Three-valued logics

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Outline

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Introduction

- Problem: assign a logical value to propositions about partial recursive functions
- Partial recursive functions $f : \mathbb{N}^n \mapsto \mathbb{N}$, can be defined only on a subset of $\mathbb{N}^n$
- Kleene defines two logics: weak and strong
- In the strong version, the third value is interpreted as *unknown*:

  > *is a category into which we can regard any proposition as falling, whose value we either do not know or choose for the moment to disregard and it does not then exclude the other two possibilities* true *and* false.

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Truth values

- Boolean truth values $\{f, t\}$, Kleene truth values $\{F, T, U\}$
- T: I know that the Boolean truth value of a proposition is true $T = \{t\}$
- F I know that the Boolean truth value of a proposition is false $F = \{f\}$
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The connectives are defined through operations on sets

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Connectives

Conjunction

$$F \sqcap T = \{f\} \sqcap \{t\} = \{f \wedge t\} = \{f\} = F$$

$$F \sqcap U = \{f\} \sqcap \{f,t\} = \{f \wedge f, f \wedge t\} = \{f\} = F$$

$$T \sqcap U = \{t\} \sqcap \{f,t\} = \{f \wedge t, t \wedge t\} = \{f,t\} = U$$

$$\ldots$$

Disjunction:

$$F \sqcup U = \{f\} \sqcup \{f,t\} = \{f \vee f, f \vee t\} = \{f,t\} = U$$

$$T \sqcup U = \{t\} \sqcup \{f,t\} = \{t \vee t, t \wedge f\} = \{t\} = T$$

$$U \sqcup U = \{f,t\} \sqcup \{f,t\} = \{f \vee f, f \vee t, t \vee t\} = \{f,t\} = U$$

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Truth tables

$\sqcap$	F	U	T
F	F	F	F
U	F	U	U
T	F	U	T

$\sqcup$	F	U	T
F	F	U	T
U	U	U	T
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Negation $U' = U$, $F' = T$, $T' = F$

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Negation $U' = U$, $F' = T$, $T' = F$

Implication

$a \rightarrow_K b := a' \sqcup b$

$\rightarrow_K$	F	U	T
F	T	T	T
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T	F	U	T

Modus ponens holds: if a is true and $a \rightarrow_K b$ is true then b is true

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Modus ponens holds: if a is true and $a \rightarrow_K b$ is true then b is true

Some comments

- If we map
  - F to 0
  - T to 1
  - U to $\frac{1}{2}$

  then $\sqcap$ corresponds to min, $\sqcup$ to max and the negation of x to $1 - x$
- there are no tautologies
  - $\nexists$ a formula P such that $\forall v$, $v(P) = T$
  - If to all variables we assign U then by we get only U (and never T)

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Kleene logic - summarizing

Kleene logic has 3 truth values $\mathbb{V}_3 = \{\mathbf{0} < \frac{\mathbf{1}}{\mathbf{2}} < \mathbf{1}\}$

- Syntax the same connectives as classical logic : $(\wedge, \vee, \neg)$
- Semantics
  - *Negation*: $\mathbf{v}(\neg p) = 1 - \mathbf{v}(p))$
  - *Conjunction*: $\mathbf{v}(p \wedge q) = \min(\mathbf{v}(p), \mathbf{v}(q))$
  - *Disjunction* : $\mathbf{v}(p \vee q) = \max(\mathbf{v}(p), \mathbf{v}(q))$
  - *Implication* : $\mathbf{v}(p \to_K q) = \max(1 - \mathbf{v}(p), \mathbf{v}(q))$ (using $p \to_K q \equiv \neg p \vee q$)
- It extends Boolean logic
- There are no tautologies
  In particular $\mathbf{v}(p \wedge \neg p) = \mathbf{v}(p \vee \neg p) = \frac{1}{2}$ when $\mathbf{v}(p) = \frac{1}{2}$

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Is "Unknown" a truth value?

