Kleene Logic
Appunti

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1 Introduction

In [4] Kleene studia le funzioni parziali ricorsive, ovvero funzioni $f : \mathbb{N}^n \mapsto \mathbb{N}$ che possono essere definite solo su un sottoinsieme di $\mathbb{N}^n$. Studiando il problema di assegnare valori di verità a proposizioni che coinvolgono funzioni parziali ricorsive arriva a definire due logiche (weak e strong) a tre valori di verità.

Nella versione *strong* Kleene assegna al terzo valore di verità il significato di *unknown*:

> is a category into which we can regard any proposition as falling, whose value we either do not know or choose for the moment to disregard and it does not then exclude the other two possibilities *true* and *false*.

Indichiamo con $\{F, T, U\}$ i valori di verità che possono assumere le proposizioni in Kleene logic. Questi valori rappresentano i possibili valori booleani che una proposizione può assumere (una volta che siano noti gli assegnamenti alle variabili al momento sconosciuti). Quindi: $T = \{t\}, F = \{f\}, U = \{f, t\}$ con f, t valori di verità booleani.

1.1 Operazioni

Per definire le operazioni, estendiamo quelle booleane su t e f agli insiemi T, F, U.

Congiunzione:

$$F \sqcap T = \{f\} \sqcap \{t\} = \{f \wedge t\} = \{f\} = F$$
$$F \sqcap U = \{f\} \sqcap \{f, t\} = \{f \wedge f, f \wedge t\} = \{f\} = F$$
$$T \sqcap U = \{t\} \sqcap \{f, t\} = \{f \wedge t, t \wedge t\} = \{f, t\} = U$$
$$\ldots$$

Disgiunzione:

$$
\begin{aligned}
&F \sqcup U = \{f\} \sqcup \{f,t\} = \{f \vee f, f \vee t\} = \{f,t\} = U \\
&T \sqcup U = \{t\} \sqcup \{f,t\} = \{t \vee t, t \wedge f\} = \{t\} = T \\
&U \sqcup U = \{f,t\} \sqcup \{f,t\} = \{f \vee f, f \vee t, t \vee t\} = \{f,t\} = U \\
&\quad \dots
\end{aligned}
$$

Si ottengono quindi le seguenti tabelle di verità.

$\sqcap$	F	U	T
F	F	F	F
U	F	U	U
T	F	U	T

$\sqcup$	F	U	T
F	F	U	T
U	U	U	T
T	T	T	T

Table 1: Tabelle di verità di congiunzione e disgiunzione in Kleene logic

Negazione $U' = U$, $F' = T$, $T' = F$

Se interpretiamo F come 0, T come 1, U come $\frac{1}{2}$, allora si vede che la congiunzione corrisponde al minimo, la disgiunzione al massimo e la negazione di x a $1 - x$.

Implicazione, se definiamo l'implicazione come $a \to_K b := a' \sqcup b$ otteniamo la seguente tabella di verità:

$\to_K$	F	U	T
F	T	T	T
U	U	U	T
T	F	U	T

Table 2: Tabelle di verità dell'implicazione in Kleene logic

Nella logica di Kleene vale il modus ponens: se a e $a \to_k b$ allora b. La logica è invece priva di tautologie. Infatti, non esiste formula P tale che $\forall u$, $i^*(P) = T$ perchè se ad ogni variabile assegniamo il valore U, allora, date le definizioni delle operazioni, il valore di verità di qualsiasi formula sarà U.

