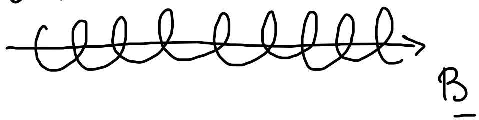
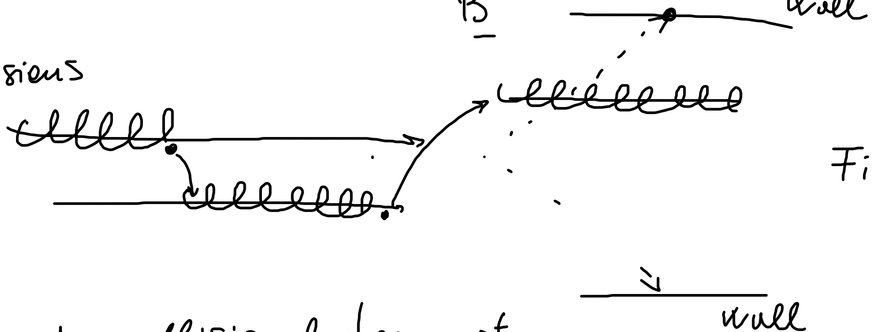
Collisional transport

w/o collisions

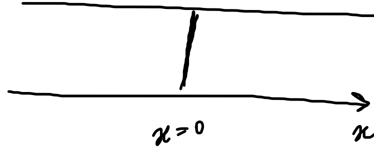


with collisions

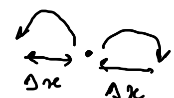


Finite confinement of time

1D model of collisional transport

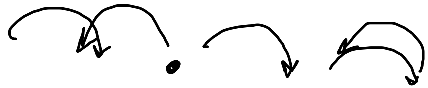


Δx : collisional step



Δt: time between collisions

$\langle x \rangle = 0$



Proof $\quad N$ collisions $\qquad t = N \Delta t$

r collisions to the right $\qquad 0 \le r \le N$

$N - r$ " " left

$$\text{Displacement} = \underbrace{r \cdot \Delta x}_{\text{Right displ.}} - \underbrace{(N-r)}_{\text{Left displ.}}\Delta x = 2r\Delta x - N\Delta x = (2r - N)\Delta x$$

$$P_r = \binom{N}{r}\left(\frac{1}{2}\right)^{r}\left(\frac{1}{2}\right)^{N-r};$$

$N = 4$

R R R L $\quad$ R R L R $\quad$ L R R R

$\langle x \rangle = \sum (\text{Displ.}) \cdot (\text{prob. of displ.})$

$$= \sum_{r=0}^{N} \underbrace{(2r-N)\Delta x}_{\text{Displ.}} \cdot \underbrace{\binom{N}{r} \frac{1}{2^N}}_{\text{Prob.}} = \frac{\Delta x}{2^N} \sum_{r=0}^{N} \binom{N}{r} (2r-N)$$

TRICK :
$$F_N(y) \overset{\text{def}}{=} \frac{(1+y)^N}{2^N y^{N/2}} = \frac{1}{2^N y^{N/2}} \sum_{r=0}^{N} \binom{N}{r} y^r = \frac{1}{2^N} \sum_{r=0}^{N} \binom{N}{r} y^{r-\frac{N}{2}}$$

$b = 1 \quad a = y$

Newton's binomial formula

$a, b \in \mathbb{R} \qquad (a+b)^N = \sum_{r=0}^{N} \binom{N}{r} a^r b^{N-r}$

$$\frac{d}{dy}F_N(y) = \frac{d}{dy}\frac{(1+y)^N}{2^N y^{N/2}}\Bigg|_{y=1} = \frac{d}{dy}\frac{1}{2^N}\sum_{r=0}^{N}\binom{N}{r}y^{r-\frac{N}{2}}$$

