Richiami di meccanica classica

$q_1 \dots q_N$ coordinate generalizzate

$\mathcal{L}(q, \dot{q}, t)$

$\uparrow$ velocità generalizzate

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \quad j = 1 \dots N$$

Invarianti esatti: se $\mathcal{L}$ non dip. q_i

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

$$\Rightarrow \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{p_i} = \text{cost}$$

p_i : momento canonico

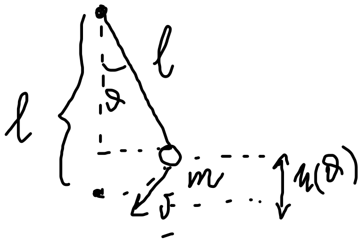
$$H = \sum_j p_j \dot{q}_j - \mathcal{L}$$

Rappresenta l'energia del sistema

$$\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$

se $\mathcal{L}$ non dip. espl. da t allora H è cost. moto

Invarianti adiabatici



$$\mathcal{L} = T - V$$

$V \to$ en. potenziale

$T \leftarrow$ en. cinetica

$q_j = \vartheta \qquad j = 1$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot\vartheta^2$$

$$v = l\dot\vartheta$$

$$V = mgh(\vartheta) = mgl(1-\cos\vartheta)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\vartheta} - \frac{\partial \mathcal{L}}{\partial \vartheta} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \vartheta} = \frac{\partial}{\partial \vartheta}\left(\cancel{\frac{1}{2} m l^2 \dot\vartheta^2} - mgl(1-\cos\vartheta)\right) = -mgl\,\mathrm{sen}\vartheta$$

$$\frac{\partial \mathcal{L}}{\partial \dot\vartheta} = \frac{1}{\cancel{2}} m l^2 \cancel{2}\dot\vartheta = m l^2 \dot\vartheta$$

$$\cancel{m} l^{\cancel{2}} \ddot\vartheta + \cancel{mg} \cancel{l}\,\mathrm{sen}\vartheta = 0$$

$$\ddot\vartheta = -\frac{g}{l}\,\mathrm{sen}\vartheta$$

$\vartheta \ll 1$

$$\ddot{\vartheta} = -\frac{g}{l}\vartheta \qquad \left[g/l\right] = s^{-2} \qquad g/l \overset{def}{=} \omega_0^2$$

$$\ddot{\vartheta} = -\omega_0^2 \vartheta \qquad \vartheta = \vartheta_0 \cos(\omega_0 t) \qquad \begin{pmatrix} \vartheta(t=0) = \vartheta_0 \\ \dot{\vartheta}(t=0) = 0 \end{pmatrix}$$

$$\vartheta = \vartheta_0 \, \mathrm{Re}\left[e^{i\omega_0 t}\right] = \vartheta_0 \, \mathrm{Re}\left[e^{i\int_0^t \omega_0 \, dt'}\right]$$

Soluzione se $l = l(t)$?

con $l \approx$ costante in un periodo di oscillazione

$$\ddot{\vartheta} = -\omega^2(t)\cdot\vartheta \qquad \omega^2(t) = g/l(t)$$

$$\vartheta(t) = \mathrm{Re}\left[\vartheta_0(t)\cdot e^{i\int_0^t \omega(t')\,dt'}\right]$$

$$\vartheta(t) = \mathrm{Re}\left[\vartheta_0(t)\ \exp\left[i\int_0^t \omega(t')\,dt'\right]\right]$$

$$\frac{d\vartheta}{dt} = \mathrm{Re}\left[\dot{\vartheta}_0\ \exp\left[i\int_0^t \omega(t')\,dt'\right] + \vartheta_0\ \exp\left[i\int_0^t \omega(t')\,dt'\right] i\omega(t)\right]$$

$$= \mathrm{Re}\left[\exp\left[i\int_0^t \omega(t')\,dt'\right]\left[\dot{\vartheta}_0 + i\omega\vartheta_0\right]\right]$$

$$\frac{d^2\vartheta}{dt^2} = \mathrm{Re}\left[i\omega\ \exp[\quad]\left[\underset{\sim}{\dot{\vartheta}_0} + i\omega\vartheta_0\right] + \exp[\quad]\cdot\left[\ddot{\vartheta}_0 + i\dot{\omega}\vartheta_0 + \underset{\sim}{i\omega}\dot{\vartheta}_0\right]\right]$$