- In practice, $\frac{1}{2}$ is used to *model the idea that the truth-value of a Boolean proposition is unknown*
- "'Unknown" is in conflict with "Known to be true" (T) and "Known to be false (F), not with "true" (t) and "false" (f)
- One must distinguish between two levels:
  - Ontic values: $v(p)$: true (t), false (f)
  - Epistemic values $\mathbf{v}(p)$:
    certainly true $T = \{t\}$, certainly false $F = \{f\}$, unknown $T = \{f, t\}$
- The three-valued $\mathbf{v}(p)$ informs about the state of knowledge about the truth $v(p)$ of a Boolean proposition

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Priest Logic

- $\frac{1}{2}$ = inconsistent, paradoxical, both true and false
- Kleene, with two designated values
  - Connectives: min, max, $\neg$
  - Designated truth values: $\frac{1}{2}, 1$ (a logical formula is considered a tautology if it evaluates to a designated truth value)
  - All Boolean tautologies are valid, no modus ponens

Kleene logic and SQL

- Let us suppose to have two relations Parts and Suppliers

S_n	City
S1	London

P_n	City
P1	NULL

- Query: select all the pairs (S_n, P_n) such that "the city of the supplier is different from the city of the part or the city of the part is different from Paris"

```
SELECT Suppliers.Sn, Parts.Pn
FROM. Suppliers, Parts
WHERE Suppliers.city <> Parts.city OR Parts.city
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Query result

- Kleene logic is used $\rightarrow$ the answer is NULL
- Let us suppose to know the value of *Parts.City* and consider the different cases:
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Three-valued logic applications

There are several situations where three valued logics arise or are imposed:

- NULL value in databases
  - Usually Kleene (min, max) logic is used (SQL)
  - Critics: does not respect the semantics in queries
  - There can be different semantic for NULL
- Rough sets: lower approximation (1), exterior region (0), boundary (1/2) (as we will discuss)
- Shadowed sets: an approximation of a fuzzy set through $\{0, [0, 1], 1\}$
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Applications

- Bipolar information
  - the most simple bipolar scale is three valued $\{-1, 0, +1\}$
  - positive/negative preferences
- Logic programming
  - Logic *here-and-there* used in Answer Set Programming (ASP)
- Aymara language
- ...

NOTATION: $\mathbf{3} = \{F, N, T\}$ with $F < N < T$

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Conjunctions

Minimal requirement: it should act as a Boolean conjunction on $\{0, 1\}$

Still $243(3^5)$ possibilities.

We adopt a more stringent definition and require monotonicity

Definition

A conjunction on $\mathbf{3} = \{F, N, T\}$ is a binary mapping $*$: $\mathbf{3} \times \mathbf{3} \mapsto \mathbf{3}$ such that

1. If $x \leq y$ then $x * z \leq y * z$ (left monotonicity)
2. If $x \leq y$ then $z * x \leq z * y$ (right monotonicity)
3. $F * F = F * T = T * F = F$ and $T * T = T$ (conformity with Boolean logic)

Note that $N * F = F * N = F$

Some conjunctions are not covered: Kleene weak logic, McCarthy logic, . . .

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$*$	F	N	T
F	F	F	F
N	F		
T	F		T

n.	$N * N$	$T * N$	$N * T$	
1	T	T	T	Sette
2	N	T	T	quasi conjunction/Sobociński
3	N	T	N	
4	N	N	T	
5	N	N	N	min/interval conjunction/Kleene
6	F	F	T	
7	F	F	N	
8	F	F	F	Bochvar external
9	F	N	F	
10	F	N	T	
11	F	N	N	Łukasiewicz
12	F	T	F	
13	F	T	N	
14	F	T	T	

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Only six are commutative

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Only five are commutative and associative

Conjunctions

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T-norms, uninorm

Implications

Conformity with Boolean logic, monotonicity

