mathbb{R}\}$.

c) $$\begin{cases} \max \displaystyle\int_0^\infty 2\sqrt{u}e^{-2t}\,\mathrm{d}t \\ \dot{x} = 2x - u \\ x(0) = 1 \\ x \geq 0 \\ u \geq 0 \end{cases}$$

In order to solve B–H–J equation for the current value function, we suggest to find the solution in the family of functions $\mathcal{F} = \{V^c(x) = A\sqrt{x},\ A \in \mathbb{R}\}$.

3 Solutions.

Exercise 1.1:

a) The optimal solution is

$$u^*(t) = \begin{cases} 0 & \text{for } 1 \le t \le 2 \\ 2 & \text{for } 2 < t \le 3 \end{cases} \qquad x^*(t) = \begin{cases} 0 & \text{for } 1 \le t \le 2 \\ 10e^{2t-4} - 4t - 2 & \text{for } 2 < t \le 3 \end{cases}$$

b) The optimal control is $u^* = \frac{2-t}{2}e^t$ and the optimal state variable is $x^* = \frac{3}{8}e^{-t} + \left(\frac{5}{8} - \frac{t}{4}\right)e^t$.

c) The optimal control is $u^* = \frac{-2}{1+2e^{-2}}e^{-3t} + \frac{2}{1+2e^{-2}}e^{3t-2}$ and the optimal state variable is $x^* = \frac{1}{1+2e^{-2}}e^{-3t} + \frac{2}{1+2e^{-2}}e^{3t-2}$.

d) The optimal control is $u^*(t) = \frac{e+1}{e^2+1}e^t - \frac{e^2-e}{e^2+1}e^{-t}$ and the optimal state variable is $x^*(t) = \frac{e+1}{e^2+1}e^t + \frac{e^2-e}{e^2+1}e^{-t} - 1$.

e) The optimal control is $u^*(t) = (1-t)/2$ and the optimal state variable is $x^*(t) = (2t - t^2)/4$.

f) The optimal control is $u^* = 2$ and the optimal state variable is $x^*(t) = (t-1)e^t/2$.

g) The optimal control is

$$u^*(t) = \begin{cases} 2 & \text{for } 0 \le t \le 2 - \log 3 \\ 0 & \text{for } 2 - \log 3 < t \le 2 \end{cases}$$

The exercise is solved in [1].

h) The optimal control is $u^* = 0$ and the optimal trajectory is $x^*(t) = 2e^{t-1}$. The exercise is solved in [1].

i) The optimal control is $u^* = 2$ and the optimal trajectory is $x^* = e^{-2}t + t^4/4 - 1/4$.

l) The optimal control is $u^* = 2$ and the optimal trajectory is $x^* = (3e^t - t - 1)/2$.

m) The optimal solution is

$$u^*(t) = \begin{cases} 0 & \text{for } -1 \le t < -1/2 \\ 2 & \text{for } -1/2 \le t < 0 \\ 0 & \text{for } 0 \le t < 1/2 \\ 2 & \text{for } 1/2 \le t \le 1 \end{cases} \qquad x^*(t) = \begin{cases} 1 & \text{for } -1 \le t < -1/2 \\ t^2 + 3/4 & \text{for } -1/2 \le t < 0 \\ 3/4 & \text{for } 0 \le t < 1/2 \\ t^2 + 1/2 & \text{for } 1/2 \le t \le 1 \end{cases}$$

n) The optimal solution does not exist.

o) The optimal solution is $u^*(t) = 0$ and $x^*(t) = 2e^{3+t}$.

p) It is a calculus of variation problem and the optimal trajectory is

$$x^*(t) = \frac{(4+\sqrt{2})e^{t/\sqrt{2}} + (4e^2 - e^2\sqrt{2})e^{-t/\sqrt{2}}}{4 + \sqrt{2} + 4e^2 - e^2\sqrt{2}}.$$

The exercise is solved in [1].