$$= \mathrm{Re}\left[\exp[\quad]\left[2i\omega\dot{\vartheta}_0 + \ddot{\vartheta}_0 + i\dot{\omega}\vartheta_0 - \omega^2\vartheta_0\right]\right]$$

$$\ddot{\vartheta} + \omega^2 \vartheta = 0$$

$$\mathrm{Re}\left[\underbrace{\left(\overset{\text{trascuro}}{\ddot{\vartheta}_0} + 2i\dot{\vartheta}_0 \omega + i\dot{\omega}\vartheta_0 \right)} \exp\left[i\int_0^t \omega(t')\,dt' \right] \right] = 0$$

$$2\cancel{i}\dot{\vartheta}_0 \omega + \cancel{i}\dot{\omega}\vartheta_0 = 0$$

$$2\frac{d\vartheta_0}{\cancel{dt}}\omega + \frac{d\omega}{\cancel{dt}}\vartheta_0 = 0 \;;\; \int_{\vartheta_0}^{\vartheta_0(t)} \frac{d\vartheta_0}{\vartheta_0} = -\frac{1}{2}\int_{\omega_0}^{\omega(t)} \frac{d\omega}{\omega}$$

$$\ln\left[\frac{\vartheta_0(t)}{\vartheta_0}\right] = -\frac{1}{2}\ln\left[\frac{\omega(t)}{\omega_0}\right] = \ln\left[\sqrt{\frac{\omega_0}{\omega(t)}}\right]$$

$$\vartheta_0(t) = \vartheta_0 \sqrt{\frac{\omega_0}{\omega(t)}}$$

$$\vartheta(t) = \mathrm{Re}\left[\vartheta_0 \sqrt{\frac{\omega_0}{\omega(t)}} \exp\left(i\int_0^t \omega(t')dt'\right)\right] = \vartheta_0 \sqrt{\frac{\omega_0}{\omega(t)}} \cos\left(\int_0^t \omega(t')dt'\right)$$

$$\omega(t) = \sqrt{g/\ell(t)}$$

$$\left|\frac{d^2\vartheta_0(t)}{dt^2}\right| << \left|\omega \frac{d\vartheta_0}{dt}\right| \qquad \vartheta_0(t) = \vartheta_0 \sqrt{\frac{\omega_0}{\omega(t)}} \quad \omega_0 \sim \omega$$

$$\frac{d\vartheta_0}{dt} = -\frac{\vartheta_0\sqrt{\omega_0}}{2} \frac{1}{\omega^{3/2}} \frac{d\omega}{dt} \sim \frac{\vartheta_0}{\omega_0}\frac{d\omega}{dt} \quad \text{trascuro } \tfrac{1}{2}$$

$$\frac{\cancel{\vartheta_0}}{\omega^2}\left(\frac{d\omega}{dt}\right)^{\cancel{2}} << \cancel{\omega} \frac{\cancel{\vartheta_0}}{\cancel{\omega}} \frac{\cancel{d}\omega}{dt}$$

$$\boxed{\frac{1}{\omega}\frac{d\omega}{dt} << \omega}$$

$$\frac{d^2\vartheta_0}{dt^2} = -\frac{\vartheta_0\sqrt{\omega_0}}{2}\left[-\frac{3}{2}\frac{1}{\omega^{5/2}}\left(\frac{d\omega}{dt}\right)^2 + O\left(\frac{d^2\omega}{dt^2}\right)\right] \sim \frac{\vartheta_0}{\omega^2}\left(\frac{d\omega}{dt}\right)^2$$

$$\frac{1}{\omega}\frac{\Delta\omega}{\Delta t} << \omega \; ; \qquad \frac{\Delta\omega}{\omega} << \Delta t \cdot \omega$$

$$\omega = \frac{2\pi}{T}$$

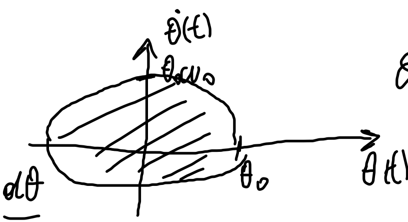
$$\frac{\Delta\omega}{\omega} << \frac{\Delta t}{T} \qquad \text{Se } \Delta t \sim T$$

