Fokker Planck equation

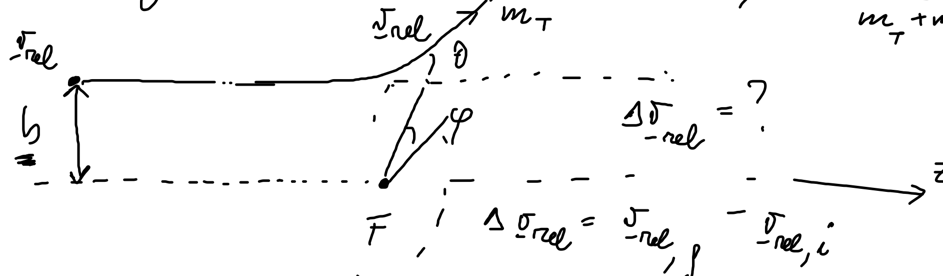
$$\frac{\partial f_T}{\partial t} = -\frac{\partial}{\partial \underline{v}}\cdot\Big(\underbrace{\Big\langle \frac{\Delta \underline{v}_T}{\Delta t}\Big\rangle_T f_T(\underline{v},t)}_{\text{friction}}\Big) + \frac{1}{2}\frac{\partial}{\partial \underline{v}}\frac{\partial}{\partial \underline{v}} : \Big(\underbrace{\Big\langle \frac{\Delta \underline{v}_T \Delta \underline{v}_T}{\Delta t}\Big\rangle_T f_T(\underline{v},t)}_{\text{"Beam spreading"}}\Big)$$

Solve in c.m. frame

q_T

q_F

$$\Delta \underline{v}_T = \frac{\mu}{m_T}\Delta \underline{v}_{rel} \qquad \mu = \frac{m_T m_F}{m_T + m_F}$$



$\Delta \underline{v}_{rel} = ?$

$\Delta \underline{v}_{rel} = \underline{v}_{rel,f} - \underline{v}_{rel,i}$

$\underline{v}_{rel,i} = v_{rel}\,\hat{\underline{z}}$

$|\underline{v}_{rel}|$ is unchanged by the collision

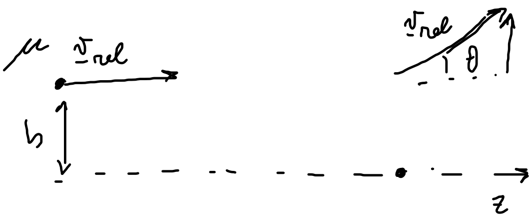
identity

$$E = E_T + E_F = \underbrace{\frac{1}{2} m_F v_F^2 + \frac{1}{2} m_T v_T^2}_{i} = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \mu v_{rel}^2$$

$M = m_T + m_F$

In c.m. $V_{cm} = 0 \Longrightarrow E_{c.m.} = \frac{1}{2}\mu v_{rel}^2 \Longrightarrow v_{rel,i} = v_{rel,f}$

In c.m. frame $\underline{v}_{rel}$ experiences is a rotation



$$\underline{v}_{rel,f} = v_{rel} \cos\theta\, \hat{\underline{z}} + v_{rel} \sin\theta \left(\cos\varphi\, \hat{\underline{x}} + \sin\varphi\, \hat{\underline{y}}\right)$$

$$\simeq v_{rel}\left(1 - \frac{\theta^2}{2}\right)\hat{\underline{z}} + \theta v_{rel}\left(\cos\varphi\, \hat{\underline{x}} + \sin\varphi\, \hat{\underline{y}}\right)$$

$\theta \ll 1$

$$\Delta \underline{v}_{rel} = \theta v_{rel}\left(\cos\varphi\, \hat{\underline{x}} + \sin\varphi\, \hat{\underline{y}}\right) - \frac{\theta^2}{2} v_{rel}\, \hat{\underline{z}}$$

$$\Delta \underline{v}_{rel} = -v_{rel}\frac{\theta^2}{2}\hat{\underline{z}} + v_{rel}\,\theta\,(\cos\varphi\,\hat{\underline{x}} + \sin\varphi\,\hat{\underline{y}})$$

