Collisions:

- Coulomb
- test particle approach
- Small angle Coulomb collisions
- $b_{\pi/2} \le b \le \lambda_D$

Test $\xrightarrow{1}$ Plasma

How is the pdf of the Test particles changed because of collisions?

Fokker-Planck

$$f_T(\underline{v}, t) \xleftarrow{\text{collisions}} f(\underline{v}-\Delta\underline{v}, \underbrace{t-\Delta t}_{\text{average time among collisions}})$$

$(\Delta\underline{v}, \Delta t)$

$F(\underline{v}, \Delta\underline{v})$: prob. density that a particle which has a velocity $\underline{v}$ receives a "kick" of $\Delta\underline{v}$

$$\int \underbrace{F(\underline{v}, \Delta\underline{v})\, d^3\Delta\underline{v}}_{\text{Prob. to have a kick of } \Delta\underline{v}} = 1$$

$$f_T(\underline{v}, t) = \int \underbrace{f_T(\underline{v} - \Delta\underline{v}, t - \Delta t)}_{\text{pdf at } t - \Delta t} \underbrace{F(\underline{v} - \Delta\underline{v}, \Delta\underline{v})}_{\text{prob. that}} d^3\Delta\underline{v}$$

$|\Delta\underline{v}| \ll |\underline{v}|$

$\Delta t \ll t$

$$\underline{v} - \Delta\underline{v} \xrightarrow[\Delta v]{} \underline{v}$$

Taylor expansion of $f_T(\underline{v} - \Delta\underline{v}, t - \Delta t) F(\underline{v} - \Delta\underline{v}, \Delta\underline{v})$

II order

centre at $(\underline{v}, t)$

$$f_T(\underline{v} - \Delta\underline{v}, t - \Delta t)\, \underline{F}(\underline{v} - \Delta\underline{v}, \Delta\underline{v}) =$$

$$= \underbrace{f_T(\underline{v}, t)\, F(\underline{v}, \Delta\underline{v})}_{0^{th}\ \text{order}} \overbrace{- \Delta t \frac{\partial}{\partial t}\Big[f_T(\underline{v}, t)\, F(\underline{v}, \Delta\underline{v}) \Big] - \Delta\underline{v} \cdot \frac{\partial}{\partial \underline{v}}\Big[f_T(\underline{v}, t)\, F(\underline{v}, \Delta\underline{v}) \Big]}^{1^{st}\ \text{order terms}}$$

$$+ \frac{1}{2} \Delta\underline{v}\,\Delta\underline{v} : \overbrace{\frac{\partial}{\partial \underline{v}} \frac{\partial}{\partial \underline{v}} \Big(f_T(\underline{v}, \Delta t)\, F(\underline{v}, \Delta\underline{v}) \Big)}^{\text{matrix}}$$

2^{nd} order term:

$$\frac{\begin{pmatrix} \Delta v_x & \Delta v_y & \Delta v_z \end{pmatrix}}{2} \begin{bmatrix} \frac{\partial^2 (f_T F)}{\partial v_x \partial v_x} & \frac{\partial^2 (f_T F)}{\partial v_x \partial v_y} & \frac{\partial^2}{\partial v_x \partial v_z} \\ \frac{\partial^2}{\partial v_y \partial v_x} & \text{---} & - \\ \frac{\partial^2}{\partial v_z \partial v_x} & \text{---} & - \ - \end{bmatrix} \cdot \begin{pmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{pmatrix}$$

$$\int \underbrace{f_T(\underline{v},t)\, F(\underline{v},\Delta\underline{v})}_{0^{\text{th}}\text{ order term}}\, d^3\Delta\underline{v} = f_T(\underline{v},t) \int \underbrace{F(\underline{v},\Delta\underline{v})\, d^3\Delta\underline{v}}_{1} = f_T(\underline{v},t)$$

$$- \int \overbrace{\Delta t\, \frac{\partial}{\partial t}\Big[f_T(\underline{v},t)\, F(\underline{v},\Delta\underline{v})}^{1^{\text{st}}\text{ order term } \left(\frac{\partial}{\partial t}\right)} \Big]\, d^3\Delta\underline{v} = -\Delta t \int \frac{\partial}{\partial t}\Big[f_T(\underline{v},t) \Big]\, F(\underline{v},\Delta\underline{v})\, d^3\Delta\underline{v}$$

$$= -\Delta t\, \frac{\partial}{\partial t} f_T \int \underbrace{F(\underline{v},\Delta\underline{v})\, d^3\Delta\underline{v}}_{1} = -\Delta t\, \frac{\partial f_T}{\partial t}$$

$$- \int \Delta\underline{v} \cdot \frac{\partial}{\partial \underline{v}} \left[f_T(\underline{v},t)\, F(\underline{v},\Delta\underline{v}) \right] d^3\Delta\underline{v} =$$

$$= - \frac{\partial}{\partial \underline{v}} \cdot \left[\int d^3\Delta\underline{v} \;\; \Delta\underline{v} \;\; f_T(\underline{v},t)\, F(\underline{v},\Delta\underline{v}) \right] =$$