Exercise 1.2:

a) The optimal control is $u^*(t) = t + 1/2$ and the optimal state variable is $x^*(t) = -t^2/2 + t/2 + 1$.

b) The optimal solution is

$$u^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le 6 \\ -1 & \text{for } 6 < t \le 11 \end{cases} \qquad x^*(t) = \begin{cases} t & \text{for } 0 \le t \le 6 \\ -t + 12 & \text{for } 6 < t \le 11 \end{cases}$$

c) The optimal solution is

$$u^*(t) = \begin{cases} 0 & \text{for } -1 \le t \le \tau \\ 3 & \text{for } \tau < t \le 1 \end{cases} \qquad x^*(t) = \begin{cases} -\frac{7}{2}e^{-2t-2} + \frac{1}{2}t - \frac{1}{4} & \text{for } -1 \le t \le \tau \\ \frac{1}{2}t - \frac{7}{4} & \text{for } \tau < t \le 1 \end{cases}$$

with $\tau = \frac{1}{2}\ln\frac{7}{3} - 1$.

d) The optimal solution is

$$u^*(t) = \begin{cases} 2 & \text{for } 0 \le t \le \ln 4 \\ 0 & \text{for } \ln 4 < t \le 4 \end{cases} \qquad x^*(t) = \begin{cases} 2(e^t - 1) & \text{for } 0 \le t \le \ln 4 \\ \frac{3}{2}e^t & \text{for } \ln 4 < t \le 4 \end{cases}$$

The solution is presented in [1].

e) The optimal solution is $u^*(t) = 3$ and $x^*(t) = 11e^{4-t} - 3t + 3$.

f) The optimal solution is $u^*(t) = 1$ and $x^*(t) = \frac{1}{e}t + \frac{1}{3}t^3 - 1 - \frac{1}{3}e^3$.

g) The optimal control is

$$u^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le \sqrt{2\ln 2} \\ 3 & \text{for } \sqrt{2\ln 2} < t \le 2 \end{cases}.$$

h) The optimal trajectory is $x^*(t) = t - e\ln t$.

i) The optimal control is

$$u^*(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 - \sqrt{2} \\ 2(t - 2 + \sqrt{2}) & \text{if } 2 - \sqrt{2} \le t \le 2 \end{cases}$$

and the optimal trajectory is

$$x^*(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 - \sqrt{2} \\ \left(t - 2 + \sqrt{2}\right)^2 & \text{if } 2 - \sqrt{2} \le t \le 2 \end{cases}$$

The exercise is solved in [1] (see a problem of inventory and production I).

l) The optimal control is $u^*(t) = \frac{1}{2(3-t)}$ with trajectory $x^*(t) = t + \frac{1}{4(3-t)} + \frac{11}{12}$.

m) The optimal control is $u^*(t) = -\frac{2}{1+e^2}e^{2-t}$ with trajectory $x^*(t) = \frac{e^t + e^{2-t}}{1+e^2}$.

n) The optimal control is

$$u^*(t) = \begin{cases} -1 & \text{if } 0 \le t \le 1/2 \\ 1 & \text{if } 1/2 < t \le 1 \end{cases}$$

and the optimal trajectory is

$$x^*(t) = \begin{cases} -2t^2 e^{2t} & \text{if } 0 \le t \le 1/2 \\ (2t^2 - 1)e^{2t} & \text{if } 1/2 < t \le 1 \end{cases}$$

o) The optimal control is $u^*(t) = -\frac{4e^{2t}}{e^4-1}$ with trajectory $x^*(t) = \frac{e^{-2t+4} - e^{2t}}{e^4-1}$.

Exercise 1.3:

a) The optimal control and the optimal trajectory are

$$u^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le 3 \\ 0 & \text{for } 3 < t \le 4 \end{cases}, \qquad x^*(t) = \begin{cases} 2e^t & \text{for } 0 \le t \le 3 \\ 2e^3 & \text{for } 3 < t \le 4 \end{cases}.$$

The solution is presented in [1] as a problem of business strategy.

b) The optimal control is

$$u^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le 4, \\ 0 & \text{for } 4 < t \le 5 \end{cases}$$

and the optimal trajectory is

$$x_1^*(t) = \begin{cases} e^{2t} & \text{for } 0 \le t \le 4, \\ e^8 & \text{for } 4 < t \le 5 \end{cases} \qquad x_2^*(t) = \begin{cases} 3 & \text{for } 0 \le t \le 4, \\ 3 + 2e^8(t-4) & \text{for } 4 < t \le 5 \end{cases}$$

The solution is presented in [1] as a two-sector model.