Definition

An implication on **3** is a binary mapping $\rightarrow$: **3** $\times$ **3** $\mapsto$ **3** such that

- I1 If $x \leq y$ then $y \rightarrow z \leq x \rightarrow z$
- I2 If $x \leq y$ then $z \rightarrow x \leq z \rightarrow y$
- I3 $F \rightarrow F = T \rightarrow T = T$ and $T \rightarrow F = F$

Other important properties

- I4 $T \rightarrow x = x$ (left neutrality or border condition)
- I5 $x \rightarrow y = T$ iff $x \leq y$ (ordering property)

Implications

Conformity with Boolean logic, monotonicity

Definition

An implication on **3** is a binary mapping $\rightarrow$: $\mathbf{3} \times \mathbf{3} \mapsto \mathbf{3}$ such that

- I1 If $x \leq y$ then $y \rightarrow z \leq x \rightarrow z$
- I2 If $x \leq y$ then $z \rightarrow x \leq z \rightarrow y$
- I3 $F \rightarrow F = T \rightarrow T = T$ and $T \rightarrow F = F$

Other important properties

- I4 $T \rightarrow x = x$ (left neutrality or border condition)
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Implications

$\rightarrow$	F	N	T
F	T	T	T
N			T
T	F		T

n.	N $\rightarrow$ N	T $\rightarrow$ N	N $\rightarrow$ F	I4	I5	
1	F	F	F	no	no	
2	N	F	F	no	no	Sobociński
3	N	F	N	no	no	
4	N	N	F	no	no	Jaśkowski
5	N	N	N	no	no	(strong) Kleene
6	T	T	F	no	no	Sette
7	T	T	N	no	no	
8	T	T	T	no	no	
9	T	N	T	yes	no	Nelson
10	T	N	F	yes	yes	Gödel
11	T	N	N	yes	yes	Łukasiewicz
12	T	F	T	no	no	Bochvar external
13	T	F	N	no	yes	
14	T	F	F	no	yes	Gaines–Rescher

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From implications to negations

Define negations as $a' := a \rightarrow 0$

Only three different negations depending on the value assigned to $\frac{1}{2}'$:

- $\frac{1}{2}' = 0$, an intuitionistic negation, implications 1,2,4,6,10,14
- $\frac{1}{2}' = \frac{1}{2}$, an involutive negation, implications 3,5,7,11,13
- $\frac{1}{2}' = 1$, a paraconsistent negation, implications 8,9,12

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Outline

Relationships

Standard ways to obtain conjunction from implications and vice versa (Dubois, Prade 1984)

Material implication $\mathcal{S}$	$a * b = \neg(a \to \neg b)$
Exchange $\mathcal{A}$	$a *' b = b * a$
Contraposition $\mathcal{V}$	$a \to' b = \neg b \to \neg a$
Residuation $\mathcal{I}$	

$$a \to' b = \begin{cases} 0 & \text{if there is no } s \text{ such that } a * s \leq b \\ \sup\{s : a * s \leq b\} & \text{otherwise} \end{cases}$$

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Relationships - case 1

Let $i \in \{1, 2, 11\}$ then

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\begin{tikzcd}
\to_i \arrow[loop right, "\mathcal{V}"] \arrow[d, bend left, "\mathcal{S}"] \\
*_i \arrow[u, bend left, "\mathcal{I}"] \arrow[loop right, "\mathcal{A}"]
\end{tikzcd}
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- Łukasiewicz logic $(\to_{11}, *_{11}, +_{11}, \neg)$
- Sobociński logic $(\to_2, *_2, +_2, \neg)$
  - $+_2$ can be defined as $a +_2 b := \neg a \to_2 b$, designated values are T, N
  - The third value means irrelevant
  - Conjunction $*_2$ is a discrete uninorm with N as neutral element and implication $\to_2$ its residuum.

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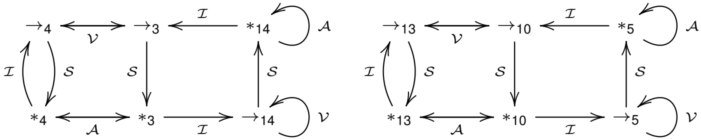
Relationships - case 2



- Gödel (intuitionistic) logic $(\rightarrow_{10}, *_5(min), max, \sim)$ on three values, also known as logic of here-and-there in logic programming
- Jaskowski logic $(\rightarrow_4, *_5(min), max, \neg)$ has been studied by several authors in the field of paraconsistent logic. The designated values are N and T.
  Third value: inconsistent, paradoxical
  It is equivalent to Sobocinski logic through the following definitions:

$$p \rightarrow_S q := (p \rightarrow_J q) \wedge (\neg q \rightarrow_J \neg p)$$
