Chapter 3 - Coulomb collisions in plasmas

1 Rosenbluth's potentials

Calculate one of the following two Rosenbluth's potentials

$$h_F(\mathbf{v}) = \frac{m}{\mu}\int \frac{f_F(\mathbf{v})}{|\mathbf{v}-\mathbf{v}'|}d\mathbf{v}' \tag{1}$$

$$g_F(\mathbf{v}) = \int |\mathbf{v}-\mathbf{v}'|f_F(\mathbf{v}')d\mathbf{v}' \tag{2}$$

when the particle field distribution is described by a Maxwellian, i.e.

$$f_F(\mathbf{v}) = n_F\left(\frac{m_F}{2\pi T}\right)^{\frac{3}{2}}\exp\left(-\frac{m_F v^2}{2T}\right) \tag{3}$$

and verify that

$$h_F(\mathbf{v}) = \frac{n_F}{v}\frac{m_T}{\mu}\mathrm{erf}(x) \tag{4}$$

$$g_F(\mathbf{v}) = \frac{n_F v_{th}}{2}\left[\frac{d}{dx}\mathrm{erf}(x) + \frac{2x^2+1}{x}\mathrm{erf}(x)\right] \tag{5}$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-y^2)dy$ is the error function and $x = v/v_{th}$. Here $v_{th} = \sqrt{2T/m_F}$ is the thermal velocity of the field species.

Hint: In order to evaluate the integrals use the variable $\xi = \mathbf{v}' - \mathbf{v}$ and observe that $d\xi = d\mathbf{v}'$. It is also convenient to use spherical coordinates such that $d\xi = \xi^2 d\xi d\phi d\cos\theta$. θ is here the angle between $\mathbf{v}$ and $\mathbf{v}'$. Note also that $v'^2 = v^2 + \xi^2 + 2v\xi\cos\theta$[1]

2 Slowing down of a fast electron beam in a plasma

A fast electron beam is in a thermal plasma. The electron beam velocity is much higher than the bulk thermal ion and electron velocities. The bulk plasma ions

[1]The following relations can be handy:

$$\int \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x) + const$$

$$\int x\exp(-x^2)dx = -\frac{\exp(-x^2)}{2} + const$$

$$\int x^2\exp(-x^2)dx = \frac{\sqrt{\pi}}{4}\mathrm{erf}(x) - \frac{1}{2}x\exp(-x^2) + const$$

have a charge $+Ze$ and the charge neutrality condition is given by $n_e = Zn_i$.

- Since the beam velocity is much higher than the bulk plasma speed, we can approximate $(v^2 - 2\mathbf{v} \cdot \mathbf{v}' + v'^2)^{\frac{1}{2}} \approx v\left(1 - \frac{\mathbf{v}\cdot\mathbf{v}'}{v^2} + \frac{v'^2}{2v^2}\right)$. Show that, in this approximation, the Rosenbluth potentials become

$$h_F(\mathbf{v}) \approx \frac{m_T}{\mu v} n_F \tag{6}$$

$$g_F(\mathbf{v}) \approx n_F \left(v + \frac{3T}{2m_F}\frac{1}{v}\right) \tag{7}$$

  where m_T is the test species mass (here the fast electrons), μ is the reduced mass and n_F the bulk plasma density. Write an explicit expression for $h_F(\mathbf{v})$ when the field species are ions and electrons.

- Assume that the electron beam has a velocity oriented along the z axis so that $v \approx v_z$. Show that the Fokker-Planck equation for the velocity distribution of the test species $F_T(v_z)$ becomes

$$\frac{\partial F_T}{\partial t} \approx \sum_{F=i,e} \frac{n_F q_T^2 q_F^2 \ln\Lambda}{4\pi\epsilon_0^2 m_T^2} \frac{\partial}{\partial v_z}\left[\frac{m_T}{\mu v_z^2} F_T + \frac{3T}{2m_F}\frac{1}{v_z^3}\frac{\partial F_T}{\partial v_z}\right] \tag{8}$$

  where the sum is over the two field species, ions (i) and electrons (e).

- Considering charge neutrality and the fact that $v_z >> v_{T_i}$, show that

$$\frac{\partial F_T}{\partial t} \approx \frac{n_e e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_e^2}\frac{\partial}{\partial v_z}\left[\frac{2+Z}{v_z^2} F_T + \frac{3T_e}{2m_e}\frac{1}{v_z^3}\frac{\partial F_T}{\partial v_z}\right] \tag{9}$$

- Finally verify that $F_T(v_z)$ is given by

$$F_T(v_z) \propto \exp\left(-\frac{(2+Z)m_e v_z^2}{3T_e}\right) \tag{10}$$

  How does the electron beam thermal velocity compare with the bulk ion and electron thermal velocities?