Serie di Fourier

$$f \in \mathcal{L}^2\left(-\frac{a}{2}, \frac{a}{2}\right) \qquad a > 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[a_n \cos\left(\frac{2\pi n x}{a}\right) + b_n \operatorname{sen}\left(\frac{2\pi n x}{a}\right)\right]$$

$$a_n = \frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\cos\left(\frac{2\pi n x}{a}\right) dx$$

$$K_n = \frac{2\pi}{a} n$$

$$\Delta K = \frac{2\pi}{a}$$

$$b_n = \frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\operatorname{sen}\left(\frac{2\pi n x}{a}\right) dx$$

$$a \longrightarrow +\infty$$

$$z_m = e^{i\frac{2\pi}{a} m x}$$

$$Re(z_m) = \cos\left(\frac{2\pi m x}{a}\right)$$

$$Im(z_m) = \operatorname{sen}\left(\frac{2\pi m x}{a}\right)$$

$$Re(z_m) = \frac{z_m + z^*_m}{2} \qquad Im(z_m) = \frac{z_m - z^*_m}{2i}$$

$*$: compl. coniugato

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \frac{z_m + z^*_m}{2} + \frac{b_m}{2i}(z_m - z^*_m)$$

$$z^*_m = z_{-m} \qquad a_m = a_{-m} \qquad b_m = -b_{-m}$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} z_m \left(\frac{a_m - ib_m}{2}\right) + z_m^* \left(\frac{a_m + ib_m}{2}\right)$$

$$= \frac{a_0}{2} + \sum_{m=1}^{\infty} z_m \left(\frac{a_m - ib_m}{2}\right) + \sum_{m=1}^{\infty} z_{-m} \left(\frac{a_{-m} - ib_{-m}}{2}\right)$$

$$m' \overset{def}{=} -m \qquad = \frac{a_0}{2} + \sum_{m=1}^{\infty} (\dots) z_m + \sum_{m'=-\infty}^{-1} z_{m'} \left(\frac{a_{m'} - ib_{m'}}{2}\right)$$

$$= \sum_{m=-\infty}^{+\infty} c_m \underbrace{e^{i\frac{2\pi}{a}mx}}_{z_m} \qquad = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\, e^{-i\frac{2\pi mx}{a}}$$

$$c_m = \frac{1}{2}\left[\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\cos\left(\frac{2\pi mx}{a}\right)dx - i\,\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\sin\left(\frac{2\pi mx}{a}\right)\right]$$

$a \longrightarrow +\infty$

$$\Delta k = \frac{2\pi}{a} \qquad k = \frac{2\pi}{a} m$$

$$\Delta k \longrightarrow dk$$

$$C_m = \frac{1}{2\pi} \cdot \frac{2\pi}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \, f(x) \, e^{-i \overbrace{\frac{2\pi}{a} m}^{k} x} = \frac{1}{\sqrt{2\pi}} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \, f(x) \, e^{-ikx} \, \frac{2\pi}{a} \cdot \frac{1}{\sqrt{2\pi}}$$

$$\frac{2\pi}{a} \to dk$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, f(x) \, e^{-ikx}}_{C(k)} \frac{dk}{\sqrt{2\pi}}$$

$$f(x) = \sum_{m=-\infty}^{+\infty} C(k) \frac{1}{\sqrt{2\pi}} dk \, e^{ikx} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \, \underline{C(k)} \, e^{ikx}$$

$C(k)$: Trasf. di Fourier di $f(x)$

Onde E.m. in vuoto

$$\underline{\nabla} \cdot \underline{E} = 0 \qquad \underline{\nabla} \times \underline{E} = -\partial \underline{B}/\partial t$$

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \underline{\nabla} \times \underline{B} = \varepsilon_0 \mu_0 \partial \underline{E}/\partial t$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\frac{\partial}{\partial t}(\underline{\nabla} \times \underline{B}) = -\frac{\partial}{\partial t}\left(\varepsilon_0 \mu_0 \partial \underline{E}/\partial t\right) = -\varepsilon_0 \mu_0 \partial^2 \underline{E}/\partial t^2$$

$$\underbrace{\cancel{\underline{\nabla}(\underline{\nabla} \cdot \underline{E})}}_{=0} - \nabla^2 \underline{E}$$

$$\nabla^2 \underline{E} - \frac{1}{c^2} \partial^2 \underline{E}/\partial t^2 = 0 \qquad \frac{1}{c^2} \overset{\text{def}}{=} \varepsilon_0 \mu_0$$

$$\underbrace{\underline{\nabla} \times (\underline{\nabla} \times \underline{B})} = \underline{\nabla} \times \left(\varepsilon_0 \mu_0 \partial \underline{E}/\partial t\right) = \varepsilon_0 \mu_0 \frac{\partial}{\partial t}(\underline{\nabla} \times \underline{E}) = -\varepsilon_0 \mu_0 \partial^2 \underline{B}/\partial t^2$$