c) The optimal control and the optimal trajectory are

$$u^*(t) = \begin{cases} 1 & \text{for } -1 \le t \le 0, \\ 0 & \text{for } 0 < t \le -1 \end{cases} \qquad x^*(t) = \begin{cases} -2e^t - 1 & \text{for } -1 \le t \le 0, \\ -3e^t & \text{for } 0 < t \le 1 \end{cases}$$

Exercise 1.4:

a) The optimal control and the optimal trajectory are

$$u^*(t) = \begin{cases} -2t & \text{for } |t| \le \frac{1}{4} \\ -\text{sgn}(t) & \text{for } \frac{1}{4} < |t| \le 1 \end{cases} \qquad x^*(t) = \begin{cases} 1 - t^2 & \text{for } |t| \le \frac{1}{4} \\ -|t| + \frac{19}{16} & \text{for } \frac{1}{4} < |t| \le 1 \end{cases}$$

b) The optimal control and the optimal trajectory are

$$u^*(t) = \begin{cases} e^t & \text{for } -1 \le t < \alpha \\ 1 & \text{for } \alpha \le t \le 1 \end{cases} \qquad x^*(t) = \begin{cases} e^t & \text{for } -1 \le t < \alpha \\ t + e^\alpha - \alpha & \text{for } \alpha \le t \le 1 \end{cases}$$

where $\alpha \in (-1, 0)$ such that $\dfrac{1}{2} + 2e^\alpha + \dfrac{1}{2}\alpha^2 - e - \alpha - \alpha e^\alpha = 0$. The solution is presented in [1].

Exercise 1.5:

a) Every constant function u is optimal and abnormal.

b) The function $\mathbf{u}^* = (u_1, u_2) = (-1, -1)$ is the optimal and abnormal control.

c) The function $u^* = 1$ is the optimal and abnormal control. The solution is presented in [1].

Exercise 1.6:

a) The optimal control is $u^*(t) = 0$ and the optimal state variable is $x^*(t) = -1$.

b) The optimal control is $u^*(t) = -2e^{-t}$ and the optimal state variable is $x^*(t) = 2e^{-t}-1$.

c) The optimal control is $u^*(t) = -6e^{-3t}$ and the optimal state variable is $x^*(t) = 2e^{-3t}$.

d) The optimal solution is $x^*(t) = \frac{1}{t^4}$.

e) The optimal solution is $u^*(t) = 12e^{-t}$ and $x^*(t) = 4e^{-t}$. The solution is presented in [1] using the current Hamiltonian in a model of optimal consumption.

f) The optimal solution is

$$u^*(t) = \begin{cases} -1, & \text{if } 0 \le t < 1 \\ -e^{1-t}, & \text{if } t \ge 1 \end{cases} \qquad x^*(t) = \begin{cases} 2-t, & \text{if } 0 \le t < 1 \\ e^{1-t}, & \text{if } t \ge 1 \end{cases}$$

The solution is presented in [1] with the current Hamiltonian.

g) The optimal solution is $u^*(t) = -2e^{-t}$ with optimal trajectory $x^*(t) = 2e^{-t}$. The solution is presented in [1] with the current Hamiltonian.

h) The optimal solution is

$$u^*(t) = \begin{cases} 1, & \text{if } 0 \le t \le \ln 2 \\ 0, & \text{if } t > \ln 2 \end{cases}.$$

Exercise 1.7:

a) If we put $\dot{x} = x_1,\ x = x_2$, we obtain the optimal time $T^* = 1 + \sqrt{6}$ and the optimal situation

	u	$x_1 = \dot{x}$	$x_2 = x$
in $\left[0, 1 + \frac{\sqrt{6}}{2}\right)$	1	$t - 1$	$\frac{1}{2}t^2 - t - 1$
in $\left[1 + \frac{\sqrt{6}}{2}, 1 + \sqrt{6}\right]$	-1	$-t + 1 + \sqrt{6}$	$-\frac{1}{2}t^2 + \left(1 + \sqrt{6}\right)t - \frac{1}{2}\left(1 + \sqrt{6}\right)^2$

See the classical example of Pontryagin in [1].

b) The optimal control is $u^* = 1$ with exit time $T^* = \ln 2$ and trajectory $x^* = 6e^t - 1$. The solution is presented in [1].

c) The optimal control is $u^* = 3$ with exit time $T^* = \ln \frac{3}{2}$ and trajectory $x^* = 2e^t - 1$.

d) The optimal control is $u^* = 3$ with exit time $T^* = \frac{1}{2} \ln \frac{13}{6}$ and trajectory $x^* = e^{2t} - \frac{1}{6}$. The solution is presented in [1].

