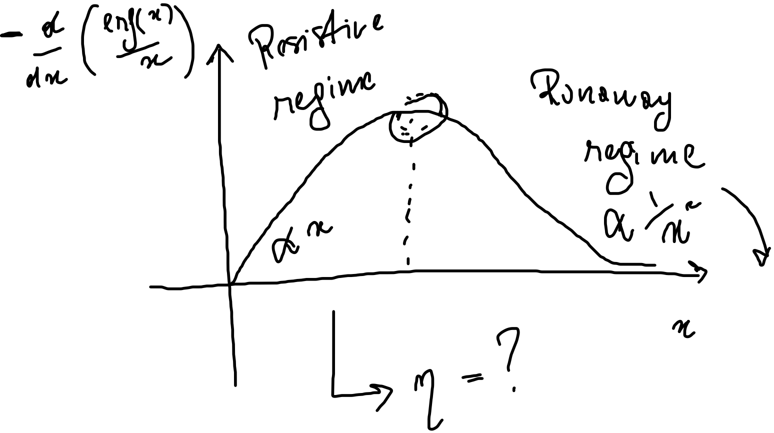
$$\frac{\partial u_T}{\partial t} = \frac{e^2 q_T^2 \ln\Lambda \, n_e}{4\pi \varepsilon_0^2 m_T^2} \left[Z \left(1 + \frac{m_T}{m_i}\right) \frac{d}{du} \left(\frac{\operatorname{erf}\left(u/v_{thi}\right)}{u} \right) + \left(1 + \frac{m_T}{m_e}\right) \frac{d}{du} \left(\frac{\operatorname{erf}\left(u/v_{the}\right)}{u} \right) \right]$$



$$x = u/v_{the}$$

What is E that leads to a runaway regime?

Resistivity η

We apply $\underline{E}$ – There is a friction from the plasma.
Balance between $\underline{E}$ and friction produces $\underline{u}_e$ $\underline{u}_i$

$$f_e(\underline{v}) = n_e \left(\frac{m_e}{2\pi T_e}\right)^{3/2} \exp\left(-\frac{(\underline{v}-\underline{u}_e)^2 m_e}{2T_e}\right)$$

$$-en_e + Zen_i = 0$$

$$f_i(\underline{v}) = n_i \left(\frac{m_i}{2\pi T_i}\right)^{3/2} \exp\left(-\frac{(\underline{v}-\underline{u}_i)^2 m_i}{2T_i}\right)$$

current density $\rightarrow$ $\underline{j} = \underset{\approx}{\eta}\, \underline{E}$

$$\underline{j} = \underline{j}_e + \underline{j}_i = \overbrace{-n_e e \underline{u}_e}^{\underline{j}_e} + \overbrace{Zen_i \underline{u}_i}^{\underline{j}_i}$$

$$= \underbrace{Zen_i}_{n_e e}(\underline{u}_i - \underline{u}_e)$$

We choose the electron ref. frame

$T_e \approx T_i$

$\underline{u}_e' = 0 \qquad \underline{u}_i' = ? = \underline{u}_i - \underline{u}_e$

$\approx -\underline{u}_e$

$u_e \gg u_i$

$v_{th_e} \gg v_{th_i}$

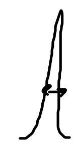
$$f_i'(\underline{v}) = n_i \left(\frac{m_i}{2\pi T_i}\right)^{3/2} \underbrace{\exp\left(-\frac{(\underline{v}-\underline{u}_i')^2 m_i}{2T_i}\right)}$$

