

# Photogrammetry & Robotics Lab

## Camera Calibration: Zhang's Method

**Cyrill Stachniss**

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The slides have been created by Cyrill Stachniss.

# 5 Minute Preparation for Today



<https://www.ipb.uni-bonn.de/5min/>

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<https://www.ipb.uni-bonn.de/5min/>

# 3D Point to Pixel: Estimating the Parameters of P

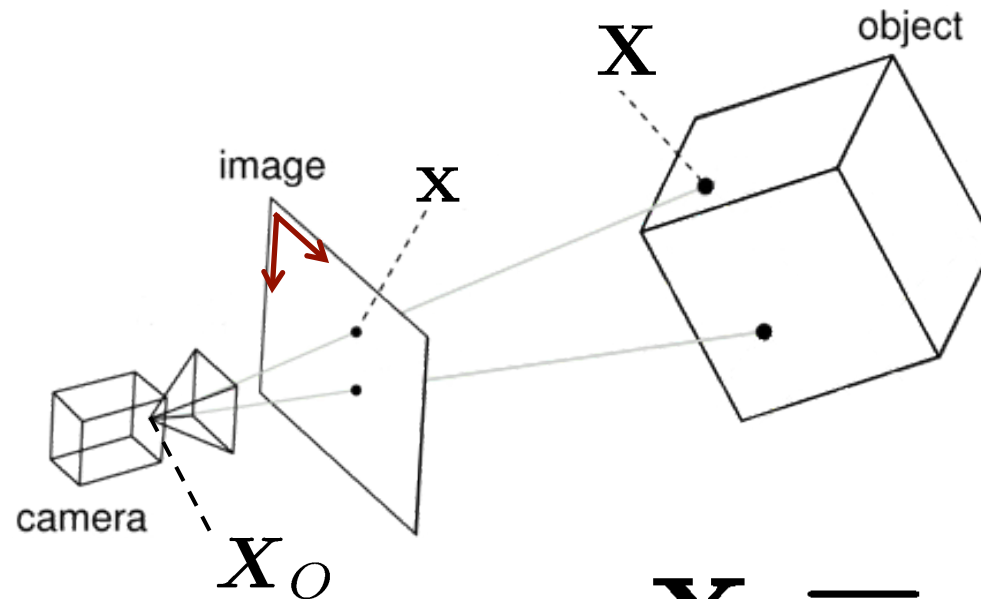
$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

pixel coordinate                      trans-formation                      world coordinate

**This time we only want the intrinsics!**

# Mapping (Recap)

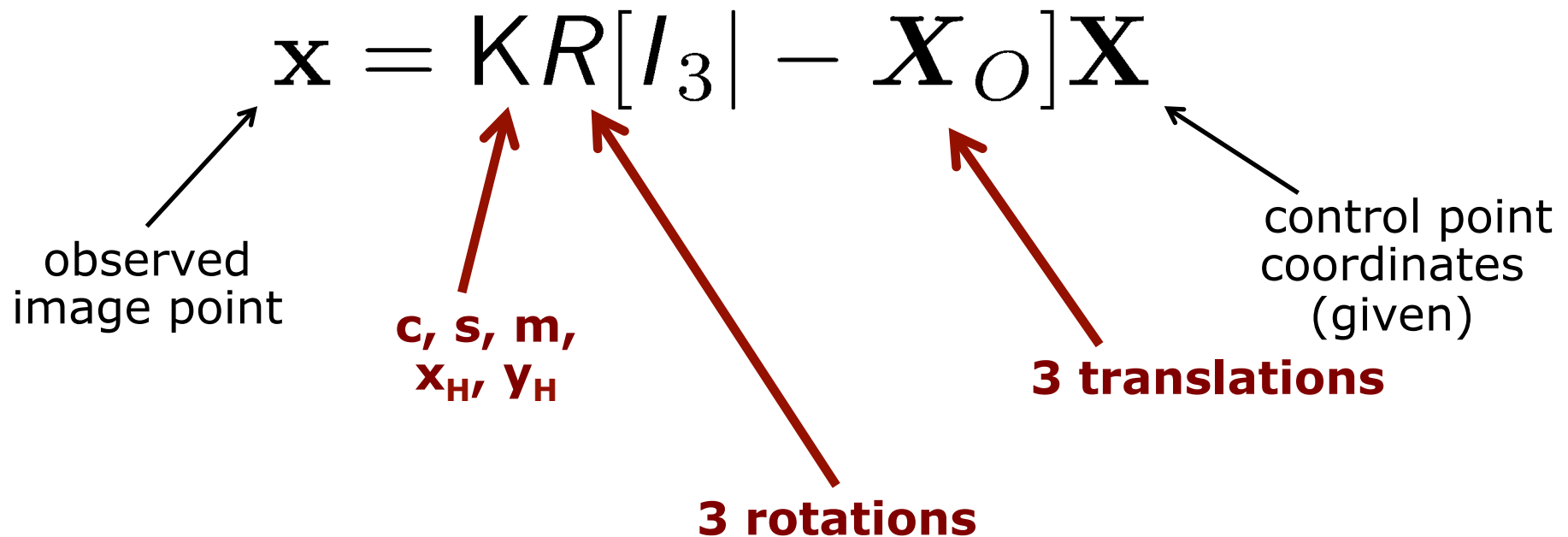
Direct linear transform (DLT) maps any object point  $\mathbf{X}$  to the image point  $\mathbf{x}$