Exercise 1.8:

a) The Lagrangian L is $L = v - x + \lambda u + \mu_1 u + \mu_2(1-u) + \mu_3(x - v^2)$. We have

	x	u	v	λ	μ_1	μ_2	μ_3
in $[0, \frac{1}{8})$	$t + \frac{1}{8}$	1	$\sqrt{t + \frac{1}{8}}$	$t - \sqrt{t + \frac{1}{8}} + \frac{3}{8}$	0	$t - \sqrt{t + \frac{1}{8}} + \frac{3}{8}$	$\frac{1}{2\sqrt{t+\frac{1}{8}}}$
in $[\frac{1}{8}, 1]$	$\frac{1}{4}$	0	$\frac{1}{2}$	0	0	0	1

Exercise proposed in [3] and solved in [1].

b) The Lagrangian is $L = x + \lambda(x+u) + \mu_1(1-u) + \mu_2(1+u) + \mu_3(2-x-u)$. We have

	x	u	λ	μ_1	μ_2	μ_3
in $[0, \ln 2)$	$e^t - 1$	1	$(4 - 2\ln 2)e^{-t} - 1$	$(4 - 2\ln 2)e^{-t} - 1$	0	0
in $[\ln 2, 1]$	$2t + 1 - 2\ln 2$	$-2t + 1 + 2\ln 2$	$1 - t$	0	0	$1 - t$

Exercise proposed and solved in [3].

c) The Lagrangian is $L = (4-t)u + \lambda u + \mu(t + 1 - x)$. We obtain the following situation:

	x	u	λ	μ
in $[0, 1)$	$2t$	2	-3	0
in $[1, 2]$	$t + 1$	1	$t - 4$	1
in $(2, 3]$	3	0	-2	0

Exercise proposed in [3] and solved in [1].

Exercise 2.1:

a) The value function is $V = -xe^t + xe^{4-t} - \frac{e^4}{2}t + \frac{1}{4}e^{2t} + \frac{3}{4}e^4$, the optimal control is $u^* = 2$ and the optimal trajectory is $x^* = -3/4e^t + te^t/2$.

b) The value function is $V = 1/3 - t/2 + x/2 + t^3/6 - xt^2/2$, the optimal control is $u^* = 1$ and the optimal trajectory is $x^* = t + 3$.

c) The value function is $V = +x - tx + t^3/3 - t^2 + t - 1/3$, the optimal control is $u^* = t - 1$ and the optimal trajectory is $x^* = -2e^{t+1} - 2t$.

d) The value function is $V = -t^5/5 + 5/6t^3 - t^2x/2 - 3/2t + x/2 + 13/15$, the optimal control is $u^* = t^2 - 1$ and the optimal trajectory is $x^* = -4/3t^3 + 5t$.

e) The value function is

$$V(t,x) = \begin{cases} -2x + 12t + 4e^{2-t} + 2xe^{2-t} + 12(\log 3 - 3) & \text{if } 0 \le t \le 2 - \log 3, \\ -2x + 2xe^{2-t} & \text{if } 2 - \log 3 < t \le 2. \end{cases}$$

the optimal control is

$$u^*(t) = \begin{cases} 2 & \text{if } 0 \le t < 2 - \log 3, \\ 0 & \text{if } 2 - \log 3 \le t \le 2. \end{cases}$$

and the optimal trajectory is

$$x^*(t) = \begin{cases} 7e^t - 2 & \text{if } 0 \le t \le 2 - \log 3, \\ (7e^2 - 6)e^{t-2} & \text{if } 2 - \log 3 < t \le 2. \end{cases}$$

The solution is presented in [1].

f) The value function is $V = \frac{45}{2} + x - 2t - \frac{3}{4}t^2 - (x + \frac{1}{2} + \frac{1}{2}t)e^{4-t}$, the optimal control is $u^* = 2$ and the optimal trajectory is $x^* = (3e^t - t - 1)/2$.

