$$\frac{d}{dt}\rho + \rho\,\underline{\nabla}\cdot\underline{u} = 0$$ (continuità)

$$\rho\,\frac{d\underline{u}}{dt} = \underline{j}\times\underline{B} - \underline{\nabla}p$$ (cons. q.tà moto)

$$\underline{E} + \underline{u}\times\underline{B} = 0$$ (legge di Ohm ideale)

$$\underline{E} + \underline{u}\times\underline{B} = \eta\,\underline{j}$$ (" " " resistiva)

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$ (eq. stato)

$$\underline{\nabla}\times\underline{E} = -\partial\underline{B}/\partial t \qquad \underline{\nabla}\cdot\underline{B} = \underline{0} \qquad \underline{\nabla}\times\underline{B} = \mu_0\,\underline{j}$$

Equilibri in MHD

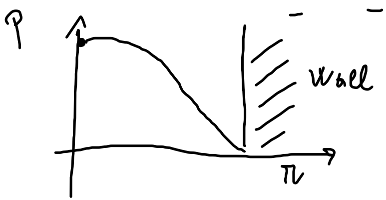
$$\frac{\partial}{\partial t} = 0 \qquad \text{stato stazionario}$$

No equilibri dinamici $\quad \underline{u} = \underline{0}$

$\underline{j} \times \underline{B} = \underline{\nabla} p$ (cons. q.tà moto) $\qquad \underline{E} = \underline{0}$ (legge di Ohm ideale)

$\rho = \text{const}$ (continuità) $\qquad \underline{B}$ non varia nel tempo (legge di Faraday)

$\underline{\nabla} \times \underline{B} = \mu_0 \underline{j}$ (th Ampere)



$$\underline{j} \times \underline{B} = \underline{\nabla} p$$

$$\underline{j} \cdot \left[\underline{j} \times \underline{B} \right] = 0 = \underline{\nabla} p \cdot \underline{j} \Rightarrow \underline{\nabla} p \perp \underline{j}$$

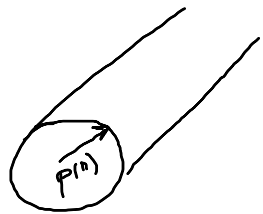
$$\underline{B} \cdot \left[\underline{j} \times \underline{B} \right] = 0 = \underline{\nabla} p \cdot \underline{B} \Rightarrow \underline{\nabla} p \perp \underline{B}$$

Sup. di livello di f sono $\perp$ al $\underline{\nabla} f$

$\underline{j}$, $\underline{B}$ si trovano sulle sup. a pressione costante

$\Downarrow$

Sup. di flusso



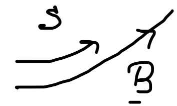
Sostituisco: $\underline{j} = \dfrac{\underline{\nabla} \times \underline{B}}{\mu_0}$

$$\frac{1}{\mu_0} \left[(\underline{\nabla} \times \underline{B}) \times \underline{B} \right] = \underline{\nabla} p$$

Proprietà

$$\underline{\nabla}\left[\underline{V}\cdot\underline{W}\right] = \underline{V}\times(\underline{\nabla}\times\underline{W}) + \underline{W}\times(\underline{\nabla}\times\underline{V}) + (\underline{V}\cdot\underline{\nabla})\underline{W} + (\underline{W}\cdot\underline{\nabla})\underline{V}$$

$$\underline{\nabla}\left(B^2\right) = 2\,\underline{B}\times(\underline{\nabla}\times\underline{B}) + 2(\underline{B}\cdot\underline{\nabla})\underline{B}$$



$\underline{V} = \underline{W} = \underline{B}$

$$\underline{B}\times(\underline{\nabla}\times\underline{B}) = \underline{\nabla}\left(\frac{B^2}{2}\right) - (\underline{B}\cdot\underline{\nabla})\underline{B}$$

$$\underline{\nabla}p = -\underline{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\underline{B}\cdot\underline{\nabla})\underline{B}$$

$$\underline{\nabla} = \underline{\nabla}_{\parallel} + \underline{\nabla}_{\perp} =$$

$$= \hat{\underline{b}}\frac{\partial}{\partial s} + \underline{\nabla}_{\perp}$$