$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

# Direct Linear Transform (Recap)

Compute the **11 intrinsic and extrinsic parameters**



# Zhang's Method

Compute the **5 intrinsic parameters**

$$\mathbf{x} = \mathbf{K} \mathbf{R} \begin{bmatrix} \mathbf{I}_3 \\ -\mathbf{X}_O \end{bmatrix} \mathbf{X}$$

observed image point

$\mathbf{c}, \mathbf{s}, \mathbf{m}, \mathbf{x}_H, \mathbf{y}_H$

3 rotations

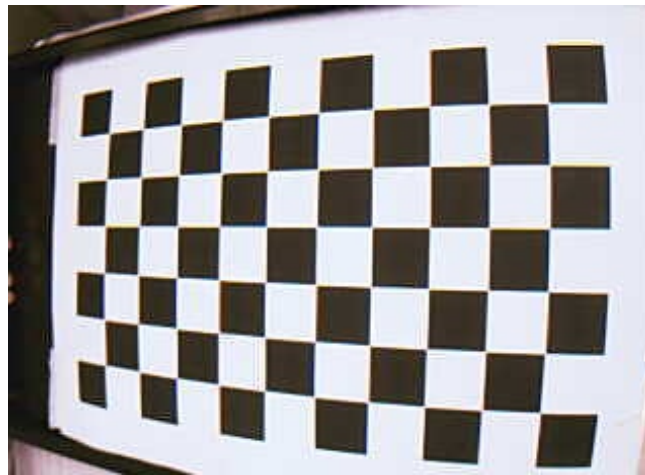
3 translations

control point

**Assumption:  
You know how DLT works!**



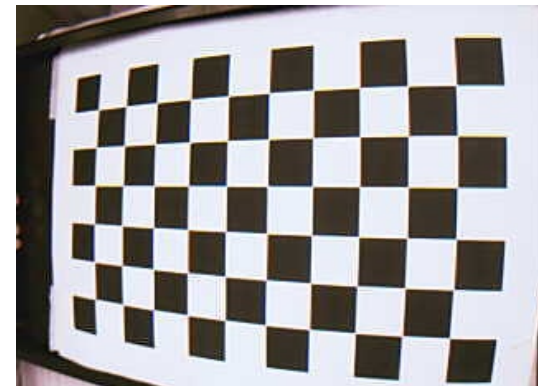
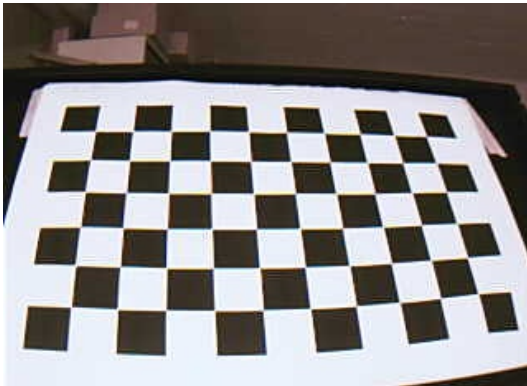
# Zhang's Method for Camera Calibration Using a Checkerboard