$$\underbrace{\cancel{\underline{\nabla}(\underline{\nabla} \cdot \underline{B})}}_{=0} - \nabla^2 \underline{B}$$

$$\nabla^2 \underline{B} - \frac{1}{c^2} \frac{\partial^2 \underline{B}}{\partial t^2} = 0$$

$$u(x,t) = u_0 \, e^{i(kx \mp \omega t)} \qquad k > 0$$

$-$: onda progressiva

$+$: " regressiva

1D: $\nabla^2 \to \dfrac{d^2}{dx^2}$

$$\nabla^2 u = u_0 \frac{d^2}{dx^2}\left[e^{i(kx \mp \omega t)} \right] = -k^2 u_0 \, e^{i(kx \mp \omega t)}$$

$$\nabla^2 \longrightarrow -k^2$$

$$\frac{d}{dx}\left[\quad \right] = ik\, u(x,t)$$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left[ik\, u(x,t) \right] = (ik)^2 u(x,t)$$

$$\frac{\partial}{\partial t}\left[u_0 \, e^{i(kx \mp \omega t)} \right] = \mp i\omega\, u(x,t) \qquad \frac{\partial^2}{\partial t^2}\left[\quad \right] = \frac{\partial}{\partial t}\left[\mp i\omega\, u(x,t) \right]$$

$$= (\mp i\omega)^2 u(x,t)$$

$$\frac{\partial^2}{\partial t^2} \longrightarrow -\omega^2 \qquad \frac{\partial}{\partial t} \longrightarrow \mp i\omega$$

$$\left(-k^2+\frac{\omega^2}{c^2}\right)u=0 \quad \begin{cases} \nearrow u=0 \\ \searrow \dfrac{\omega^2}{c^2}=k^2 \end{cases}$$

$k>0 \qquad k=\omega/c \;;\; \omega=kc$

$\dfrac{\omega}{k}=c$ vel. fase

Diel. $v=\dfrac{1}{\sqrt{\varepsilon\mu}} \longrightarrow \omega/k=v$

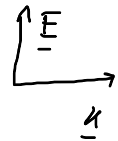
$$u(x,t)=\int_{-\infty}^{+\infty}\underbrace{dk\,\tilde{u}(k)}\,e^{ik(x-vt)}$$

$$\underline{E}(x,t) = \underline{\varepsilon}_1 E_0 e^{i(kx-\omega t)}$$

$$\underline{B}(x,t) = \underline{\varepsilon}_2 B_0 e^{i(kx-\omega t)}$$

$$\underline{\nabla}\cdot\left[\underline{E}\right] = i\underline{k}\cdot\left[\underline{\varepsilon}_1 E_0 e^{i(kx-\omega t)}\right] = 0$$

$$\Rightarrow \underline{k}\cdot\underline{\varepsilon}_1 = 0$$



$$\underline{\nabla}\cdot\underline{B} = i\underline{k}\cdot\left[\underline{\varepsilon}_2 B_0 e^{i(kx-\omega t)}\right] = 0$$

$$\Rightarrow \underline{k}\cdot\underline{B} = 0$$

Onde trasversali: $\underline{k} \perp \underline{E}$

" longitudinali: $\underline{k} \parallel \underline{E}$

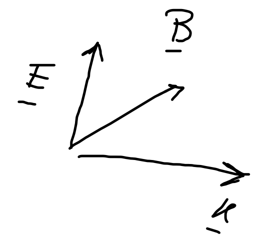
$$\underline{\nabla}\times\underline{E} = i\underline{k}\times\underline{E}$$

$$\underline{\nabla}\times\underline{E} = -\partial\underline{B}/\partial t \Rightarrow i\underline{k}\times\underline{E} = +i\omega\underline{B}$$

$$i \underline{k} \times \underline{E} = i\omega \underline{B}$$

$$i \underline{k} \times \underline{\varepsilon}_1 E_0 e^{i(kx-\omega t)} = i\omega \underline{\varepsilon}_2 B_0 e^{i(kx-\omega t)}$$

$$i e^{i(kx-\omega t)} \left[E_0 (\underline{k} \times \underline{\varepsilon}_1) - \omega \underline{\varepsilon}_2 B_0 \right] = 0$$



$$E_0 (\underline{k} \times \underline{\varepsilon}_1) - \omega \underline{\varepsilon}_2 B_0 = \underline{0}$$

$$\underline{\varepsilon}_2 = \frac{E_0}{B_0} \frac{\underline{k} \times \underline{\varepsilon}_1}{\omega}$$

Moduli: $1 = \frac{E_0}{B_0} \frac{k}{\omega} \Rightarrow \overset{c}{\frac{\omega}{k}} = \frac{E_0}{B_0} \Rightarrow \frac{E_0}{B_0} = c$

Verso: $\underline{k} \cdot \underline{\varepsilon}_2 = 0 \Rightarrow \underline{k} \perp \underline{\varepsilon}_2$

$\underline{k} \perp \underline{\varepsilon}_1$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \tilde{u}(k)\, e^{i(kx - \omega(k)t)}$$

Mezzo dispersivo:

$$\frac{\omega}{k} = v(k)$$

$$\Rightarrow \omega = k \cdot v(k)$$

$\omega = k \cdot (\text{cost})$ mezzo non disp.