Centre: $u_e \sim v_{th_e}$

Width: $\propto \sqrt{\frac{2T_i}{m_i}} = v_{th_i} \Rightarrow$

$$\exp\left(-\frac{(\underline{v}-\underline{u}_e)^2}{\sigma^2}\right)$$

$f_i' \sim$ Dirac delta at $\underline{u}_i' \sim \underline{u}_e$



Ion eq. of motion in the el. ref. frame

$$m_i \frac{\partial \underline{u}_{rel}}{\partial t} = (\text{friction}) + Ze\underline{E} =$$

$$m_i \frac{\partial \underline{u}_{rel}}{\partial t} = \underbrace{\frac{n_e Z^2 e^2 \cdot e^2 \ln\Lambda}{4\pi\varepsilon_0^2 \mu_e} \left[\frac{\partial}{\partial \underline{v}} \left(\frac{1}{v} \operatorname{erf}\left(\frac{v}{v_{th}} \right) \right) \right]}_{\text{friction}} + Ze\underline{E}$$

$\underline{v} = \underline{u}_{rel}$

Steady state solution: $\frac{\partial \underline{u}_{rel}}{\partial t} = 0$

Resistive regime: $\frac{d}{dx}\left(\frac{\operatorname{erf}(x)}{x} \right) \approx \frac{-4}{3\sqrt{\pi}} x$ $\quad x = v/v_{th}$

$$\frac{d}{dv} = \frac{d}{dx}\frac{dx}{dv} = \frac{1}{v_{th}}\frac{d}{dx}$$

$$\frac{1}{v_{th}} \frac{d}{dv} \left(\frac{1}{\underset{x}{\frac{v}{v_{th}}}} \operatorname{erf}\left(\frac{v}{v_{th}} \right) \right) = \frac{1}{v_{th}} \frac{d}{dv} \left(\frac{\operatorname{erf}(x)}{x} \right) = \frac{1}{v_{th}^2} \frac{d}{dx} \left(\frac{\operatorname{erf}(x)}{x} \right)$$

$$\approx \frac{-4}{3\sqrt{\pi}} \frac{x}{v_{th}^2} \Bigg|_{u_{rel}} = \frac{-4\, u_{rel}}{3\sqrt{\pi}\, v_{th}^3}$$

$$\cancel{Z}e\,\underline{E} = \frac{n_e Z e^{\cancel{2}\,\cancel{4}\,3} \ln\Lambda}{4\pi\varepsilon_0^2\, \mu_e}\, \frac{4}{3\sqrt{\pi}} \underbrace{\left(\frac{m_e}{2T_e}\right)^{3/2}}_{1/v_{th_e}^3}\, \underline{u}_{rel}$$

$$\underline{E} = \eta\, \underline{j}$$

$$= \eta\, n_e\, e\, \underline{u}_{rel}$$

$$\eta = \frac{Z e^2 \ln\Lambda}{\cancel{4}\pi\varepsilon_0^2\, \cancel{\mu_e}}\, \frac{\cancel{4}}{3\sqrt{\pi}} \left(\frac{m_e}{2T_e}\right)^{3/2} = \boxed{\frac{Z e^2 \ln\Lambda\, m_e^{\,T_2}}{3\pi^{3/2}\varepsilon_0^2\, (2T_e)^{3/2}}}$$

$\pi^{3/2}$

$\mu_e \approx m_e$

$$\eta \propto \frac{1}{T_e^{3/2}}$$

Hydrogen plasma $Z = 1$

$T = 1\ \text{keV}$ $\qquad 1\ \text{eV} \rightarrow 12000\ \text{K}$

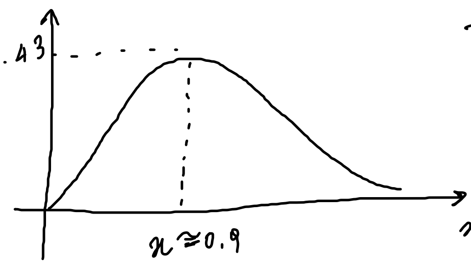
$T \sim$ several millions K

$$\eta \approx 7 \cdot 10^{-8}\ \Omega \cdot m$$

Tokamak

$$\eta_{\text{Copper}} \approx 1.7 \cdot 10^{-8}\ \Omega \cdot m \qquad I \sim \text{MA}$$