Zhang, A Flexible New Technique for  
Camera Calibration, MSR-TR-98-71

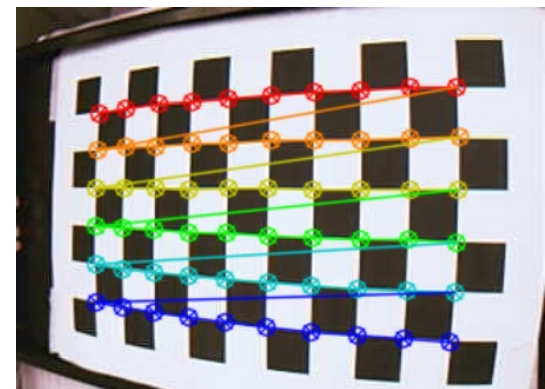
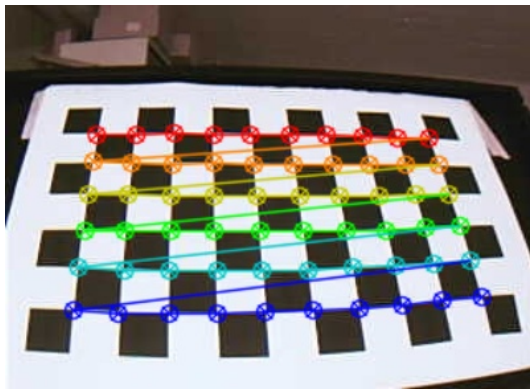
# Camera Calibration Using a 2D Checkerboard

- Observed 2D pattern (checkerboard)
- **Known size and structure**



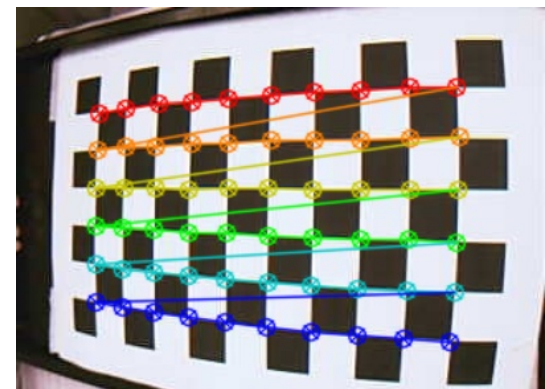
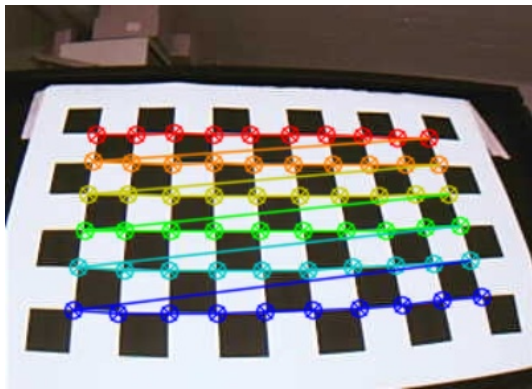
# Trick for Checkerboard Calibration

- **Set the world coordinate system to the corner of the checkerboard**



# Trick for Checkerboard Calibration

- **Set the world coordinate system to the corner of the checkerboard**
- All points on the checkerboard lie in the X/Y plane, i.e.,  $Z=0$



# Simplification

- The Z coordinate of each point on the checkerboard is equal to zero

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ \boxed{Z} \\ 1 \end{bmatrix}$$

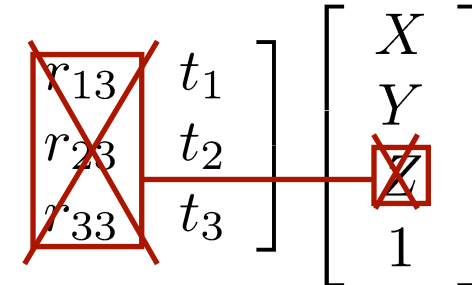
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- We can delete the 3<sup>rd</sup> column of the extrinsic parameter matrix

# Simplification

- The Z coordinate of each point on the checkerboard is equal to zero
- Deleting the 3<sup>rd</sup> column of the extrinsic parameter matrix leads to

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

**One point observed on the checkerboard generates such an equation**



# Setting Up the Equations for Determining the Parameter

$$H = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[\mathbf{r}_1, \mathbf{r}_2, \mathbf{t}]}$$

# Setting Up the Equations for Determining the Parameter

$$H = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[r_1, r_2, t]}$$

One point generates the equation:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{K}[r_1, r_2, t] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