$$\frac{\Delta\omega}{\omega} << 1$$

$l = \text{const}$

$$\theta(t) = \theta_0 \cos(\omega_0 t)$$
$$\dot\theta(t) = -\theta_0 \omega_0 \,\text{sen}(\omega_0 t)$$

$$\left[\frac{\theta(t)}{\theta_0}\right]^2 + \frac{\dot\theta^2}{\theta_0^2\omega_0^2} = 1$$



$$\text{Area} = \pi\theta_0\,\theta_0\omega_0 = \pi\theta_0^2\omega_0$$

$$\frac{d\theta}{dt} \;\text{a}\; p_\theta \qquad p_\theta = \frac{\partial L}{\partial\dot\theta} = ml^2\frac{d\theta}{dt}$$

$$\int_{\text{ciclo}} \frac{d\theta}{dt}\, d\theta$$

$$\int_{\text{ciclo}} p\, dq$$ Integrale di azione

$$J = \int_{\text{ciclo}} \frac{d\theta}{dt}\frac{d\theta}{dt}\, dt = \int_0^T \left(\frac{d\theta}{dt}\right)^2 dt$$

$$\frac{d}{dt}\left(\theta \frac{d\theta}{dt}\right) = \left(\frac{d\theta}{dt}\right)^2 + \theta \frac{d^2\theta}{dt^2}$$

$$= \int_0^T \left[\frac{d}{dt}\left(\theta\frac{d\theta}{dt}\right) - \theta \frac{d^2\theta}{dt^2}\right] dt = \underbrace{\cancel{\theta\frac{d\theta}{dt}\Big|_0^T}}_{=0} + \int_0^T \omega^2\theta^2\, dt$$

$$= \int_0^T \omega^{\cancel{2}} \frac{\omega_0}{\cancel{\omega}} \theta_0^2 \cos^2\left(\int_0^t \omega(t')\,dt'\right) dt$$

$$x = \int_0^t \omega(t')\,dt' \Rightarrow \frac{dx}{dt} = \omega$$

$$dx = \omega\, dt$$

$$= \theta_0^2 \omega_0 \int_0^{2\pi} \cos^2(x)\, dx = \omega_0 \theta_0^2 \pi$$

$t = 0 \Rightarrow x = 0$

$t = T$

$$x = \int_0^T \frac{d\theta}{\cancel{dt'}}\, \cancel{dt'} = \int_0^{2\pi} d\theta = 2\pi$$

Lagrangiana di una part. q in $\underline{E}(\underline{x},t)$ e $\underline{B}(\underline{x},t)$

$$\mathcal{L} = \frac{1}{2} m v^2 + q\, \underline{v} \cdot \underline{A}(\underline{x},t) - q\,\phi(\underline{x},t)$$

Pot. vettore Pot. scalare

$$\underline{E} = -\underline{\nabla}\phi - \partial \underline{A}/\partial t \qquad \underline{B} = \underline{\nabla} \times \underline{A}$$

$x, y, z \qquad x_i \qquad i = 1,2,3$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_j} - \frac{\partial \mathcal{L}}{\partial x_j} = 0 \iff \underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_j} = \frac{\partial}{\partial \dot{x}_j}\left[\frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 + q \sum_{i=1}^{3} \dot{x}_i A_i(\underline{x},t) - \cancel{q\,\phi(\underline{x},t)}\right] = \frac{1}{\cancel{2}} m \cancel{2} \dot{x}_j + q A_j(\underline{x},t)$$

$$= m\dot{x}_j + q A_j(\underline{x},t)$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_j}\right] = m\ddot{x}_j + q \sum_{i=1}^{3} \frac{\partial A_j}{\partial x_i} \cdot \dot{x}_i + q \frac{\partial A_j}{\partial t}$$