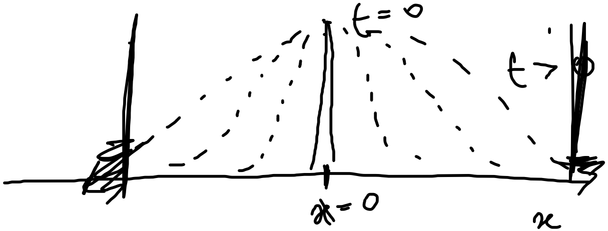
$$= \frac{1}{2^N}\sum_{r=0}^{N}\binom{N}{r}\left(r-\frac{N}{2}\right)y^{r-\frac{N}{2}-1}$$

$$= \frac{1}{2^{N+1}}\sum_{r=0}^{N}\binom{N}{r}(2r-N)\,y^{r-\frac{N}{2}-1}\Bigg|_{y=1}$$

$$= \frac{1}{2^{N+1}}\sum_{r=0}^{N}\binom{N}{r}(2r-N)$$

$$\| \quad \cdots \quad \| \quad 0$$

$$\langle x \rangle = 0$$



σ : st. deviation

$\equiv$

$$\sigma = \left[\langle x^2\rangle - \langle x\rangle^2\right]^{\frac{1}{2}}$$

$$\sigma^2 = \langle x^2\rangle - \underbrace{\cancel{(\langle x\rangle)^2}}_{=\,0} = \langle x^2\rangle$$

Displ: $(2r-N)\Delta x$

r coll. to R

$N-r$ " " L

Prob: $\binom{N}{r}\frac{1}{2^N}$

$$\langle x^2\rangle = \sum_{\text{Displ.}} (\text{Displ.})^2 \times (\text{Prob.})$$

$$= \sum_{r=0}^{N} (2r-N)^2\, \Delta x^2 \binom{N}{r}\frac{1}{2^N}$$

$$= \frac{(\Delta x)^2}{2^N} \sum_{r=0}^{N} (2r-N)^2 \binom{N}{r}$$

Trick:

$$F_N(y) = \underbrace{\frac{(1+y)^N}{2^N y^{N/2}}}_{\text{unexpanded form}} = \underbrace{\frac{1}{2^N y^{N/2}} \sum_{r=0}^{N} \binom{N}{r} y^r}_{\text{expanded form}}$$

$$y \frac{d}{dy}\left(y F_N(y)\right)\Big|_{y=1} = (\text{expanded}) = (\text{coeff}) \cdot \sum_{r=0}^{N} (2r-N)^2 \binom{N}{r}$$

$$\| \quad (\text{unexpanded}) \quad \| \quad \frac{N}{4}$$

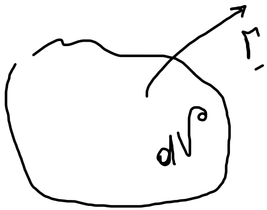
Result: $\langle x^2 \rangle = N \cdot (\Delta x)^2 \neq 0$

$\underline{t} = N \cdot \Delta t \Rightarrow N = t/\Delta t$

$$\sigma^2 = \langle x^2 \rangle = t \cdot \frac{(\Delta x)^2}{\Delta t}$$

$$\sigma \propto \sqrt{t}$$

Diffusion equation



$\underline{\Gamma}$: net flux of particles $\quad \dfrac{\#\ \text{particles}}{m^2 s}$

$$\frac{dN}{dt} = -\oint \underline{\Gamma}\cdot d\underline{S}$$

$$\underbrace{\underline{\Gamma}\cdot d\underline{S}}_{\substack{\#\ \text{particles that} \\ \text{go through the surface } d\underline{S}}} > 0$$

n: density of particles in dV

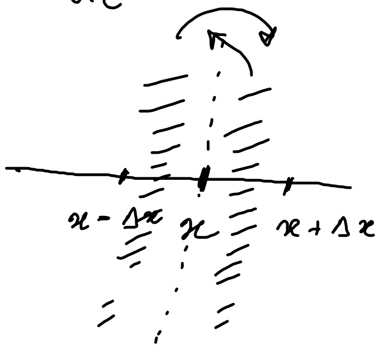
$$N = \int n\, dV$$

$$\frac{d}{dt}\int_V n\, dV = -\oint \underline{\Gamma}\cdot d\underline{S} \underset{\text{divergence theorem}}{=} -\int (\underline{\nabla}\cdot\underline{\Gamma})\, dV$$

$$\int \frac{dn}{dt}\, dV = \int -\underline{\nabla}\cdot\underline{\Gamma}\, dV \Rightarrow \int_V dV \left[\frac{dn}{dt} + \underline{\nabla}\cdot\underline{\Gamma}\right] = 0$$

$$\Rightarrow \quad \frac{dn}{dt} + \underline{\nabla} \cdot \underline{\Gamma} = 0$$