# Setting Up the Equations for Determining the Parameter

- For multiple observed points on the checkerboard (in the same image), we obtain

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \underset{3 \times 3}{\mathbf{H}} \begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix} \quad i = 1, \dots, I$$

**How to proceed?**

# Setting Up the Equations for Determining the Parameter

- For multiple observed points on the checkerboard (in the same image), we obtain

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \underset{3 \times 3}{\mathbf{H}} \begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix} \quad i = 1, \dots, I$$

- Analogous to first **steps of the DLT**

# DLT-Like Estimation

- We estimate a 3x3 homography instead of a 3x4 projection matrix

- Rest is identical

- We use  $a_{x_i}^\top h = 0$   
 $a_{y_i}^\top h = 0$

- with

$$h = (h_k) = \text{vec}(H^\top)$$

$$a_{x_i}^\top = (-X_i, -Y_i, -\cancel{Z_i}, -1, 0, 0, \cancel{0}, 0, x_i X_i, x_i Y_i, x_i \cancel{Z_i}, x_i)$$

$$a_{y_i}^\top = (0, 0, \cancel{0}, 0, -X_i, -Y_i, -\cancel{Z_i}, -1, y_i X_i, y_i Y_i, y_i \cancel{Z_i}, y_i)$$

# DLT-Like Estimation

- We estimate a 3x3 homography instead of a 3x4 projection matrix
- Rest is identical
- We use  $\mathbf{a}_{x_i}^\top \mathbf{h} = 0$   
 $\mathbf{a}_{y_i}^\top \mathbf{h} = 0$
- with

$$\begin{aligned}\mathbf{h} &= (h_k) = \text{vec}(\mathbf{H}^\top) \\ \mathbf{a}_{x_i}^\top &= (-X_i, -Y_i, -1, 0, 0, 0, x_i X_i, x_i Y_i, x_i) \\ \mathbf{a}_{y_i}^\top &= (0, 0, 0, -X_i, -Y_i, -1, y_i X_i, y_i Y_i, y_i)\end{aligned}$$

# DLT-Like Estimation

- Solving the system of linear equations leads to an estimate of  $H$
- **How many points are needed to estimate  $H$ ?**

# DLT-Like Estimation

- Solving the system of linear equations leads to an estimate of  $H$
- We need to identify at least 4 points as  $H$  has 8 DoF and each point consists of 2 observations ( $x$  and  $y$ )

**This provides an estimate of  $H$**



## DLT-Like Estimation

- Solving the system of linear equations leads to an estimate of  $H$
- We need to identify at least 4 points as  $H$  has 8 DoF and each point consists of 2 observations ( $x$  and  $y$ )

**After we have estimated  $H$ ,  
we need to compute  $K$  from  $H$**

# Computing K Given H

$$H = [h_1, h_2, h_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_K \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[r_1, r_2, t]}$$



**no rotation matrix, thus  
QR decomposition is not  
applicable as for DLT**

**How to obtain the matrix decomposition?**

# Computing $K$ Given $H$

- We need to extract  $K$  from the matrix  $H = K[r_1, r_2, t]$  we computed via SVD

# Computing $K$ Given $H$

- We need to extract  $K$  from the matrix  $H = K[r_1, r_2, t]$  we computed via SVD

## Four step procedure:

1. Exploit constraints about  $K, r_1, r_2$
2. Define a matrix  $B = K^{-T}K^{-1}$
3. This  $B$  can be computed by solving another homogeneous linear system
4. Decompose matrix  $B$

# Computing K Given H is Different from the DLT Solution

- Homography H has only 8 DoF
- No direct decomposition as in DLT
- Exploit constraints on the parameters

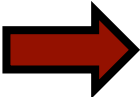
$$H = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[\mathbf{r}_1, \mathbf{r}_2, \mathbf{t}]}$$

$$[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = \mathbf{K}[\mathbf{r}_1, \mathbf{r}_2, \mathbf{t}]$$

# Exploiting Constraints for Determining the Parameter

$$H = [h_1, h_2, h_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_K \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[r_1, r_2, t]}$$

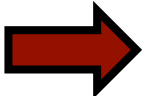
$$[r_1, r_2, t] = K^{-1}[h_1, h_2, h_3]$$