$$= - \frac{\partial}{\partial \underline{v}} \cdot \left[f_T(\underline{v},t) \underbrace{\int d^3\Delta\underline{v} \;\; \Delta\underline{v}\, F(\underline{v},\Delta\underline{v})}_{\overset{\text{def}}{=}\ \langle \Delta\underline{v} \rangle} \right]$$

$$= - \frac{\partial}{\partial \underline{v}} \cdot \left(f_T \langle \Delta\underline{v} \rangle \right)$$

$$+\frac{1}{2}\int d^3\Delta \underline{v}\;\Delta\underline{v}\Delta\underline{v} : \frac{\partial}{\partial\underline{v}}\frac{\partial}{\partial\underline{v}}\Big(f(\underline{v},t)F(\underline{v},\Delta\underline{v})\Big) =$$

$$= \frac{1}{2}\frac{\partial}{\partial\underline{v}}\frac{\partial}{\partial\underline{v}} : \left[\int d^3\Delta\underline{v}\;\Delta\underline{v}\Delta\underline{v}\, f(\underline{v},t)\,F(\underline{v},\Delta\underline{v})\right]$$

$$\langle \Delta\underline{v}\Delta\underline{v}\rangle$$

$$= \frac{1}{2}\frac{\partial}{\partial\underline{v}}\frac{\partial}{\partial\underline{v}} : \left[f(\underline{v},t)\overbrace{\int d^3\Delta\underline{v}\;\Delta\underline{v}\Delta\underline{v}\,F(\underline{v},\Delta\underline{v})}\right]$$

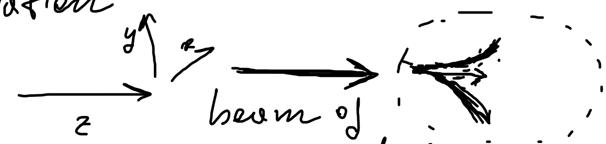
$$= \frac{1}{2}\frac{\partial}{\partial\underline{v}}\frac{\partial}{\partial\underline{v}} : \left[f(\underline{v},t)\langle\Delta\underline{v}\Delta\underline{v}\rangle\right] \qquad \begin{bmatrix} \Delta v_x\Delta v_x & \Delta v_x\Delta v_y & \cdots \\ \Delta v_y\Delta v_x & \cdots & \end{bmatrix} \qquad \int d^3\Delta\underline{v}\;\Delta v_x\Delta v_x F(\underline{v},\Delta\underline{v})$$

$$\cancel{f_T(\underline{v}, t)} = \cancel{f_T(\underline{v}, t)} - \underbrace{\Delta t \frac{\partial f}{\partial t}} - \frac{\partial}{\partial \underline{v}} \cdot \left(f_T \langle \Delta \underline{v} \rangle \right) + \frac{1}{2} \frac{\partial}{\partial \underline{v}} \frac{\partial}{\partial \underline{v}} : \left(f_T \langle \Delta \underline{v} \Delta \underline{v} \rangle \right)$$

friction term   isotropy term

$$\frac{\partial f_T}{\partial t} = - \frac{\partial}{\partial \underline{v}} \cdot \left(f_T \frac{\langle \Delta \underline{v} \rangle}{\Delta t} \right) + \frac{1}{2} \frac{\partial}{\partial \underline{v}} \frac{\partial}{\partial \underline{v}} : \left(f_T \frac{\langle \Delta \underline{v} \Delta \underline{v} \rangle}{\Delta t} \right)$$

Fokker-Planck equation



beam of test particles

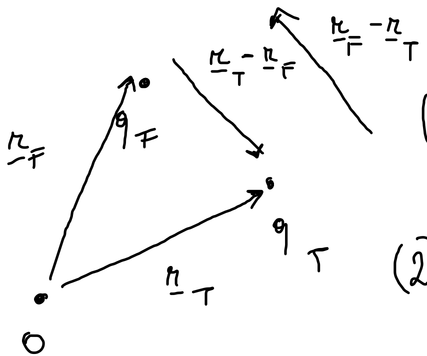
v_z $v_z \downarrow$ $v_x \uparrow$ $v_y \uparrow$

$\left.\begin{array}{l}\langle \Delta \underline{v}_T \rangle ? \\ \langle \Delta \underline{v}_T \Delta \underline{v}_T \rangle ?\end{array}\right\}$ for a plasma?

field particle (F): a particle of the bulk plasma

In c.m. frame: $\left\{\begin{array}{l}\text{one particle at rest} \\ \text{other is moving with } \underline{v}_{rel}\end{array}\right.$

How to convert $\Delta \underline{v}$ from c.m. frame to lab?



$$(1)\quad m_T \ddot{\underline{r}}_T = \frac{1}{4\pi\varepsilon_0} \frac{q_T q_F}{|\underline{r}_T - \underline{r}_F|^3} (\underline{r}_T - \underline{r}_F)$$

$$(2)\quad m_F \ddot{\underline{r}}_F = \frac{1}{4\pi\varepsilon_0} \frac{q_T q_F}{|\underline{r}_T - \underline{r}_F|^3} (\underline{r}_F - \underline{r}_T)$$