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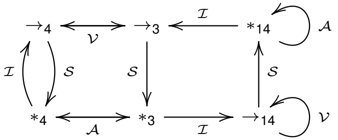

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{*_4} \arrow[u, bend left, "\mathcal{I}"] \arrow[r, leftrightarrow, "\mathcal{A}"'] & {*_3} \arrow[r, "\mathcal{I}"'] & {\to_{14}} \arrow[u, "\mathcal{S}"'] \arrow[loop right, "\mathcal{V}"]
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\begin{tikzcd}
{\to_{13}} \arrow[r, leftrightarrow, "\mathcal{V}"'] \arrow[d, bend left, "\mathcal{S}"] & {\to_{10}} \arrow[d, "\mathcal{S}"'] & {*_5} \arrow[l, "\mathcal{I}"'] \arrow[loop right, "\mathcal{A}"] \\
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Relationships - case 3 and 4

$$\begin{tikzcd}
\to_{12} \arrow[r, leftrightarrow, "\mathcal{V}"'] \arrow[d, bend left, "\mathcal{S}"] & \to_{6} \arrow[d, "\mathcal{S}"'] & \\
*_{12} \arrow[u, bend left, "\mathcal{I}"] \arrow[r, leftrightarrow, "\mathcal{A}"'] & *_{6} \arrow[r, "\mathcal{I}"'] & \text{"out"}
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- Bochvar logic $(\to_{12}, *_8, +_8, \flat)$ where $x +_8 y$ is T if at least one of a and b is equal to T and F in all other cases.
  Third value N stands for meaningless
- Sette paraconsistent logic $(\to_6, *_1, \flat)$ where $x +_1 y$ takes the value F if $x = y = F$ and T otherwise and designated values are N and T

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Connections among groups

Other way to define an implication given another implication (all these equalities hold in Boolean logic)

- $p \rightarrow_{new} q = (p \rightarrow q) \wedge (\neg q \rightarrow \neg p)$
- $p \rightarrow_{new} q = q \vee (p \rightarrow q)$
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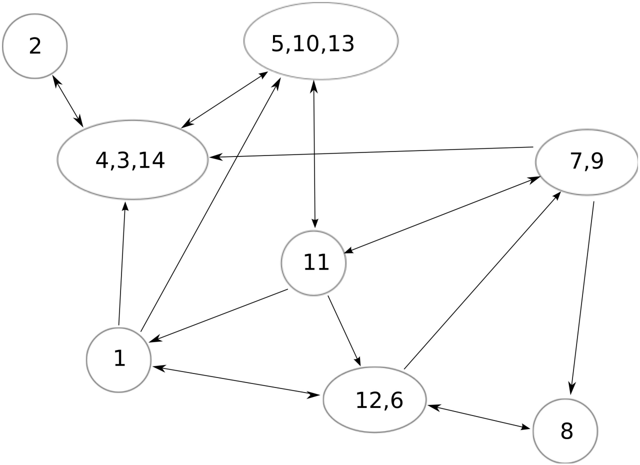
Connections among groups - diagram



3-valued logic: one or many?

Relationship among connectives

All the 14 implications and conjunctions can be defined starting from a single logical system

- $(\{0, \frac{1}{2}, 1\}, \rightarrow_L, 0)$ where $\rightarrow_L$ is Łukasiewicz implication
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Functional Completeness

All functions with Boolean results on Boolean values are definable in Łukasiewicz logic

Ciucci, Dubois, "A map of dependencies among three-valued logics", Information Sciences 250, 162-177 (2013)

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Feasibility $\neq$ simplicity

Implications in Łukasiewicz logic
$\Delta(a) = a \odot a$, $\nabla(a) = a \oplus a$, $J(a) = \nabla(\neg a \wedge \neg\nabla(\neg a \wedge a))$

n	$a \to_n b$	
1	$\Delta(\neg a) \vee \Delta(b)$	
2	$(b \vee (a \to_1 b)) \wedge (\neg a \vee (\neg b \to_1 \neg a))$	Sobociński
3	$\neg a \vee [(b \vee (a \to_1 b)) \wedge (\neg a \vee (\neg b \to_1 \neg a))]$	
4	$b \vee (\Delta(\neg a) \vee \Delta(b))$	Jaśkowski
5	$\neg a \vee (\Delta(\neg a) \vee \Delta(b))$	(strong) Kleene
6	$J(b) \to_L J(a)$	Sette
7	$\neg b \to_L (\neg b \to_L \neg a)$	
8	$a \to_L (a \to_L b)) \vee (\neg b \to_L (\neg b \to_L \neg a))$	
9	$a \to_L (a \to_L b)$	Nelson
10	$b \vee [(J(b) \to_L J(a)) \wedge (J(\neg a) \to_L J(\neg b))]$	Gödel
12	$J(\neg a) \to_L J(\neg b)$	Bochvar external
13	$\neg a \vee [(J(\neg a) \to_L J(\neg b)) \wedge (J(b) \to_L J(a))]$	
14	$(J(\neg a) \to_L J(\neg b)) \wedge (J(b) \to_L J(a))$	Gaines–Rescher

Translations in modal logic

The translation of Implications

Implication	Translation $\mathcal{T}_1(a \to b)$
1–5 (Sobocinski, Jaskowski,Kleene)	$\Diamond a \Rightarrow \Box b$
6,7 (Sette)	$\Diamond a \Rightarrow \Diamond b$
8	$\Box a \Rightarrow \Diamond b$
9,12 (Nelson, Bochvar)	$\Box a \Rightarrow \Box b$
10,11,13,1 (Gödel, Łukasiewicz, Gaines-Rescher)	$(\Box a \Rightarrow \Box b) \wedge (\Diamond a \Rightarrow \Diamond b)$

- The interpretation of each logic is clear