  $r_1 = K^{-1}h_1 \quad r_2 = K^{-1}h_2$

# Exploiting Constraints for Determining the Parameter

$$H = [h_1, h_2, h_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_K \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[r_1, r_2, t]}$$

$$[r_1, r_2, t] = K^{-1}[h_1, h_2, h_3]$$

  $r_1 = K^{-1}h_1 \quad r_2 = K^{-1}h_2$

As  $r_1, r_2, r_3$  form an orthonormal basis

  $r_1^T r_2 = 0 \quad \|r_1\| = \|r_2\| = 1$

# Exploiting Constraints

$$r_1 = K^{-1}h_1 \quad r_2 = K^{-1}h_2$$

$$r_1^T r_2 = 0$$



$$h_1^T K^{-T} K^{-1} h_2 = 0$$



# Exploiting Constraints

$$\mathbf{r}_1 = \mathbf{K}^{-1}\mathbf{h}_1 \quad \mathbf{r}_2 = \mathbf{K}^{-1}\mathbf{h}_2$$

$$\mathbf{r}_1^T \mathbf{r}_2 = 0$$



$$\mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2 = 0$$

$$\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = 1$$



$$\mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_1 = \mathbf{h}_2^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2$$

# Exploiting Constraints

$$\mathbf{r}_1 = \mathbf{K}^{-1}\mathbf{h}_1 \quad \mathbf{r}_2 = \mathbf{K}^{-1}\mathbf{h}_2$$

$$\mathbf{r}_1^T \mathbf{r}_2 = 0$$



$$\mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2 = 0$$

$$\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = 1$$



$$\mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_1 = \mathbf{h}_2^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2$$

$$\mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_1 - \mathbf{h}_2^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2 = 0$$

# Exploiting Constraints

$$r_1 = K^{-1}h_1 \quad r_2 = K^{-1}h_2$$

$$h_1^T K^{-T} K^{-1} h_2 = 0$$

$$h_1^T K^{-T} K^{-1} h_1 = h_2^T K^{-T} K^{-1} h_2$$

$$h_1^T K^{-T} K^{-1} h_1 - h_2^T K^{-T} K^{-1} h_2 = 0$$

# Exploiting Constraints

$$h_1^T \underline{K^{-T} K^{-1}} h_2 = 0$$

$$\underline{h_1^T K^{-T} K^{-1} h_1} - \underline{h_2^T K^{-T} K^{-1} h_2} = 0$$

- Define symmetric and positive definite matrix  $B := K^{-T} K^{-1}$

# Exploiting Constraints

$$h_1^T \underline{B} h_2 = 0$$

$$h_1^T \underline{B} h_1 - h_2^T \underline{B} h_2 = 0$$

- Define symmetric and positive definite matrix  $B := K^{-T} K^{-1}$

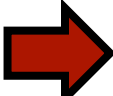
# Exploiting Constraints

$$h_1^T \underline{B} h_2 = 0$$

$$h_1^T \underline{B} h_1 - h_2^T \underline{B} h_2 = 0$$

- Define symmetric and positive definite matrix  $B := K^{-T} K^{-1}$
- From B, the calibration matrix can be recovered through Cholesky decomp.

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix} \quad \text{chol}(B) = AA^T$$

  $A = K^{-T}$

 **If we know B, then we can compute K**

# Exploiting Constraints

$$h_1^T \underline{B} h_2 = 0$$

$$h_1^T \underline{B} h_1 - h_2^T \underline{B} h_2 = 0$$

- Define symmetric and positive definite matrix  $B := K^{-T} K^{-1}$
- If we know  $B$ , we can compute  $K$
- Inspect equations above:
  - $B$  consists of the unknowns
  - $h$  are known
  - Two equations that relate  $B$  and  $h$

## Next Step: Compute B

- Define a vector  $b = (b_{11}, b_{12}, b_{13}, b_{22}, b_{23}, b_{33})$  of unknowns

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$$



# Exploiting Constraints

- Define a vector  $b = (b_{11}, b_{12}, b_{13}, b_{22}, b_{23}, b_{33})$  of unknowns