$\underline{\nabla}_{\parallel}p = 0 \qquad \underline{\nabla}p = \underline{\nabla}_{\perp}p$

$$\underline{\nabla}\left(\frac{B^2}{2\mu_0}\right) = \hat{\underline{b}}\frac{\partial}{\partial s}\left(\frac{B^2}{2\mu_0}\right) + \underline{\nabla}_{\perp}\left(\frac{B^2}{2\mu_0}\right)$$

$$\underline{B} = B\,\hat{\underline{b}}$$

$$\beta = \frac{p}{B^2/2\mu_0} = \frac{2\mu_0 p}{B^2}$$

$$(\underline{B}\cdot\underline{\nabla})\underline{B} = (B\,\underbrace{\hat{\underline{b}}\cdot\underline{\nabla}}_{\frac{\partial}{\partial s}})\left[B\,\hat{\underline{b}}\right] = B\frac{\partial}{\partial s}\left[B\,\hat{\underline{b}}\right] =$$

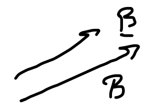
$$= \underbrace{B\frac{\partial B}{\partial s}}_{\frac{\partial}{\partial s}\left(\frac{B^2}{2}\right)}\hat{\underline{b}} + \overbrace{B^2\frac{\partial}{\partial s}\hat{\underline{b}}}^{(\hat{\underline{b}}\cdot\underline{\nabla})\hat{\underline{b}}}$$

Se $\beta > 1$: $p > B^2/2\mu_0$

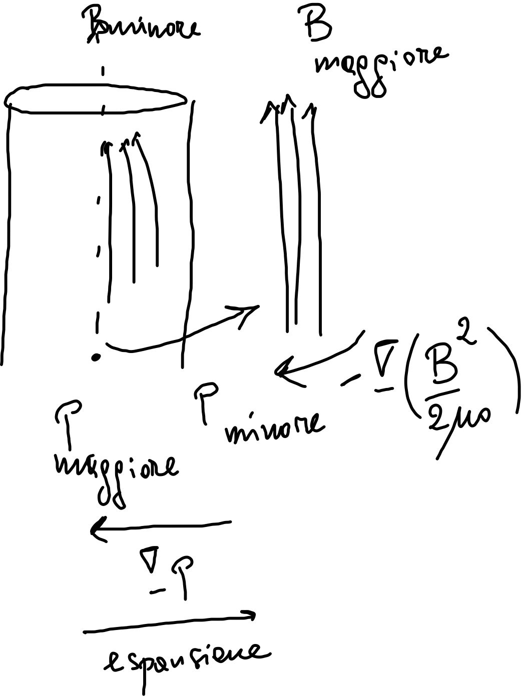
$$\underline{\nabla}_\perp p = \cancel{-\frac{\partial}{\partial s}\left(\frac{B^2}{2\mu_0}\right)\hat{\underline{b}}} - \underline{\nabla}_\perp\left(\frac{B^2}{2\mu_0}\right) + \cancel{\hat{\underline{b}}\frac{\partial}{\partial s}\left(\frac{B^2}{2\mu_0}\right)} + \frac{B^2}{\mu_0}\frac{\partial}{\partial s}\hat{\underline{b}}$$

$$\boxed{\underline{\nabla}_\perp\left(p + \frac{B^2}{2\mu_0}\right) - \frac{B^2}{\mu_0}\hat{\underline{\kappa}} = 0}$$

$$\hat{\underline{\kappa}} = (\hat{\underline{b}}\cdot\underline{\nabla})\hat{\underline{b}}$$



$$W = \int \frac{B^2}{2\mu_0}\,dV$$



$(\hat{\underline{b}}\cdot\underline{\nabla})\hat{\underline{b}}$ Coordinate cilindriche

Se $\hat{\underline{b}} = \hat{\underline{e}}_z$ $\quad \frac{\partial}{\partial z}(1)\,\hat{\underline{e}}_z = 0$