ω non è una funzione lineare di k mezzo disp.

Condizioni iniziali

$$u(x, t=0) = u_0(x)$$

Ipotesi

$$\frac{\partial}{\partial t} u(x, t=0) = v_0(x) = 0$$

$$u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \tilde{u}(k) \cdot e^{ikx}$$

$\tilde{u}(k)$ è la T.F. di $u_0(x)$

$u(x,t)$ è reale?

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \tilde{u}(k)\, e^{i(kx-\omega t)}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{+\infty} dk\, \tilde{u}(k)\, e^{i(kx-\omega t)} + \int_{-\infty}^{0} dk\, \tilde{u}(k)\, e^{i(kx-\omega t)} \right]$$

$$k' \longrightarrow -k$$

$u_0(x)$ è reale

$$\Longrightarrow \tilde{u}(-k) = \tilde{u}^*(k)$$

$$- \int_{+\infty}^{0} dk'\, \tilde{u}(-k')\, e^{i(-k'x-\omega t)} = \int_{0}^{+\infty} dk\, \tilde{u}^*(k)\, e^{-ikx}\, e^{-i\omega t}$$

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty} dk\left[\underline{\tilde{u}(k)e^{ikx}+\tilde{u}^*(k)e^{-ikx}}\right]\underline{\underline{e^{-i\omega t}}}$$

Numero reale

Non è una sol. reale

Forziamo:

$$u(x,t)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} dk\,\tilde{u}(k)e^{i(kx-\omega t)}+c.c.\right]$$

$u_0(x)\neq 0 \quad v_0(x)\neq 0$

$$u_0(x)=\frac{1}{\sqrt{2\pi}}\frac{1}{2}\left[\int_{-\infty}^{+\infty} dk\left(\tilde{u}(k)e^{ikx}+\tilde{u}^*(k)e^{-ikx}\right)\right]$$

$$v_0(x)=\frac{1}{\sqrt{2\pi}}\frac{1}{2}\left[\int_{-\infty}^{+\infty} dk\left(-i\omega\tilde{u}(k)e^{ikx}+i\omega\tilde{u}^*(k)e^{-ikx}\right)\right.$$

$$\int_{-\infty}^{+\infty} dk\, u^*(k)\, e^{-ikx} = \int_{-\infty}^{+\infty} dk'\, u^*(-k)\, e^{ikx}$$

$$k' = -k$$

$$u_0(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{+\infty} dk \left[\tilde{u}(k) + \tilde{u}^*(-k) \right] e^{ikx} \qquad (1)$$

$$v_0(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{+\infty} dk\, \left(-i\omega(k)\right) \left(\tilde{u}(k) - \tilde{u}^*(-k) \right) e^{ikx} \qquad (2)$$

$$(1): \quad \frac{\tilde{u}(k) + \tilde{u}^*(-k)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, u_0(x)\, e^{-ikx}$$

(2) : $$-\frac{i\omega}{2}\left[\tilde{u}(k)-\tilde{u}(-k)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\, v_0(x)\, e^{-ikx}$$

$(1) + -\frac{2}{i\omega}(2)$:

$$\tilde{u}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\left[u_0(x)+\frac{i}{\omega(k)}v_0(x)\right]e^{ikx}$$

<u>Velocità di fase e di gruppo</u>

Se $\exists\, k_0$ "dominante"

$\tilde{u}(k_0)\, e^{-} >> \tilde{u}(k) \quad k\neq k_0$

$$\frac{\omega}{k} = v$$

$$\omega(k) \approx \omega(k_0) + (k-k_0)\underbrace{\left.\frac{d\omega}{dk}\right|_{k=k_0}}_{v_g} + \dots$$

Mezzo dispersivo

$$u(x,t) \simeq \mathrm{Re}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\, \tilde{u}(k)\, \exp\left[i\left[(k-k_0)x + k_0 x - \omega(k_0)t - (k-k_0)v_g t\right]\right]\right]$$

$$\simeq \mathrm{Re}\left[\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\, \tilde{u}(k)\, \exp\left[ik(x-v_g t)\right]\right]}_{u_0(x-v_g t)} e^{i(k_0 v_g t - \omega(k_0)t)}\right]$$

$$= u_0(x-v_g t)\cos\left[(k_0 v_g - \omega(k_0))t\right]$$