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$$

- Construct a system of linear equations  $Vb = 0$  exploiting the prev constraints:

$$v_{12}^T b = 0$$

(first constraint)

$$r_1^T r_2 = 0$$

$$v_{11}^T b - v_{22}^T b = 0$$

(second constraint)

$$\|r_1\| = \|r_2\| = 1$$

# The Matrix V

- The matrix  $V$  is given as

$$V = \begin{pmatrix} & v_{12}^T \\ v_{11}^T & -v_{22}^T \end{pmatrix} \quad \text{with} \quad v_{ij} = \begin{bmatrix} h_{1i}h_{1j} \\ h_{1i}h_{2j} + h_{2i}h_{1j} \\ h_{3i}h_{1j} + h_{1i}h_{3j} \\ h_{2i}h_{2j} \\ h_{3i}h_{2j} + h_{2i}h_{3j} \\ h_{3i}h_{3j} \end{bmatrix}$$

- For one image, we obtain

$$\begin{pmatrix} & v_{12}^T \\ v_{11}^T & -v_{22}^T \end{pmatrix} b = 0$$

↑  
elements of H

# The Matrix $V$

- For multiple images, we stack the matrices to a  $2n \times 6$  matrix

$$\begin{array}{l} \text{image 1} \\ \text{image n} \end{array} \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \left( \begin{array}{ccc} & v_{12}^T & \\ v_{11}^T & - & v_{22}^T \\ & \dots & \\ & v_{12}^T & \\ v_{11}^T & - & v_{22}^T \end{array} \right) b = 0$$

- We need to solve the linear system  $Vb = 0$  to obtain  $b$  and thus  $K$

# Solving the Linear System

- The system  $Vb = 0$  has a trivial solution which (invalid matrix  $B$ )
- Impose additional constraint  $\|b\| = 1$

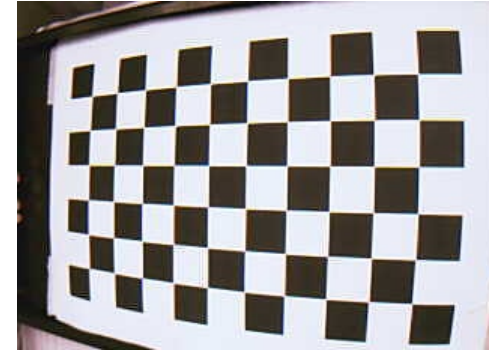
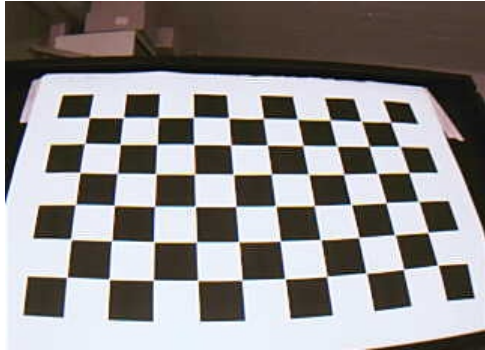
# Solving the Linear System

- The system  $Vb = 0$  has a trivial solution which (invalid matrix  $B$ )
- Impose additional constraint  $\|b\| = 1$
- Real measurements are noisy
- Find the solution that minimizes the squares error

$$b^* = \arg \min_b \|Vb\| \text{ with } \|b\| = 1$$

- Solve as in DLT computation

# What is Needed?



- We need at least **4 points per plane** to compute the matrix  $H$
- Each **plane** gives us **two equations**
- Since  $B$  has 5/6 DoF, we need at least **3 different views of a plane**
- Solve  $Vb = 0$  to compute  $K$