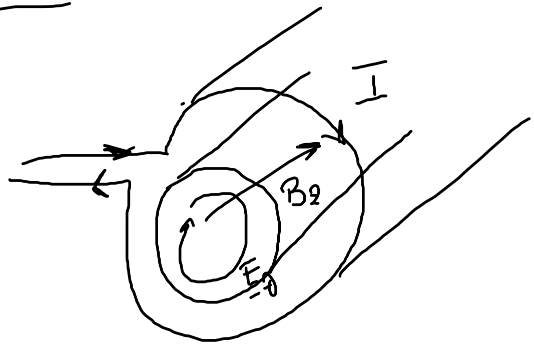
$\hat{\underline{b}} = \hat{r}$ $\quad \frac{\partial}{\partial r}\left(\cos\theta\,\hat{\underline{e}}_x + \text{sen}\,\theta\,\hat{\underline{e}}_y\right) = 0$

$\hat{\underline{b}} = \hat{\theta}$ $\quad \frac{1}{r}\frac{\partial}{\partial \theta}\left(-\text{sen}\,\theta\,\hat{\underline{e}}_x + \cos\theta\,\hat{\underline{e}}_y\right)$

$\hat{\underline{b}}$ $\quad = -\frac{1}{r}\left(\cos\theta\,\hat{\underline{e}}_x + \text{sen}\,\theta\,\hat{\underline{e}}_y\right)$

$= -\hat{\underline{r}}/r \neq 0$

θ pinch



$$\underbrace{I \uparrow}_{} \Rightarrow \underbrace{B_z \uparrow}_{}$$

Per legge Faraday: $E_\theta \neq 0$

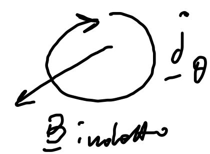
$$j_\theta = \underbrace{\sigma E_\theta}_{} \qquad \underline{j} \times \underline{B} \neq \underline{0} \qquad \underline{\nabla} p$$

$\underline{B}$ lungo z, no curvatura: $\hat{\underline{k}} = 0$

$$\underline{\nabla}_\perp \left(p + \frac{B^2}{2\mu_0} \right) = 0 \ ; \qquad \frac{\partial}{\partial r} \left(p + \frac{B_z^2(r)}{2\mu_0} \right) \hat{\underline{e}}_r = 0$$

Appena fuori dal plasma: $B = B_0$, $p = 0$

$$p + \frac{B_z^2(r)}{2\mu_0} = \text{const}$$



$$p(r) + \frac{B_z^2(r)}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

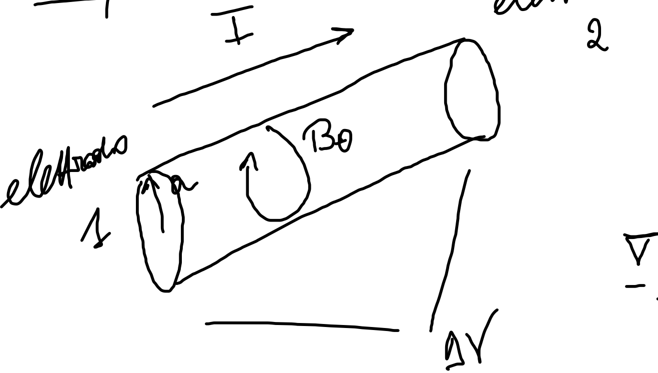
Nel centro : $r = 0$

$$p(0) + \frac{B_z^2(0)}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

$$\beta = \frac{p(0)}{B_0^2} 2\mu_0 = \frac{2\mu_0 p(0)}{2\mu_0 p(0) + B_z^2(0)}$$

$$\boxed{0 < \beta < 1}$$

Z pinch



$$\underline{\nabla}_\perp \left(p + \frac{B^2}{2\mu_0}\right) - \frac{B^2}{\mu_0}\,\hat{\underline{\kappa}} = 0$$

$$\underline{\nabla}_\perp = \frac{\partial}{\partial r}\,\hat{\underline{e}}_r + \cancel{\frac{\partial}{\partial z}\,\hat{\underline{e}}_z}$$

$$\hat{\underline{b}} = \hat{\underline{\theta}}$$

$$\frac{\partial}{\partial r}\left(p(r) + \frac{B_\theta^2(r)}{2\mu_0}\right)\hat{\underline{e}}_r + \frac{B_\theta^2}{\mu_0 r}\,\hat{\underline{e}}_r = 0$$

$$\hat{\underline{\kappa}} = (\hat{\underline{b}}\cdot\underline{\nabla})\,\hat{\underline{b}} = \frac{1}{r}\frac{\partial}{\partial \theta}\left(-\mathrm{sen}\,\theta\,\hat{\underline{e}}_x + \cos\theta\,\hat{\underline{e}}_y\right) = -\hat{\underline{e}}_r/r$$

