$\mathcal{L}(q_j, \dot{q}_j, t)$ $\qquad j = 1 \dots N$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \qquad j = 1 \dots N$$

Exact invariants : if $\mathcal{L}$ does <u>not</u> depend on q_i $\Rightarrow \frac{\partial \mathcal{L}}{\partial q_i} = 0$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \Rightarrow p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{const}$$

conjugate momentum

If there is toroidal symmetry : $\mathcal{L}$ cannot depend on $\underline{\varphi}$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \text{const}$$

Strategy: 1) Write down or guess for $\mathcal{L}$

2) Show that $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$ is the Lorentz force

$$\mathcal{L} = \frac{mv^2}{2} + q\,\underline{v}\cdot\underline{A}(\underline{x},t) - q\,\phi(\underline{x},t)$$

$\underline{A}$: vector potential

q_j : x, y, z

$$\underline{E} = -\underline{\nabla}\phi - \partial \underline{A}/\partial t$$

ϕ : scalar potential

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

$q_j : x, y, z \qquad \dot{q}_j : \dot{x}, \dot{y}, \dot{z}$

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial}{\partial x_j} \sum_{i=1}^{3} q v_i A_i(\underline{x}, t) - q \frac{\partial \Phi}{\partial x_j} =$$

$$= q \sum_{i=1}^{3} v_i \frac{\partial A_i}{\partial x_j} - q \frac{\partial \Phi}{\partial x_j}$$

$$\frac{\partial \mathcal{L}}{\partial v_j} = \frac{m}{2} \frac{\partial}{\partial v_j} \sum_{i=1}^{3} v_i^2 + q \frac{\partial}{\partial v_j} \sum_{i=1}^{3} v_i A_i = \frac{m}{2} 2 v_j + q A_j =$$

$$= m v_j + q A_j(\underline{x}, t)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v_j} = m \dot{v}_j + q \sum_{i=1}^{3} \frac{\partial A_j}{\partial x_i} \dot{x}_i + q \frac{\partial A_j}{\partial t}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v_j} - \frac{\partial L}{\partial x_j} = 0$$

$$m\dot{v}_j + q\sum_{i=1}^{3}\frac{\partial A_j}{\partial x_i}\dot{x}_i + q\frac{\partial A_j}{\partial t} - q\sum_{i=1}^{3} v_i\frac{\partial A_i}{\partial x_j} + \frac{\partial \phi}{\partial x_j}q = 0$$

$$m\underline{a} = q\left(\underline{E} + \underline{v}\times\underline{B}\right)$$

$$m\dot{v}_j = \underbrace{-q\frac{\partial \phi}{\partial x_j} - q\frac{\partial A_j}{\partial t}}_{+q\left[-\frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t}\right]} + q\sum_{i=1}^{3}\left(v_i\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}v_i\right)$$

$$+q\left[-\frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t}\right] \rightarrow E_j \qquad \underline{E} = -\underline{\nabla}\phi - \frac{\partial A}{\partial t}$$

$$m\dot{v}_j = qE_j + \underbrace{q\sum_{i=1}^{3} v_i \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)}$$

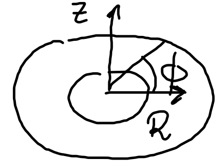
$$q(\underline{v} \times \underline{B})_j = q\left[\underline{v} \times (\underline{\nabla} \times \underline{A})\right]_j$$

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

$$\underline{v} \times (\underline{\nabla} \times \underline{A}) = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \left[\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\right] = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$$

$$\begin{pmatrix} v_y(\partial_x A_y - \partial_y A_x) - v_z(\partial_z A_x - \partial_x A_z) \end{pmatrix}$$

$$q\sum_{i=1}^{3} v_i \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) \qquad j=1$$



$$= q\, v_x \cancel{\frac{\partial A_x}{\partial x}} - \cancel{\frac{\partial A_x}{\partial x}} + q\, v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z q\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)$$

If there is toroidal symm.

$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = \text{const}$$

$$\mathcal{L} = \frac{m v^2}{2} + q\,\underline{v}\cdot\underline{A}(\underline{x}, t) - q\phi(\underline{x}, t)$$

$$(R, \varphi, z)$$

$$\mathcal{L} = \frac{m}{2}\left(v_R^2 + v_z^2 + v_\varphi^2\right) + q\left(v_R A_R + v_z A_z + v_\varphi A_\varphi\right)$$

$$-q\phi$$

$$\mathcal{L} = \frac{m}{2}\left(v_R^2 + v_z^2 + v_\varphi^2\right) + q\left(A_R v_R + A_z v_z + A_\varphi v_\varphi\right) - q\phi$$

toroidal component

$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = \frac{m}{\cancel{2}} R^2 \cancel{2}\dot\phi + qRA_\varphi \Rightarrow p_\varphi = \underline{mR^2}\dot\phi + q\underline{RA}_\varphi$$

$$v_\varphi = R\cdot\dot\phi$$

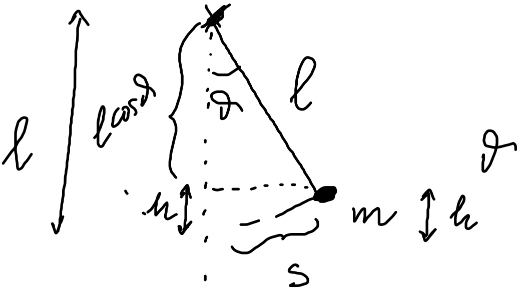
$$[p_\varphi] = [mR\cdot v_\varphi] = \text{angular momentum}$$

canonical

$$\text{arch} = R\phi \qquad v_\varphi = \frac{d(\text{arch})}{dt} = R\dot\phi$$

$$v_\varphi^2 = R^2\dot\phi^2 \Rightarrow \frac{d}{d\dot\phi}v_\varphi^2 = 2\dot\phi R^2$$