2 Algebra

Definition 2.1 *A* de Morgan lattice *is a structure* $\langle \mathcal{L}, \wedge, \vee, {}', 0 \rangle$ *where*

- $\langle \mathcal{L}, \wedge, \vee, 0 \rangle$ *is a lattice with minimum element* 0 *and maximum element* $1 := 0'$*;*
- $'$ *is a unary operation on* $\mathcal{L}$*, called* de Morgan complementation*, which satisfies the following conditions for arbitrary* $a, b \in \mathcal{L}$*:*

(oc-1) $a = a''$
(dM-1) $a' \wedge b' = (a \vee b)'$

Remark 1 *We have adopted the convention of defining de Morgan lattices those de Morgan complemented structures based on a non necessarily distributive lattice. This choice is made also by Kalman in [3] where de Morgan lattices are called* i–lattices *and distributive Kleene lattices* normal i–lattices. *On the contrary, other authors consider only distributive lattices (see for instance [1, 2]). Our choice is due to the desire of being more general as possible, since there exist de Morgan lattices which are not distributive.*

Example 2.1 *For example, let us consider the de Morgan lattice of figure 1.*

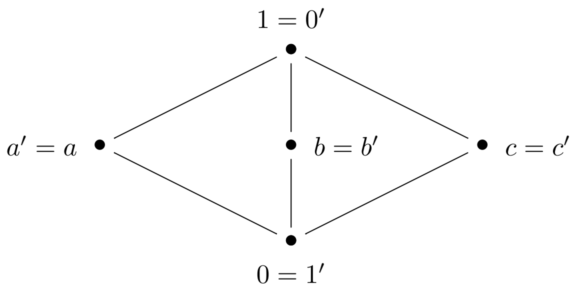

Figure 1: An example of de Morgan lattice which is not distributive.

This lattice is not distributive, since

$$a \vee (b \wedge c) = a \neq 1 = (a \vee b) \wedge (a \vee c)$$

Proposition 2.1 *In a de Morgan lattice, the following are equivalent*

1. $a' \wedge b' = (a \vee b)'$
2. $a' \vee b' = (a \wedge b)'$
3. *If* $a \leq b$ *then* $b' \leq a'$
4. *If* $a' \leq b'$ *then* $b \leq a$

Proof — Conditions 1 and 2 are equivalent due to double negation (dM1). Similarly, for conditions 3 and 4.
From condition 1 to condition 3.
In any lattice we have that: $a \leq a \vee b$ and $b \leq a \vee b$. Thus, by condition 1 we get $(a \vee b)' \leq a'$ and $(a \vee b)' \leq b'$, that is $(a \vee b)'$ is a lower bound of both a' and b'. We need to show that it is the greatest lower bound. By absurdity, we suppose that there exists c lower bound of a' and b', that is $c \leq a'$ and $c \leq b'$. By condition 1 and double negation we get $a'' = a \leq c'$ and $b'' = b \leq c'$, hence, $a \vee b \leq c'$. Again, by condition 1 and double negation $c'' = c \leq (a \vee b)'$ proving

the thesis.
From condition 3 to condition 1.
Let us suppose that $a \leq b$, then we get $a \vee b = b$. Then $b' = (a \vee b)'$. Applying condition 3, we obtain $b' = a' \wedge b'$, that is $b' \leq a'$. ■

Definition 2.2 *A Kleene lattice is a de Morgan lattice satisfying also the Kleene condition:*

$$(K) \qquad a \wedge a' \leq b \vee b'$$

Trivially, any Kleene lattice is a de Morgan lattice too, but there exist examples of de Morgan lattices which cannot be Kleene lattices, as can be seen in the following example.

Example 2.2 *Let us consider the de Morgan lattice of figure 2.*

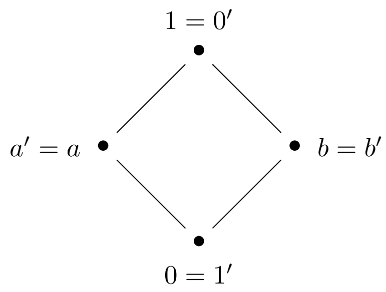

Figure 2: The de Morgan algebra **T**, which is not a Kleene lattice.