(1) + (2):

$$m_T \ddot{\underline{r}}_T + m_F \ddot{\underline{r}}_F = 0$$

c.m. coordinate

$$\underline{R} = \frac{m_T \underline{r}_T + m_F \underline{r}_F}{m_T + m_F} \Rightarrow \ddot{\underline{R}} = 0$$

$$\Rightarrow \dot{\underline{R}} = \overrightarrow{const}$$

Relative position $\underline{r} = \underline{r}_T - \underline{r}_F$

$$\ddot{\underline{r}} = \ddot{\underline{r}}_T - \ddot{\underline{r}}_F = \overbrace{\left(\frac{1}{m_T}\right)\frac{1}{4\pi\varepsilon_0}\frac{q_T q_F}{|\underline{r}_T - \underline{r}_F|^3}(\underline{r}_T - \underline{r}_F)}^{\ddot{\underline{r}}_T} \overbrace{- \frac{1}{m_F}\frac{1}{4\pi\varepsilon_0}\frac{q_T q_F}{4\pi\varepsilon_0}\underbrace{(\underline{r}_T - \underline{r}_F)}_{\underline{r}}}^{-\ddot{\underline{r}}_F}$$

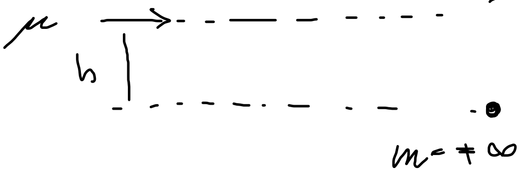
$$= \left(\frac{1}{m_T} + \frac{1}{m_F}\right) \cdot \frac{1}{4\pi\varepsilon_0} q_T q_F (\underline{r}_T - \underline{r}_F)$$

Relative mass

$$\frac{1}{\mu} = \frac{1}{m_T} + \frac{1}{m_F}$$

$$\mu \ddot{\underline{r}} = \frac{1}{4\pi\varepsilon_0} \frac{q_T q_F}{r^3} \underline{r}$$

eg. a particle with mass μ interacts by Coulomb collision with a part. with $m = +\infty$



$$\underbrace{(\underline{r}_T, \underline{r}_F)}_{\text{LAB frame}} \longrightarrow (\underline{r}, \underline{R})$$

only 1 particle with $m = \mu$ is moving

$\dot{\underline{R}} = \vec{\text{const}}$

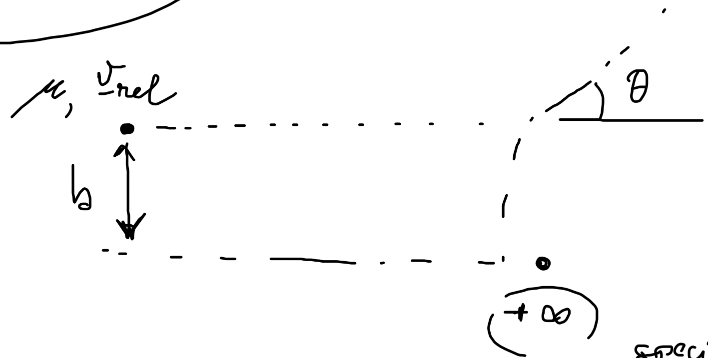
$$\begin{cases} \underline{R} = \dfrac{m_T \underline{r}_T + m_F \underline{r}_F}{m_T + m_F} \\ \underline{r} = \underline{r}_T - \underline{r}_F \end{cases} \Rightarrow \begin{cases} \underline{r}_F = \underline{R} - \dfrac{\mu}{m_F} \underline{r} \\ \underline{r}_T = \underline{R} + \dfrac{\mu}{m_F} \underline{r} \end{cases}$$

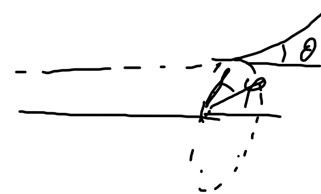
$$\underline{v}_T = \dot{\underline{r}}_T = \dot{\underline{R}} + \frac{\mu}{m_F} \underbrace{\dot{\underline{r}}}_{\underline{v}_{rel}}$$

$$\Delta \underline{v}_T = \underbrace{\cancel{\Delta \dot{\underline{R}}}}_{=0} + \frac{\mu}{m_F} \Delta \underline{v}_{rel}$$

$$\Delta \underline{v}_T = \frac{\mu}{m_T} \Delta \underline{v}_{rel}$$

$\langle \Delta \underline{v}_{rel} \rangle = ?$





$\Delta \underline{v}_{rel}$ in a given collision → specific value of $\underline{v}_{rel}$
→ " " of b

$\Delta \underline{v}_{rel} = \Delta \underline{v}_{rel}(\underline{v}_{rel}, b, \varphi)$ Average over all possible $\left\{ \begin{array}{l} b \\ \varphi \\ \underline{v}_{rel} \end{array} \right.$