$\underline{\Gamma} = ?$ for a 1D collisional process



$$\Gamma = \left(\text{particles}/_{m^2/s} \text{ in } (x - \Delta x, x) \text{ that go to the right}\right)$$

$$- \left(\text{particles}/_{m^2/s} \text{ in } (x, x + \Delta x) \text{ that go to the left}\right)$$

$$= \Gamma_+ - \Gamma_-$$

$$\Gamma_+ = \frac{1}{2 S \Delta t} \int_{x-\Delta x}^{x} n \, dV = \frac{1}{2\Delta t} \int_{x-\Delta x}^{x} n(x') \, dx' = \frac{1}{2\Delta t} \int_{x-\Delta x}^{x} \left[n(x) + \left.\frac{dn}{dx'}\right|_x (x' - x) \right] dx'$$

$$dV = S \cdot dx$$

$$n(x') \simeq n(x) + \left.\frac{dn}{dx'}\right|_x (x' - x)$$

$$\Gamma_+ = \frac{1}{2\Delta t} \int_{x-\Delta x}^{x} \left[n(x) + \left.\frac{dn}{dx'}\right|_x (x'-x) \right] dx' =$$

$$= \frac{1}{2\Delta t} \left[n(x) \cdot \Delta x + \frac{1}{2} \frac{dn}{dx'} (x'-x)^2 \Big|_{x-\Delta x}^{x} \right]$$

$$= \frac{1}{2\Delta t} \left[n(x) \Delta x - \frac{1}{2} \frac{dn}{dx'} (\Delta x)^2 \right]$$

$$\Gamma_- = \frac{1}{2\Delta t} \int_{x}^{x+\Delta x} n(x')\, dx' = \frac{1}{2\Delta t} \left[n(x) \cdot \Delta x + \frac{1}{2} \frac{dn}{dx'} (\Delta x)^2 \right]$$

$$n(x') = n(x) + \left.\frac{dn}{dx'}\right|_x (x'-x)$$

$$\Gamma = \Gamma_+ - \Gamma_- = \frac{1}{2\Delta t}\left(-\frac{1}{2}\frac{dn}{dx}(\Delta x)^2 - \frac{1}{2}\frac{dn}{dx}(\Delta x)^2\right) = -\frac{dn}{dx}\,\frac{(\Delta x)^2}{2\Delta t}$$

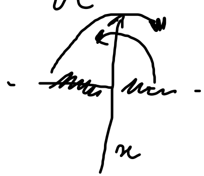
$$[D] \quad \frac{m^2}{s}$$

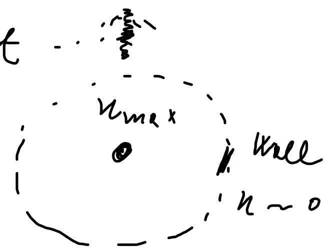
$$D \overset{def}{=} \frac{(\Delta x)^2}{2\Delta t} \qquad \sigma^2 = t \cdot D$$

$$\frac{\partial n}{\partial t} + \frac{d}{dx}\left(-D\frac{dn}{dx}\right) = 0 \; ; \qquad \frac{\partial n}{\partial t} - D\frac{\partial^2 n}{\partial x^2} = 0$$

diffusion equation

If $\frac{dn}{dx} = 0$ then $\frac{\partial n}{\partial t} = 0 \Rightarrow n = \text{const}$





L : size of system

$$\frac{\partial n}{\partial t} - D\frac{d^2 n}{dx^2} = 0$$

$$\frac{d^2 n}{dx^2} \approx \frac{n}{L^2}$$

τ : confinement time

$$\frac{\partial n}{\partial t} \approx \frac{n}{\tau}$$

$$\frac{\cancel{n}}{\tau} - \frac{D\cancel{n}}{L^2} \approx 0$$

$$\tau \approx \frac{L^2}{D}$$

$D \uparrow \quad \tau \downarrow$

$$\tau \propto L^2$$