This structure is denoted by **T** *in [1] and it is the smallest de Morgan algebra which is not a Kleene algebra, since*

$$a \wedge a' = a \not\leq b = b \vee b'$$

Proposition 2.2 *Kleene condition is equivalent to the following* regularity *condition:*

$$(R) \qquad a \leq a' \text{ and } b' \leq b \text{ implies } a \leq b.$$

Proof — $(K) \Rightarrow (R)$. By hypothesis, $a \leq a'$ and $b' \leq b$. Then, by definition of meet and join in a lattice, we get $a = a \wedge a'$ and $b = b \vee b'$. Thus, from (K) we get the thesis $a \leq b$.
$(R) \Rightarrow (K)$. By definition of meet and join: $a \wedge a' \leq a \leq a \vee a' = (a \wedge a')'$, the last equality derives from de Morgan properties. So, we get $a \wedge a' \leq (a \wedge a')'$. Similarly, $b' \wedge b'' \leq b' \leq b \vee b'$, from which it follows that $(b \vee b')' \leq b \vee b'$. Now applying (R): $a \wedge a' \leq b \vee b'$. ■

Definition 2.3 *A de Morgan (resp. Kleene) lattice is a de Morgan (resp. Kleene) algebra if the underlying lattice is distributive.*

2.1 Logic

La formalizzazione della logica di Kleene non prevede quindi assiomi (come detto non ci possono essere tautologie) ma 16 regole di deduzione.

Definition 2.4 *The* Kleene three-valued logic *is defined by no axiom and the following set of rules*

$$(R1): \frac{a \wedge b}{a} \qquad (R2): \frac{a \wedge b}{b} \qquad (R3): \frac{a \quad b}{a \wedge b} \qquad (R4): \frac{a}{a \vee b}$$

$$(R5): \frac{a \vee b}{b \vee a} \qquad (R6): \frac{a \vee a}{a} \qquad (R7): \frac{a \vee (b \vee c)}{(a \vee b) \vee c}$$

$$(R8): \frac{a \vee (b \wedge c)}{(a \vee b) \wedge (a \vee c)} \qquad (R9): \frac{(a \vee b) \wedge (a \vee c)}{a \vee (b \wedge c)} \qquad (R10): \frac{a \vee c}{\neg\neg a \vee c}$$

$$(R11): \frac{\neg(a \vee b) \vee c}{(\neg a \wedge \neg b) \vee c} \qquad (R12): \frac{\neg(a \wedge b) \vee c}{(\neg a \vee \neg b) \vee c} \qquad (R13): \frac{\neg\neg a \vee c}{a \vee c}$$

$$(R14): \frac{(\neg a \wedge \neg b) \vee c}{\neg(a \vee b) \vee c} \qquad (R15): \frac{(\neg a \vee \neg b) \vee c}{\neg(a \wedge b) \vee c}$$

$$(R16 :): \frac{a \vee (b \wedge \neg b)}{a}$$

The syntactic inference $\Gamma \vdash_B p$*, where* $\Gamma \subseteq \mathcal{L}$ *is a set of formulas, means that* p *can be derived from* Γ *using the above inference rules.*

These rules express that conjunction is idempotent, and it yields a more specific proposition than each of the conjuncts (R1, R2, R3); disjunction is idempotent and it yields a less specific proposition than its disjuncts (R4, R5, R6). Disjunction is associative (R7), conjunction is distributive on disjunction (R8) and conversely (R9). Negation is involutive (R10, R13) and De Morgan Laws are satisfied (R11, R12, R14, R15). It makes it clear that the underlying algebra is a De Morgan algebra. Finally, (R16) characterizes the de Morgan algebra as a Kleene one.

References

[1] R. Cignoli, *Injective de Morgan and Kleene algebras*, Proceedings of the American Mathematical Society **47** (1975), 269–278.

[2] J. M. Dunn, *Relevance logic and entailment*, Handbook of Philosophical Logic (D. Gabbay and F. Guenther, eds.), vol. 3, Kluwer, 1986, pp. 117–224.

[3] J.A. Kalman, *Lattices with involution*, Transactions of the American Mathematica Society **87** (1958), 485–491.

[4] S. C. Kleene, *Introduction to metamathematics*, North–Holland Pub. Co., Amsterdam, 1952.