Simple pendulum



$\mathcal{L}$ = Kinetic en. − Potent. energy

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$v = \frac{ds}{dt} = \frac{d(l\theta)}{dt} = l\dot{\theta}$$

$$U = mgh = mgl(1-\cos\theta)$$

$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta) = \mathcal{L}(\theta, \dot{\theta})$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\ddot{\theta}$$

$$\cancel{m} l^{\cancel{2}} \ddot{\vartheta} + \cancel{m} g \cancel{l} \sin\vartheta = 0 \quad ; \quad \ddot{\vartheta} = -\frac{g}{l}\sin\vartheta$$

$\vartheta \ll 1$

$$\ddot{\vartheta} \simeq -\frac{g}{l}\vartheta \qquad \vartheta(t) = \vartheta_0 \cos(\omega t) \qquad \omega = \sqrt{\frac{g}{l}}$$

Initial conditions: $\begin{cases} \vartheta(0) = \vartheta_0 \\ \dot{\vartheta}(0) = 0 \end{cases}$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

$$\vartheta(t) = \mathrm{Re}\left[\vartheta_0 \, e^{i\omega t}\right] = \mathrm{Re}\left[\vartheta_0 \, e^{i\int_0^t \omega \, dt'}\right]$$

$$l = l(t) \qquad \ddot{\vartheta} \simeq -\frac{g}{l(t)}\sin\vartheta \quad \Rightarrow \quad \omega(t) = \sqrt{\frac{g}{l(t)}}$$

Guess

$$\vartheta(t) = \mathrm{Re}\left[\vartheta_0(t)\, e^{i\int_0^t \omega(t')\,dt'}\right] \qquad \omega(t) = \sqrt{g/\ell(t)}$$

$$\ddot\vartheta = -\frac{g}{\ell}\,\vartheta$$

$$\frac{d\vartheta}{dt} = \mathrm{Re}\left[\dot\vartheta_0\, e^{i\int_0^t \omega(t')\,dt'} + \vartheta_0\, e^{i\int_0^t \omega(t')\,dt'}\, i\omega(t)\right]$$

$$\frac{d^2\vartheta}{dt^2} = \mathrm{Re}\left[\ddot\vartheta_0\, e^{[\ ]} + \dot\vartheta_0\, e^{[\ ]}\, i\omega + \dot\vartheta_0\, e^{[\ ]}\, i\omega + \vartheta_0\, e^{[\ ]}(i\omega)^2 + \vartheta_0\, e^{[\ ]}\, i\,\frac{d\omega}{dt}\right] = \mathrm{Re}\left[e^{[\ ]}\left(\ddot\vartheta_0 + 2i\omega\dot\vartheta_0 - \omega^2\vartheta_0 + i\vartheta_0\frac{d\omega}{dt}\right)\right]$$

$$\mathrm{Re}\left[e^{[\ ]} \left(\underbrace{\ddot{\vartheta}_0}_{\ddot{\ } \simeq 0} + 2i\dot{\vartheta}_0 \omega + i\dot{\omega}\vartheta_0 \cancel{- \omega^2 \vartheta_0} \right) \right] = \cancel{-\omega^2 \vartheta_0}$$

$$\vartheta_0 = \vartheta_0(t)$$

$$\ddot{\vartheta}_0 + 2i\dot{\vartheta}_0\omega + i\dot{\omega}\vartheta_0 = 0$$

$$\omega_0 = \sqrt{g/l_0}$$

$$2\cancel{i}\dot{\vartheta}\omega = -\cancel{i}\dot{\omega}\vartheta_0$$

$$\omega(t) = \sqrt{g/l(t)}$$

$$\int_{\vartheta_0(0)}^{\vartheta_0(t)} \frac{d\vartheta_0}{\vartheta_0} = -\int_{\omega_0}^{\omega(t)} \frac{d\omega}{2\omega} = \ln\left[\sqrt{\frac{\omega_0}{\omega(t)}}\right]$$

$$\ln\left[\frac{\vartheta_0(t)}{\vartheta_0(0)}\right] = -\frac{1}{2} \ln\left[\frac{\omega(t)}{\omega_0}\right]$$

$$\boxed{\vartheta_0(t) = \vartheta_0(0)\sqrt{\frac{\omega_0}{\omega(t)}}}$$

$$\vartheta(t) = \mathrm{Re}\left[\underbrace{\vartheta_0(0)\sqrt{\frac{\omega_0}{\omega(t)}}}\, e^{i\int_0^t \omega(t')\,dt'}\right] =$$

$$\vartheta(t) = \vartheta_0(0)\sqrt{\frac{\omega_0}{\omega(t)}}\cos\left(\int_0^t \omega(t')\,dt'\right)$$