# Non-Linear Parameters?

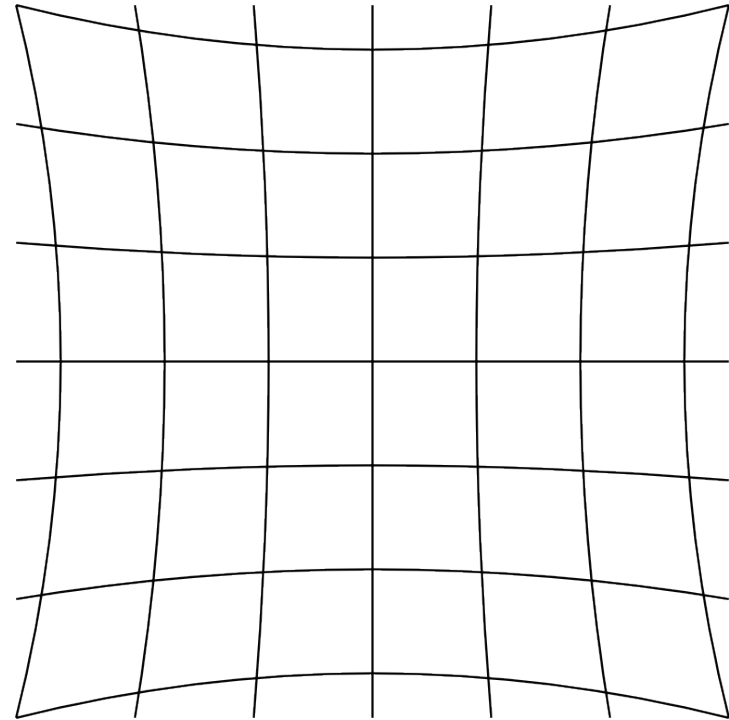
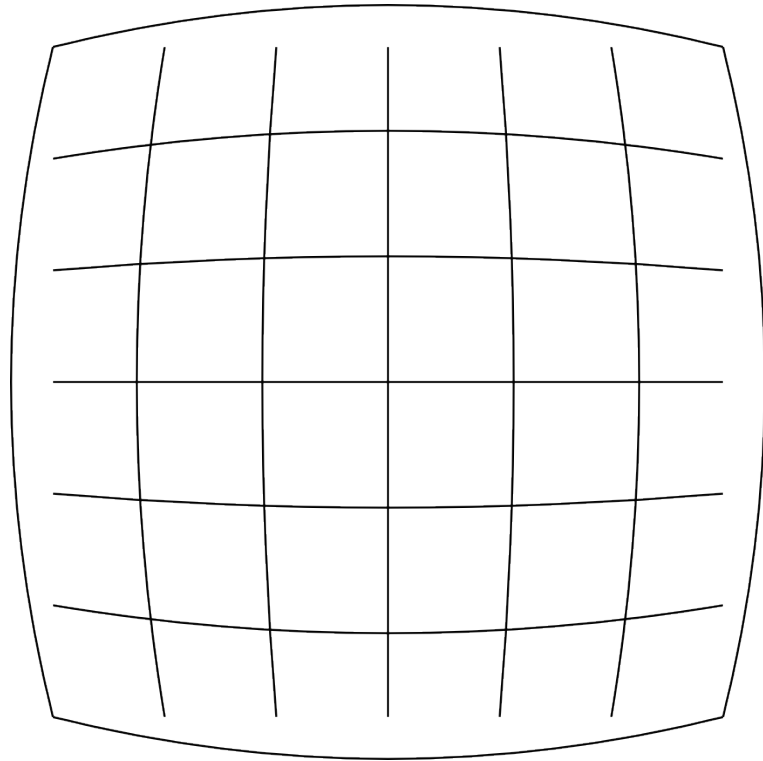
# General Calibration Matrix

- General calibration matrix is obtained by combining the one of the affine camera with the general mapping

$$\begin{aligned} {}^a K(\mathbf{x}, \mathbf{q}) &= {}^a H_s(\mathbf{x}, \mathbf{q}) K \\ &= \begin{bmatrix} 1 & 0 & \Delta x(\mathbf{x}, \mathbf{q}) \\ 0 & 1 & \Delta y(\mathbf{x}, \mathbf{q}) \\ 0 & 0 & 1 \end{bmatrix} K \end{aligned}$$



# Lens Distortion Example



# Example: Barrel Distortion

- A standard approach for wide angle lenses is to model the barrel distortion

$${}^a x = x(1 + q_1 r^2 + q_2 r^4)$$

$${}^a y = y(1 + q_1 r^2 + q_2 r^4)$$

- with  $[x, y]^T$  being point as projected by an ideal pin-hole camera
- with  $r$  being the distance **of the pixel** in the image to the principal point
- Additional non-linear parameters  $q_1, q_2$