Average over all possible → b
→ φ
→ v_{rel} (at fixed v_T)

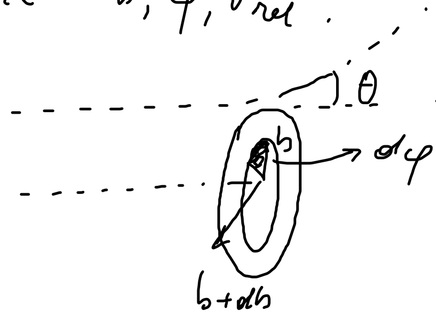
$$v_{rel} = |\underline{v}_T - \underline{v}_F|$$

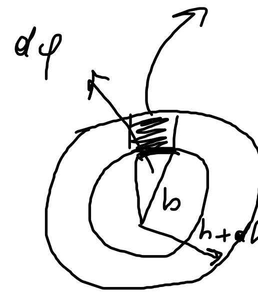
$d\text{Area} = b\,d\varphi\,db$

# of collisions that occur at b, φ, v_{rel}?

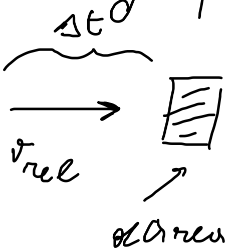
Those particles that impinge on $d\text{Area}$

are → deflected by φ
→ " " $\theta(b)$





How many particles enter dArea?



collisions $= \underbrace{v_{rel}\,\Delta t\,dArea}_{\text{volume}}$ (density of scatterers)

$$n_{\underline{F}} \longrightarrow \int f(\underline{v}_F)\,d^3\underline{v}_F$$

$$\#\text{collisions} = v_{rel}\,\Delta t\, f(\underline{v}_F)\,b\,db\,d\varphi\,d^3\underline{v}_F$$

v_{rel} fixed

$\underline{v}_T$ is fixed

$$v_{rel} = |\underline{v}_T - \underline{v}_F|$$

$$n_F = \int f(\underline{v}_F)\,d^3\underline{v}_F$$

$$\left\langle \frac{\Delta \underline{v}_{rel}}{\Delta t} \right\rangle = \int_{b,\varphi,\underline{v}_F} \Big(\overbrace{-v_{rel}\frac{\theta^2}{2}\hat{\underline{z}} + v_{rel}\,\theta\,(\cos\varphi\,\hat{\underline{x}} + \sin\varphi\,\hat{\underline{y}})}^{\Delta \underline{v}_{rel}} \Big)\, v_{rel}\, f(\underline{v}_F)\, b\,db\,d\varphi\,d^3\underline{v}_F$$

$$\int_0^{2\pi}\cos\varphi\,d\varphi = 0 = \int_0^{2\pi}\sin\varphi\,d\varphi$$

$$\left\langle \frac{\Delta \underline{v}_{rel}}{\Delta t} \right\rangle = -\hat{\underline{z}} \int_{b,\, \underline{v}_F} v_{rel}^2 \, \frac{\theta^2}{2} \, f(\underline{v}_F) \, b \, db \, d^3\underline{v}_F \cdot 2\pi$$

$$tg\left(\frac{\theta}{2}\right) = \frac{q_T q_F}{4\pi \varepsilon_0 b \mu v_{rel}^2} \qquad \theta \ll 1$$

$$\theta \simeq \frac{q_T q_F}{2\pi \varepsilon_0 b \mu v_{rel}^2}$$

$$= -\hat{\underline{z}} \, \frac{1}{\cancel{2}} \, \frac{q_T^2 q_F^2 \cdot \cancel{2\pi}}{4\pi \varepsilon_0^2 \mu^2} \int_{b,\, \underline{v}_F} \cancel{v_{rel}^2} \, \frac{1}{v_{rel}^{\cancel{4}\, 2}} \, \cancel{b} \, \frac{1}{b^{\cancel{2}}} \, f(\underline{v}_F) \, d^3\underline{v}_F \, db$$

$$= -\hat{\underline{z}} \, \frac{q_T^2 q_F^2 \ln\Lambda}{4\pi \varepsilon_0^2 \mu^2} \int d^3\underline{v}_F \, \frac{f(\underline{v}_F)}{|\underline{v}_T - \underline{v}_F|^2}$$

$$\int_{b_{\pi/2}}^{\lambda_D} \frac{db}{b} = \ln\Lambda = \ln\left[\frac{\lambda_D}{b_{\pi/2}}\right]$$

