$\underline{B} \neq 0$ Equazioni della MHD

$$\rho \left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \underline{u} \right] = \underline{j} \times \underline{B} - \underline{\nabla} p$$

$\exists$ un equilibrio: $\underline{u}_0 = 0$ $\rho = \rho_0$ $p = p_0$

$$\underline{B} = B_0 \hat{\underline{e}}_z \qquad \underline{j} = 0$$

Perturbazione: $\underline{\xi}(\underline{r}) = \underline{\xi}_1 \exp\left(i\left[\underline{k} \cdot \underline{r} - \omega t\right]\right)$ ξ_1 piccolo

$\underline{\xi}$ → spostamento

Obiettivo: trovare $\omega = \omega(\underline{k})$

Linearizzare eq. cons. mom. lineare

$$\left[\rho_0+\cancel{\rho_1}\right]\left[\underbrace{\frac{\partial}{\partial t}\left(\cancel{\underline{u}_0}+\underline{u}_1\right)}_{\text{ordine } 1}+\left[\underbrace{\left(\cancel{\underline{u}_0}+\underline{u}_1\right)\cdot\underline{\nabla}\right]\left(\cancel{\underline{u}_0}+\underline{u}_1\right)}_{\text{ordine } 2}\right]=\left[\cancel{\underline{j}_0}+\underline{j}_1\right]\times\left[\underline{B}_0+\cancel{\underline{B}_1}\right]-\underline{\nabla}\left(p_0+p_1\right)$$

$$\underline{\nabla} p_0 = 0$$

$$\rho_0 \frac{\partial \underline{u}_1}{\partial t} = \underline{j}_1 \times \underline{B}_0 - \underline{\nabla} p_1$$

$$\underline{u}_1 \overset{\text{def}}{=} \frac{\partial \underline{\xi}_1}{\partial t} = -i\omega\, \underline{\xi}_1$$

modi normali

$$\frac{\partial \underline{u}_1}{\partial t} = -i\omega \frac{\partial \underline{\xi}_1}{\partial t} = (-i\omega)^2 \underline{\xi}_1 = -\omega^2 \underline{\xi}_1$$

$$-\rho_0 \omega^2 \underline{\xi}_1 = \underline{j}_1 \times \underline{B}_0 - \underline{\nabla} p_1$$

Eq. stato in MHD

$$\frac{d}{dt}\left[\frac{P}{\rho^{\gamma}}\right] = 0 \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + (\underline{u}\cdot\underline{\nabla})$$

$$\left[\frac{\partial}{\partial t} + \cancel{(\underline{u}_0 + \underline{u}_1)}\cdot\underline{\nabla}\right]\left[\frac{P_0 + P_1}{(\rho_0+\rho_1)^{\gamma}}\right] = 0$$

$$\frac{P_0+P_1}{(\rho_0+\rho_1)^{\gamma}} = \frac{P_0+P_1}{\rho_0^{\gamma}\left[1+\frac{\rho_1}{\rho_0}\right]^{\gamma}} = \frac{P_0+P_1}{\rho_0^{\gamma}}\left(1-\gamma\frac{\rho_1}{\rho_0}\right) = \frac{P_0}{\rho_0^{\gamma}} - \frac{\gamma\rho_1 P_0}{\rho_0^{\gamma+1}}$$

$$\frac{1}{(1+\varepsilon)^{\gamma}} = (1+\varepsilon)^{-\gamma} \simeq 1-\gamma\varepsilon \qquad + \frac{P_1}{\rho_0^{\gamma}} + O(\rho_1 P_1$$

$$\frac{\partial}{\partial t}\left[\frac{p_0}{\cancel{\rho_0^{\gamma}}} - \frac{\gamma \rho_1 p_0}{\rho_0^{\cancel{\gamma}+1}} + \frac{p_1}{\cancel{\rho_0^{\gamma}}}\right] = 0$$

$$-i\omega\left[-\frac{\gamma \rho_1 p_0}{\rho_0} + p_1\right] = 0 \qquad p_1 = \frac{\gamma p_0}{\rho_0}\rho_1$$

Eq. continuità in MHD

$$\frac{d}{dt}\rho + \rho \underline{\nabla}\cdot\underline{u} = 0 \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla} \simeq \frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t}(\cancel{\rho_0} + \rho_1) + [\rho_0 + \cancel{\rho_1}]\,\underline{\nabla}\cdot(\cancel{\underline{u}_0}^{\,=0} + \underline{u}_1) = 0$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \underline{\nabla}\cdot\underline{u}_1 = 0$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \underline{\nabla} \cdot \underline{u}_1 = 0$$

modi normali

$$-i\omega \rho_1 + \rho_0 \, i \, \underline{k} \cdot \underline{u}_1 = 0 \qquad \underline{u}_1 = \frac{\partial \underline{\xi}_1}{\partial t} = -i\omega \underline{\xi}_1$$