E_{Dreicer} $\quad -\frac{d}{dx}\left(\frac{1}{x}\,\text{erf}(x)\right)$



$$E_{\text{Dreicer}} = \frac{n_e \cancel{Z} e^3 \ln\Lambda}{4\pi \varepsilon_0^2 \cancel{m_e}} \left(\frac{\cancel{m_e}}{2T_e}\right) \underbrace{\left[\frac{d}{dx}\left(\frac{\text{erf}(x)}{x}\right)\right]}_{\max \approx 0.43}$$

$$= 0.43\, \frac{n_e Z e^3 \ln\Lambda}{\cancel{4}8\pi \varepsilon_0^2 T_e}$$

Power density = $\underline{j} \cdot \underline{E}$

dV $\quad \underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$

$P = \underline{F} \cdot \underline{v} = q \underline{E} \cdot \underline{v}$ } for 1 charge

n density of charges in dV $\quad P_{TOT} = n q \underline{E} \cdot \underline{v} = \underline{j} \cdot \underline{E}$ $\quad [P_{TOT}] = W/m^3$

$= \eta j^2$

$\underline{E} = \eta \underline{j}$

Transfer of energy to electrons and/or ions?

$v_{thi} << v_{the}$

$(u << v_{the}, v_{thi})$ $\quad v_{thi} < u < v_{the}$ $\quad (u > v_{the}, v_{thi})$

$v_{thi} << u << v_{the}$

$u/v_{thi} >> 1$

$x_i = u/v_{thi}$

$$\underbrace{\frac{d}{dx_i}\left[\frac{\operatorname{erf}(\cancel{u}/v_{thi})}{x_i}\right]}_{\text{ion contribution}} \approx -1/x_i^2.$$

$u/v_{the} << 1$

$x_e = u/v_{the}$

$$\underbrace{\frac{d}{dx_e}\left[\frac{\operatorname{erf}(x_e)}{x_e}\right]}_{\text{electron contribution}} \approx -\frac{4}{3\sqrt{\pi}}x_e$$

$$\frac{du}{dt} = - \frac{e^2 q_T^2 \ln\Lambda \, n_e}{4\pi \varepsilon_0^2 m_T^2} \cdot$$

$$\left[\left(1+\frac{m_T}{m_e}\right) \frac{m_e}{2T_e} \frac{4 x_e}{3\sqrt{\pi}} + Z \left(1+\frac{m_T}{m_i}\right) \frac{m_i}{2T_i} \frac{1}{x_i^2} \right]$$

$$\underbrace{\left(1+\frac{m_T}{m_e}\right)\left(\frac{m_e}{2T_e}\right)^{\frac{3}{2}} \frac{4}{3\sqrt{\pi}} u}_{\text{electron}} + \underbrace{Z\left(1+\frac{m_T}{m_i}\right) \frac{1}{u^2}}_{\text{ion}}$$

If u is "large": mostly collisions against electrons

If u is "small": mostly collisions against ions

What is the u_{cr} so that (electron friction) = (ion friction)?

$$\left(1+\frac{m_T}{m_e}\right)\left(\frac{m_e}{2T_e}\right)^{3/2}\frac{4}{3\sqrt{\pi}}u = Z\left(1+\frac{m_T}{m_i}\right)\frac{1}{u^2};$$

$$u^3 = \frac{Z\left(1+\frac{m_T}{m_i}\right)3\sqrt{\pi}\left(\frac{2T_e}{m_e}\right)^{3/2}}{4\underbrace{\left(1+\frac{m_T}{m_e}\right)}_{\frac{m_T}{m_e}}}$$

T = ion

$m_T = m_i$ $\quad Z=1$ (Hydrogen)

$$E_{cr} = \frac{1}{2}m_T u_T^2 = \left(\qquad\right)\cdot T_e \approx 24\cdot T_e$$

$20 \div 30\ T_e$

$E_b > E_{cr}$: $\quad E_{th} \qquad E_{cr} \qquad E_b$

mostly ions | mostly electrons

$E_b < E_{cr}$: $\quad E_{th} \qquad E_b \qquad E_{cr}$

mostly ions |