Coulomb logarithm

$15 \div 30$

$$\left\langle \frac{\Delta \underline{v}_T}{\Delta t} \right\rangle = \frac{\mu}{m_T} \left\langle \frac{\Delta \underline{v}_{rel}}{\Delta t} \right\rangle = -\hat{\underline{z}} \frac{q_F^2 q_T^2 \ln\Lambda}{4\pi\varepsilon_0^2 \mu m_T} \int \frac{f_F(\underline{v}_F)\, d^3 \underline{v}_F}{|\underline{v}_T - \underline{v}_F|^2}$$

$$\frac{1}{|\underline{v}_T - \underline{v}_F|^2} \hat{\underline{z}} = \frac{(\underline{v}_T - \underline{v}_F)}{|\underline{v}_T - \underline{v}_F|^3} \qquad \hat{\underline{z}} = \frac{(\underline{v}_T - \underline{v}_F)}{|\underline{v}_T - \underline{v}_F|}$$

$$= -\frac{\partial}{\partial \underline{v}_T} \frac{1}{|\underline{v}_T - \underline{v}_F|}$$

$$\boxed{\left\langle \frac{\Delta \underline{v}_T}{\Delta t} \right\rangle = \frac{q_F^2 q_T^2}{4\pi\varepsilon_0^2 \mu m_T} \ln\Lambda \frac{\partial}{\partial \underline{v}_T} \int d^3 \underline{v}_F \frac{f_F(\underline{v}_F)}{|\underline{v}_T - \underline{v}_F|}}$$

$$\int_0^{2\pi} \cos^2\varphi \, d\varphi = \int_0^{2\pi} \sin^2\varphi \, d\varphi = \pi$$

$$\Delta \underline{v}_{rel} = -\frac{\theta^2}{2} v_{rel} \, \hat{\underline{z}} + v_{rel} \, \theta \left(\cos\varphi \, \hat{\underline{x}} + \sin\varphi \, \hat{\underline{y}} \right)$$

$$\Delta \underline{v}_{rel} \Delta \underline{v}_{rel} = \begin{pmatrix} v_{rel}^2 \theta^2 \cos^2\varphi & \alpha \sin\varphi \cos\varphi & \alpha \cos\varphi \\ \alpha \sin\varphi \cos\varphi & v_{rel}^2 \theta^2 \sin^2\varphi & \alpha \sin\varphi \\ \alpha \cos\varphi & \alpha \sin\varphi & \frac{\theta^4}{4} v_{rel}^2 \simeq 0 \end{pmatrix}$$

$$\left\langle \frac{\Delta \underline{v}_{rel} \Delta \underline{v}_{rel}}{\Delta t} \right\rangle = $$

$$= \int \frac{\Delta \underline{v}_{rel} \Delta \underline{v}_{rel}}{\Delta t} \cdot \underbrace{\left(\# \text{collisions at } \underline{v}_F, b, \varphi \text{ fixed} \right)}_{v_{rel} \Delta t \, b \, db \, d\varphi \, d^3 \underline{v}_F}$$

After integration over φ

$$\left\langle \frac{\Delta \underline{v}_{rel}\, \Delta \underline{v}_{rel}}{\Delta t} \right\rangle = \pi \int v_{rel}\, b\, db\, d^3\underline{v}_F\, v_{rel}^2\, \theta^2 \begin{pmatrix} 1 & 0 & \\ & 1 & \\ 0 & & 0 \end{pmatrix} f_F(\underline{v}_F) \qquad \theta = \frac{q_T q_F}{2\pi \varepsilon_0 b \mu v_{rel}^2}$$