$$-i\omega \rho_1 + \underbrace{(-i\omega)\rho_0 \, i \, \underline{k}} \cdot \underline{\xi}_1 = 0$$

$$-\cancel{i\omega}\rho_1 + \cancel{\omega}\rho_0 \, \underline{k} \cdot \underline{\xi}_1 = 0 \qquad \rho_1 = -i\rho_0 \left(\underline{k} \cdot \underline{\xi}_1 \right)$$

$$\underline{\nabla} p_1 = \underline{\nabla} \left(\frac{\gamma p_0}{\rho_0} \rho_1 \right) = \underline{\nabla} \left(\frac{\gamma p_0}{\cancel{\rho_0}} \left(-i \cancel{\rho_0} \left(\underline{k} \cdot \underline{\xi}_1 \right) \right) \right) = -i \gamma p_0 \overset{i\underline{k}}{\underline{\nabla}} \left(\underline{k} \cdot \underline{\xi}_1 \right)$$

$$\underline{\nabla} p_1 = \gamma p_0 \underline{k} \left(\underline{k} \cdot \underline{\xi}_1 \right)$$

$\underline{j}_1 \times \underline{B}_0 = ?$

Th Ampere in MHD $\quad \underline{\nabla} \times \underline{B} = \mu_0 \underline{j}$

$$\underline{\nabla} \times (\cancel{\underline{B}_0} + \underline{B}_1) = \mu_0 (\cancel{\underline{j}_0} + \underline{j}_1)$$

modi normali

$$\underline{\nabla} \times \underline{B}_1 = \mu_0 \underline{j}_1 \, ; \quad i \underline{k} \times \underline{B}_1 = \mu_0 \underline{j}_1 \, ;$$

Th Faraday $\quad \underline{\nabla} \times \underline{E}_1 = -\frac{\partial \underline{B}_1}{\partial t}$ $\qquad \underline{j}_1 = \frac{i}{\mu_0} \underline{k} \times \underline{B}_1$

modi normali $\quad \cancel{i} \underline{k} \times \underline{E}_1 = \cancel{i} \omega \underline{B}_1 \, ; \quad \underline{B}_1 = \frac{\underline{k} \times \underline{E}_1}{\omega}$

eq. Ohm in MHD ideale $\quad \underline{E} + \underline{u} \times \underline{B} = 0 \quad \leftarrow \eta \underline{j} \, \eta = 0$

$$(\cancel{\underline{E}_0} + \underline{E}_1) + (\cancel{\underline{u}_0} + \underline{u}_1) \times (\underline{B}_0 + \cancel{\underline{B}_1}) = 0$$

$$\underline{E}_1 = -\underline{u}_1 \times \underline{B}_0$$

$$\underline{E}_1 = -\underline{u}_1 \times \underline{B}_0 = -\frac{\partial \underline{\xi}_1}{\partial t} \times \underline{B}_0 = i\omega\, \underline{\xi}_1 \times \underline{B}_0$$

modi norm.

$$\underline{B}_1 = \frac{\underline{k} \times \underline{E}_1}{\omega} = \frac{\underline{k} \times (i\cancel{\omega}\, \underline{\xi}_1 \times \underline{B}_0)}{\cancel{\omega}} = i\, \underline{k} \times (\underline{\xi}_1 \times \underline{B}_0)$$

$$\underline{j}_1 = \frac{i}{\cancel{\mu_0}}\, \underline{k} \times \underline{B}_1 = \frac{i}{\cancel{\mu_0}}\, \underline{k} \times \left(i\, \underline{k} \times (\underline{\xi}_1 \times \underline{B}_0)\right) = -\frac{1}{\cancel{\mu_0}}\, \underline{k} \times \left(\underline{k} \times (\underline{\xi}_1 \times \underline{B}_0)\right)$$

$$\underline{j}_1 \times \underline{B}_0 = -\frac{B_0^2}{\cancel{\mu_0}}\, \underline{k} \times \left[\left(\underline{k} \times (\underline{\xi}_1 \times \hat{\underline{e}}_z)\right)\right] \times \hat{\underline{e}}_z$$

$$\underline{B}_0 = B_0\, \hat{\underline{e}}_z$$

Eq. cons. momento lin.

$$\underbrace{\omega^2 \underline{\xi}_1}_{\frac{\partial \underline{u}_1}{\partial t}} = \underbrace{\frac{B_0^2}{\rho_0 \mu_0}\left[\underline{k} \times \left(\underline{k} \times (\underline{\xi}_1 \times \hat{\underline{e}}_z)\right)\right] \times \hat{\underline{e}}_z}_{\underline{j}_1 \times \underline{B}_0} + \underbrace{\frac{\gamma p_0}{\rho_0} \underline{k}\, (\underline{k} \cdot \underline{\xi}_1)}_{\underline{\nabla} p_1}$$

$$\frac{B_0^2}{\rho_0 \mu_0} \to v_A^2 \qquad \frac{\gamma p_0}{\rho_0} \to c_s^2$$