# Error Minimization

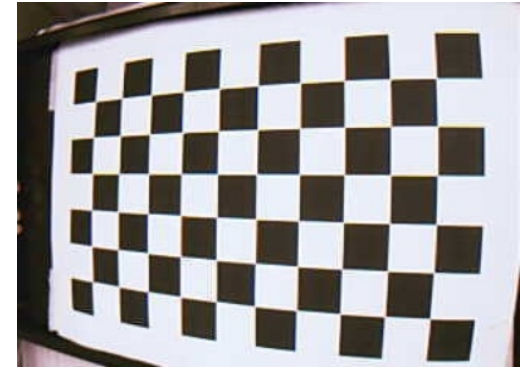
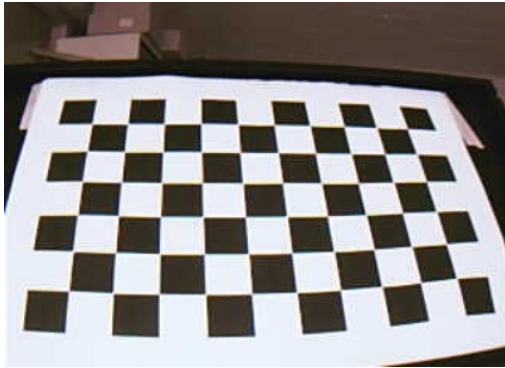
Lens distortion can be calculated by minimizing a non-linear error function

$$\min_{(K, \mathbf{q}, R_n, \mathbf{t}_n)} \sum_n \sum_i \|\mathbf{x}_{ni} - \hat{\mathbf{x}}(K, \mathbf{q}, R_n, \mathbf{t}_n, \mathbf{X}_{ni})\|^2$$

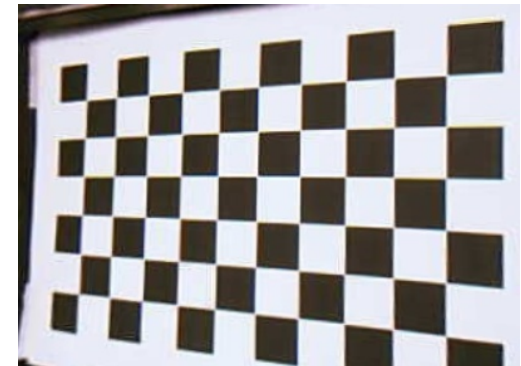
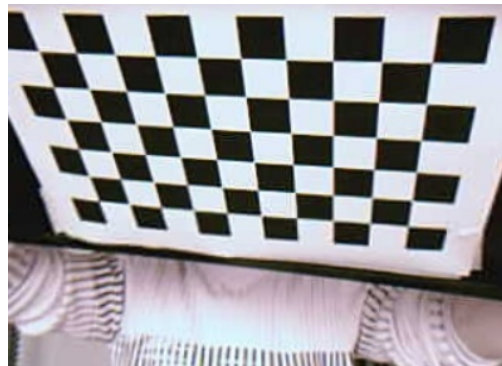
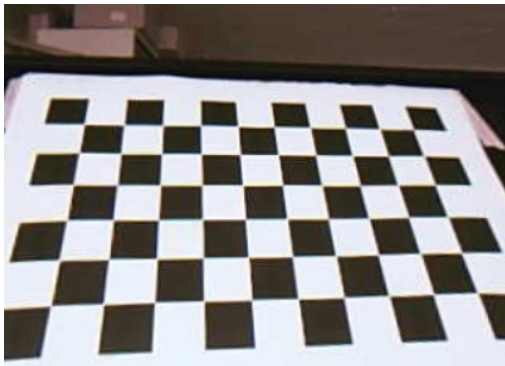
**...linearize to obtain a quadratic function, compute derivative, set it to 0, solve linear system, iterate...**  
(solved using Levenberg-Marquardt, K by Zhang's m. as initial value)

# Example Results

- Before calibration:



- After calibration:



# Summary on Camera Calibration Using a Checkerboard

- Pinhole camera model (first step)
- Non-linear model for lens distortion (second step)
- Approach to camera calibration that
  - accurately determines the camera parameters
  - is relatively easy to realize in practice

# Camera Calibration Summary

- Calibration means estimating the (intrinsic) parameters of a camera
- Linear and non-linear errors
- Linear: 5 parameters
- Zhang: estimate the 5 linear parameters using a checkerboard
- Estimate non-linear parameters in a second step

# Literature

- Zhang, A Flexible New Technique for Camera Calibration, MSR-TR-98-71 (uses a slightly different notation)
- Förstner & Wrobel, Photogrammetric Computer Vision, Chapter 11.2

# Slide Information

- The slides have been created by Cyrill Stachniss as part of the photogrammetry and robotics courses.
- **I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.**
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.
- If you are a university lecturer, feel free to use the course material. If you adapt the course material, please make sure that you keep the acknowledgements to others and please acknowledge me as well. To satisfy my own curiosity, please send me email notice if you use my slides.