$j_0 = j = \text{const}$ (Ipotesi)

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{j}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) = \mu_0 j$$

$$\int_0^r \frac{\partial}{\partial r}(r B_\theta)\, dr = \mu_0 j \int_0^r r\, dr$$

$$r B_\theta(r) = \mu_0 j \, r^2/2 \; ; \quad B_\theta(r) = \frac{\mu_0 j r}{2}$$

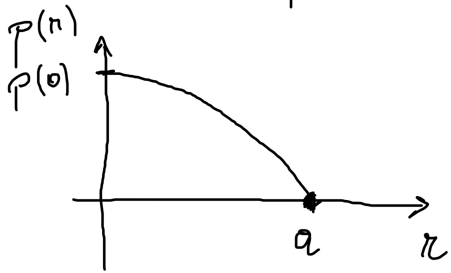
$$\frac{\partial}{\partial r}\left[p(r) + \frac{\mu_0^{\cancel{2}} j^2 r^2}{8 \cancel{\mu_0}} \right] + \frac{\mu_0^{\cancel{2}} j^2 r^{\cancel{2}}}{4 \cancel{r} \cancel{\mu_0}} = 0$$

$$\frac{\partial p}{\partial r} + \frac{\mu_0}{\cancel{8}\, 4} j^2 \cancel{2} r + \frac{\mu_0 j^2}{4} r = 0 \; ;$$

$$\frac{\partial p}{\partial r} = -\frac{\mu_0 j^2}{2} r$$

$$\int_a^r \frac{\partial p}{\partial r}\, dr = -\frac{\mu_0 j^2}{2} \int_a^r r\, dr$$

$$p(r) - \cancel{p(a)} = -\frac{\mu_0 j^2}{4}\left(r^2 - a^2\right) = \frac{\mu_0 j^2}{4}\left(a^2 - r^2\right)$$



Se $r = 0$: $p(0) = \frac{\mu_0 j^2 a^2}{4}$

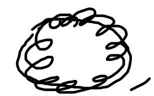
$$\beta = \frac{\text{max pressione}}{\text{max p magn.}} = \frac{\cancel{\mu_0 j^2 a^2}}{\cancel{4}} \cdot \frac{2\cancel{\mu_0}}{\cancel{\mu_0^2 j^2 a^2}} \cdot \cancel{4} = 2$$

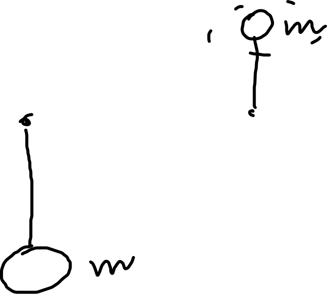
θ pinch

perdite . . .

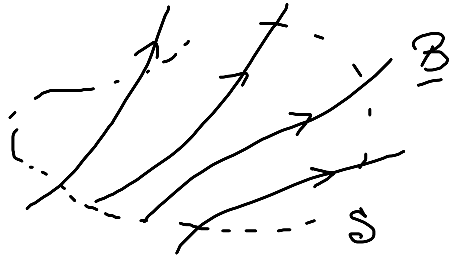
. . . perdite

Z pinch





Conservazione del flusso magnetico in MHD ideale (legge di gelo)



$$\Phi_S(\underline{B}) = \text{const}$$

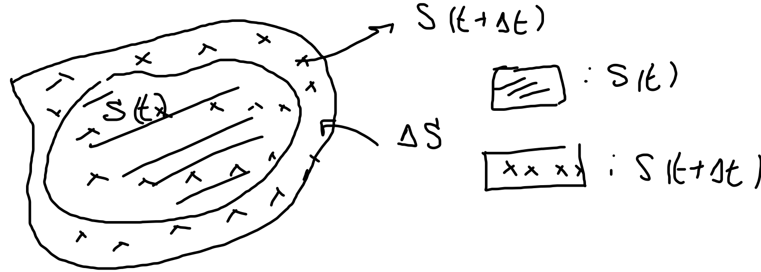
$$\Phi = \int_S \underline{B} \cdot d\underline{S}$$

$$\frac{d\Phi}{dt} = \int_C \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S} + \int_C \underline{B} \cdot d\left(\frac{\Delta \underline{S}}{\Delta t}\right)$$