$$\underline{\underline{M}} \cdot \underline{\xi}_1 = 0 \qquad \text{Rel. disp. : } \det \underline{\underline{M}} = 0$$

$$\underline{\xi}_1 = \underline{0} \quad \text{sol. poco int.}$$

$$\underline{k} = k_{\parallel} \hat{\underline{e}}_z + k_{\perp} \hat{\underline{e}}_y \qquad \underline{k} \times (\underline{\xi}_1 \times \hat{\underline{e}}_z) = k_{\parallel} \underline{\xi}_{1\perp} - (\underline{k} \cdot \underline{\xi}_1)\, \hat{\underline{e}}_z$$

$$\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\,\underline{B} - (\underline{A} \cdot \underline{B})\,\underline{C}$$

$$= (\underline{C} \times \underline{B}) \times \underline{A}$$

$$\underline{k} \times \left[\underline{k} \times \left(\underline{\xi}_1 \times \hat{\underline{e}}_z \right) \right] = \underline{k} \times \left[k_\parallel \underline{\xi}_1 - \left(\underline{k} \cdot \underline{\xi}_1 \right) \hat{\underline{e}}_z \right] =$$

$$= k_\parallel \left(\underline{k} \times \underline{\xi}_1 \right) - \left(\underline{k} \cdot \underline{\xi}_1 \right) \underbrace{\underline{k} \times \hat{\underline{e}}_z}_{\left(k_\parallel \hat{\underline{e}}_z + k_\perp \hat{\underline{e}}_y \right) \times \hat{\underline{e}}_z = k_\perp \hat{\underline{e}}_x}$$

$$\underline{k} \times \underline{\xi}_1 = \left(k_\parallel \hat{\underline{e}}_z + k_\perp \hat{\underline{e}}_y \right) \times \left(\xi_{1x} \hat{\underline{e}}_x + \xi_{1y} \hat{\underline{e}}_y + \xi_{1z} \hat{\underline{e}}_z \right) =$$

$$= + k_\parallel \xi_{1x} \hat{\underline{e}}_y - k_\parallel \xi_{1y} \hat{\underline{e}}_x - k_\perp \xi_{1x} \hat{\underline{e}}_z + k_\perp \xi_{1z} \hat{\underline{e}}_x$$

$$\left[\underline{k} \times \left[\underline{k} \times \left(\underline{\xi}_1 \times \hat{\underline{e}}_z \right) \right] \right] \times \hat{\underline{e}}_z = k_\parallel^2 \xi_{1x} \hat{\underline{e}}_x + k_\parallel^2 \xi_{1y} \hat{\underline{e}}_y \cancel{- k_\parallel k_\perp \xi_{1z} \hat{\underline{e}}_y} + \overbrace{\left(k_\parallel \xi_{1z} + k_\perp \xi_{1y} \right)}^{\underline{k} \cdot \underline{\xi}_1} k_\perp \hat{\underline{e}}_y$$

Eq. di cons. momento per componenti

$$v_A^2 = \frac{B_0^2}{\mu_0 \rho_0}$$

vel. Alfven

$$c_s^2 = \frac{\gamma P_0}{\rho_0} \qquad \underline{\xi} = \begin{pmatrix} \xi_{1x} \\ \xi_{1y} \\ \xi_{1z} \end{pmatrix}$$

(x) $\omega^2 \xi_{1x} = k_{=}^2 v_A^2 \xi_{1x}$;

$$\left(\omega^2 - k_{=}^2 v_A^2\right) \xi_{1x} = 0$$

(y) $\omega^2 \xi_{1y} = v_A^2 \left(k_{=}^2 \xi_{1y} + k_\perp^2 \xi_{1y}\right) + c_s^2 \left(k_\perp^2 \xi_{1y} + k_{=} k_\perp \xi_{1z}\right)$

$$\left[\omega^2 - v_A^2\left(k_{=}^2 + k_\perp^2\right) - c_s^2 k_\perp^2\right] \xi_{1y} - k_{=} k_\perp c_s^2 \xi_{1z} = 0$$

$$\underline{\underline{M}} \cdot \underline{\xi} = 0$$

(z) $\omega^2 \xi_{1z} = c_s^2 k_{=}^2 \xi_{1z} + c_s^2 k_{=} k_\perp \xi_{1y}$; $\quad \left(-c_s^2 k_{=} k_\perp\right) \xi_{1y} + \xi_{1z}\left(\omega^2 - c_s^2 k_{=}^2\right) = 0$

$$\underline{\underline{M}} = \begin{pmatrix} \omega^2 - v_A^2 k_\parallel^2 & 0 & 0 \\ 0 & \omega^2 - k^2 v_A^2 - c_s^2 k_\perp^2 & - k_\parallel k_\perp c_s^2 \\ 0 & - c_s^2 k_\parallel k_\perp & \omega^2 - c_s^2 k_\parallel^2 \end{pmatrix}$$