$A_j(x_1(t), x_2(t), x_3(t), t)$

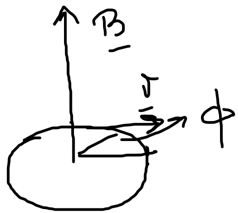
$$\frac{\partial L}{\partial x_j} = \frac{\partial}{\partial x_j}\left[q \sum_{i=1}^{3} \dot{x}_i A_i(\underline{x},t) - q\phi(\underline{x},t) \right]$$

$$= q \sum_{i=1}^{3} \dot{x}_i \frac{\partial A_i}{\partial x_j} - q \frac{\partial \phi}{\partial x_j}$$

$$m\ddot{x}_j + q\frac{\partial A_j}{\partial t} + \sum_{i=1}^{3} \frac{\partial A_j}{\partial x_i} \dot{x}_i q - q \sum_{i=1}^{3} \dot{x}_i \frac{\partial A_i}{\partial x_j} + q \frac{\partial \phi}{\partial x_j} = 0$$

$$m a_j = +q \Bigg[\overbrace{\underbrace{-\frac{\partial \phi}{\partial x_j}}_{(\underline{\nabla}\phi)_j} \underbrace{- \frac{\partial A_j}{\partial t}}_{\left(\frac{\partial \underline{A}}{\partial t}\right)_j}}^{E_j} \Bigg] + \underbrace{q \sum_{i=1}^{3} \dot{x}_i \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right)}_{[q\, \underline{v} \times (\underbrace{\underline{\nabla} \times \underline{A}}_{\underline{B}})]_j}$$

Moto di girazione



$$\mathcal{L} = \frac{1}{2} m v^2 + q\, \underline{v} \cdot \underline{A}$$

$$v^2 = v_r^2 + v_z^2 + v_\phi^2 \qquad v_\phi = r_L \dot{\phi}$$

girazione

$$\mathcal{L} = \frac{1}{2} m \left(v_r^2 + v_z^2 + r_L^2 \dot{\phi}^2 \right) + q \left(v_r A_r + v_z A_z + \underbrace{v_\phi A_\phi}_{r_L \dot{\phi} A_\phi} \right)$$

$$\phi \longrightarrow p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \cancel{\frac{1}{2}} m r_L^2 \cancel{2} \dot{\phi} + q r_L A_\phi = m r_L^2 \dot{\phi} + q r_L A_\phi$$

$$J = \int_0^{2\pi} p_\phi \, d\phi = \int_0^T m r_L^2 \frac{d\phi}{dt} \frac{d\phi}{dt} dt + \int_0^{2\pi} q r_L A_\phi \, d\phi = m v_L^2 \cdot \int_0^T dt + $$

$$r_L \cdot \frac{d\phi}{dt} = v_L \qquad m v_L^2 \cdot T$$

$A_\phi = ?$

$$\int_{\text{cerchio di Larmor}} \underline{B} \cdot d\underline{S} = \pi r_L^2 B \qquad \mathcal{J} = \frac{6\pi m}{q} \mu$$

[μ è invariante adiabatico]

$$\parallel \qquad \underline{B} = \underline{\nabla} \times \underline{A}$$

$$\int_{\text{cerchio di Larmor}} (\underline{\nabla} \times \underline{A}) \cdot d\underline{S} \overset{\text{th Stokes}}{=} \oint_{\text{circonferenza}} \underline{A} \cdot d\underline{\ell} = \oint A_\phi \, d\ell = 2\pi r_L A_\phi$$

$$\cancel{\pi} r_L^{\cancel{2}} B = \cancel{2\pi} \cancel{r_L} A_\phi ; \qquad \mu = \frac{m v_L^2}{2B}$$

$$A_\phi = \frac{r_L}{2} B$$

$$\int_0^{2\pi} q r_L \frac{r_L}{2} B \, d\phi = \frac{q r_L^2}{2} B \cdot 2\pi$$

$$\mathcal{J} = m v_L^2 \frac{2\pi m}{qB} + \frac{\cancel{2}\pi q}{\cancel{2}} B r_L^2 = \frac{4\pi m}{q} \mu + \pi \cancel{q} \cancel{B} \frac{m^2 v_L^2}{q^{\cancel{2}} B^{\cancel{2}}}$$

$$= \frac{4\pi m}{q} \mu + \frac{2\pi m}{q} \mu = \frac{6\pi m}{q} \mu$$