$\Delta \underline{S}$: variazione di S in Δt

$$\phi(t) = \int_{S(t)} \underline{B}(t) \cdot d\underline{S}$$

$$\phi(t+\Delta t) = \int_{S(t+\Delta t)} \underline{B}(t+\Delta t) \cdot d\underline{S}$$

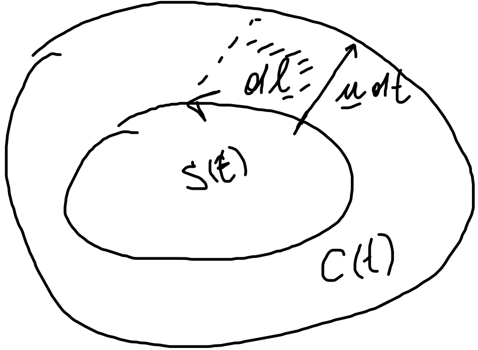


$$\approx \int_{S(t+\Delta t)} \underline{B}(t) \cdot d\underline{S} + \int_{S(t+\Delta t)} \frac{\partial \underline{B}}{\partial t} \Delta t \cdot d\underline{S} = \int_{S} \underline{B}(t) \cdot d\underline{S} + \underbrace{\int_{\Delta S} \underline{B}(t) \cdot d\underline{S}}_{\underline{B}(t) \cdot \Delta \underline{S}} +$$

Serie di Taylor
I ordine per $\underline{B}(t+\Delta t)$

$$+ \Delta t \int_{S} \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S} + O(\Delta t \, \Delta S)$$

$$\frac{d\phi}{dt} = \lim_{\Delta t \to 0} \frac{\phi(t+\Delta t) - \phi(t)}{\Delta t} = \int_{S} \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S} + \int \underline{B}(t) \frac{d(\Delta \underline{S})}{dt}$$



$$d\underline{S} = \underline{u}_\perp dt \times d\underline{l}$$

$$\frac{d\phi}{dt} = \int_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S} + \int_C \underline{B}(t) \cdot (\underline{u}_\perp \times d\underline{l})$$

Legge di FNL: $\underline{\nabla} \times \underline{E} = -\dfrac{\partial \underline{B}}{\partial t}$

$$\underline{E} + \underline{u}_\perp \times \underline{B} = 0 \qquad \text{(MHD ideale)}$$

$$\underline{E} = -\underline{u}_\perp \times \underline{B}$$

$$\underline{\nabla} \times (\underline{u}_\perp \times \underline{B}) = \frac{\partial \underline{B}}{\partial t}$$

$$\frac{d\phi}{dt} = \int_S \underline{\nabla} \times (\underline{u}_\perp \times \underline{B}) \cdot d\underline{S} + \int_C \underline{B}(t) \cdot (\underline{u}_\perp \times d\underline{l})$$

$$= \int_C (\underline{u}_\perp \times \underline{B}) \cdot d\underline{l} + \int_C \underline{B} \cdot (\underline{u}_\perp \times d\underline{l})$$

Th Stokes

$$\underline{A}\cdot(\underline{B}\times\underline{C}) = (\underline{A}\times\underline{B})\cdot\underline{C} = \underline{B}\cdot(\underline{C}\times\underline{A})$$

$$(\underline{u}_\perp \times \underline{B})\cdot d\underline{l} = \underline{B}\cdot(d\underline{l}\times\underline{u}_\perp) = -\underline{B}\,(\underline{u}_\perp \times d\underline{l})$$

$\times$ è anticomm.

$$\Longrightarrow \frac{d\phi}{dt} = 0$$