g) The optimal control and the optimal trajectory are

$$u^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le 2 \\ 0 & \text{for } 2 < t \le 3 \end{cases}, \qquad x^*(t) = \begin{cases} e^t & \text{for } 0 \le t \le 2 \\ e^2 & \text{for } 2 < t \le 3 \end{cases}.$$

The solution is presented in [1] as a problem of business strategy.

h) The value function is $V = -\frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{4}t + x - xt + \frac{1}{12}$, the optimal control is $u^* = (1-t)/2$ and the optimal trajectory is $x^* = (2t - t^2)/4 + 2$. The solution is presented in [1].

i) The value function is $V = x^2(e^{4-2t} - 1)/2$, for $x \ge 0$ and the optimal control is $u^* = 0$ and the optimal trajectory is $x^* = 2e^t$. The solution is presented in [1].

l) The value function is $V = t^3/3 - xt^2 - 4t + 4x + 16/3$, the optimal control is $u^* = 0$ and the optimal trajectory is $x^* = t$.

m) The value function is

$$V(t,x) = -x^2 \frac{e^{\sqrt{2}t} - e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{\sqrt{2}(4-t)}}, \qquad \forall (t,x) \in [0,2] \times (-\infty, 0).$$

The optimal control is

$$u^* = -2\frac{e^{\sqrt{2}t} - e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1) + (\sqrt{2}-1)e^{4\sqrt{2}}}$$

and the optimal trajectory is

$$x^* = -2\frac{(\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1) + (\sqrt{2}-1)e^{4\sqrt{2}}}.$$

The solution is presented in [1].

Exercise 2.2:

a) The value function is $V = \frac{2x^2}{1+e^{2t-2}}$, the optimal control is $u^* = -\frac{2}{1+e^2}e^{2-t}$, and the optimal trajectory is $x^* = \frac{e^t+e^{2-t}}{1+e^2}$.

b) The value function is $V = 4 - 3t + \frac{1}{2}t^2 - \frac{1}{4}\ln(3-t) + (3-t)x$, the optimal control is $u^*(t) = \frac{1}{2(3-t)}$ with trajectory $x^*(t) = t + \frac{1}{4(3-t)} + \frac{11}{12}$.

c) In this case we obtain that

$$V(t,x) = \begin{cases} \infty & \text{if } 0 \le t < 2 \text{ and } x > 2 \\ \infty & \text{if } t = 2 \text{ and } x \neq 2 \\ 0 & \text{if } t = 2 \text{ and } x = 2 \\ 4x(2-t) + \dfrac{8}{3}\sqrt{(2-x)^3} & \text{if } 0 \le t < 2,\ x < 2 \\ & \quad \text{and } x \ge 2 - (t-2)^2 \\ \dfrac{1}{3}(t-2)^3 - 2(x+2)(t-2) - \dfrac{(x-2)^2}{t-2} & \text{if } 0 \le t < 2,\ x < 2 \\ & \quad \text{and } x < 2 - (t-2)^2 \end{cases}$$

Here $\tau = 2 - \sqrt{2-A}$ and the optimal trajectory is

$$x^*(t) = \begin{cases} A & \text{for } t \in [0,\tau] \\ (t-\tau)^2 + A & \text{for } t \in (\tau, 2] \end{cases}$$

The optimal control is given by

$$u^*(t) = \begin{cases} 0 & \text{for } t \in [0,\tau] \\ 2(t-\tau) & \text{for } t \in (\tau, 2] \end{cases}$$

The solution is presented in [1].

d) The optimal control is $u^* = 0$. The solution is presented in [1].

e) The optimal control is $u^*(t) = -\dfrac{\sqrt{2}e^{2-t}}{e^2+1}$. The solution is presented in example 2.7 in [2] and in [1].

f) The value function is $V(t,x) = \dfrac{(x-1)^2}{1-t}$ and optimal control is $u^*(t) = 1$.

Exercise 2.3:

a) The current value function is $V^c(x) = x^2$, the optimal control is $u^* = -e^{-t}$ and the optimal trajectory is $x^* = e^{-t}$. The solution is presented in [1].

b) The optimal control is $u^* = 2e^{-t}$ and the optimal trajectory is $x^* = e^{-t}$. The solution is presented in [1] as a model of optimal consumption.

c) The optimal control is $u^* = 2$ and the optimal trajectory is $x^* = 1$. The solution is presented in [1] as a model of optimal consumption.

References

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