$$= \cancel{\pi} \frac{q_T^2 q_F^2}{4\cancel{\pi^2} \varepsilon_0^2 \mu^2} \int db\, \cancel{b} \frac{\cancel{v_{rel}}}{v_{rel}^{\cancel{4}}} \frac{1}{b^{\cancel{2}}} \cancel{v_{rel}^2} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} d^3\underline{v}_F\, f_F(\underline{v}_F)$$

$$\int_{b_{\pi/2}}^{\lambda_D} \frac{db}{b} = \ln\Lambda$$

$$= \frac{q_T^2 q_F^2 \ln\Lambda}{4\pi \varepsilon_0^2 \mu^2} \int d^3\underline{v}_F\, \frac{f_F(\underline{v}_F)}{|\underline{v}_T - \underline{v}_F|} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

$$\begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix} = \mathbb{1} - \begin{pmatrix}0&&\\&0&\\&&1\end{pmatrix} =$$

$$= \mathbb{1} - \frac{\underline{v}_{rel}\,\underline{v}_{rel}}{v_{rel}^2} = \frac{v_{rel}^2\,\mathbb{1} - \underline{v}_{rel}\,\underline{v}_{rel}}{v_{rel}^2}$$

$$\frac{1}{v_{rel}}\begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix} = \frac{v_{rel}^2\,\mathbb{1} - \underline{v}_{rel}\,\underline{v}_{rel}}{v_{rel}^3} = \frac{\partial}{\partial \underline{v}_T}\underbrace{\frac{\partial \overbrace{v_{rel}}^{\text{scalar}}}{\partial \underline{v}_T}}_{\text{vector}} = \overbrace{\frac{\partial}{\partial \underline{v}_1}}^{\text{gradient}}\ \overbrace{\frac{\underline{v}_{rel}}{v_{rel}}}^{\text{vector}}$$

$$\frac{\partial}{\partial \underline{x}}\,\underline{A}(\underline{x}) \overset{def}{=} \underline{\underline{M}} \qquad M_{ij} \overset{def}{=} \frac{\partial}{\partial x_i} A_j(\underline{x})$$

$$\left\langle \frac{\Delta \underline{v}_T \Delta \underline{v}_T}{\Delta t} \right\rangle = \left(\frac{\mu}{m_T} \right)^2 \cdot \left\langle \frac{\Delta \underline{v}_{rel} \, \Delta \underline{v}_{rel}}{\Delta t} \right\rangle$$

$$= \frac{q_T^2 q_F^2}{4\pi \varepsilon_0^2 \cancel{\mu^2}} \frac{\cancel{\mu^2}}{m_T^2} \int d^3 \underline{v}_F \, f_F(\underline{v}_F) \frac{\partial}{\partial \underline{v}_T} \frac{\partial}{\partial \underline{v}_T} \left| \underline{v}_T - \underline{v}_F \right|$$

$$= \frac{q_T^2 q_F^2}{4\pi \varepsilon_0^2 m_T^2} \frac{\partial}{\partial \underline{v}_T} \int d^3 \underline{v}_F \, f_F(\underline{v}_F) \frac{\partial}{\partial \underline{v}_T} \left| \underline{v}_T - \underline{v}_F \right|$$

Rosenbluth potentials

$$g_F(\underline{v}) = \int |\underline{v} - \underline{v}'| \, f_F(\underline{v}') \, d^3\underline{v}'$$

$$h_F(\underline{v}) = \frac{m_T}{\mu} \int \frac{f_F(\underline{v}')}{|\underline{v} - \underline{v}'|} \, d^3\underline{v}'$$

$$\frac{\partial f_T}{\partial t} = \sum_{F=i,e} \frac{q_T^2 q_F^2}{4\pi\varepsilon_0^2 m_T^2} \ln\Lambda \left[-\frac{\partial}{\partial \underline{v}} \cdot \left(f_T \frac{\partial h_F}{\partial \underline{v}} \right) + \frac{1}{2} \frac{\partial}{\partial \underline{v}} \frac{\partial}{\partial \underline{v}} : \left(f_T \frac{\partial^2 g_F}{\partial \underline{v} \partial \underline{v}} \right) \right]$$

Fokker-Planck equation for a plasma