Orthopairs: Knowledge Representation

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Uncertainty in Computer Science

Outline

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Orthopair: a set with uncertainty

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- Boundary $Bnd = (P \cup N)^c \longrightarrow$ a tri-partition of the universe
- Remark: also (P, Bnd) and (N, Bnd) are orthopairs!

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- $(\emptyset, \emptyset)$, Bnd $=$ X
- $\{x_1, x_2, x_3\}$ Boolean variables, x_1 is true, x_2 is false
  $(\{x_1\}, \{x_2\})$ Bnd $= \{x_3\}$, x_3 is unknown

A few definitions

- A set S is consistent with an orthopair $O = (P, N)$ if

$$x \in P \rightarrow x \in S \text{ and } x \in N \rightarrow x \notin S.$$

Example: $O = (\{1, 3\}, \{2, 4\})$
$S_1 = \{1, 3\}$, $S_2 = \{1, 3, 5\}$ are consistent with O
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- O_1, O_2 are disjoint if
  - $P_1 \cap P_2 = \emptyset$
  - $P_1 \cap Bnd_2 = \emptyset$ and $Bnd_1 \cap P_2 = \emptyset$

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- Bipolar Information: positive/negative preferences, trust/distrust (in Social Network Analysis), ...

Generalizations

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- Fuzzy orthopairs = (Atanassov) Intuitionistic Fuzzy Sets
  IFSs are pairs of fuzzy sets $f_P, f_N : X \mapsto [0, 1]$ such that for all $x \in X$, $f_P(x) + f_N(x) \leq 1$
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- Possibility Theory: set of valuations E are arbitrary
  The class of orthopairs coincides with the particular class of hyper-rectangular Boolean possibility distributions on the space $\{0, 1\}^n$

3-Valued Logic and Orthopairs

- Let f be a three valued set on the universe X, $f : X \mapsto \{0, \frac{1}{2}, 1\}$

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  From f to an orthopair (A_1, A_0)

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- From an orthopair (A, B) to a three valued set f

$$f(x) = \begin{cases} 1 & x \in A \\ 0 & x \in B \\ \frac{1}{2} & \text{ortherwise} \end{cases}$$

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All three-valued connectives can be translated to orthopairs

Conjunctions on Orthopairs

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n	$(P_1, N_1) * (P_2, N_2)$
1	$(N_1^c \cap N_2^c, N_1 \cup N_2)$
2	$((P_1 \cap N_2^c) \cup (P_2 \cap N_1^c), N_1 \cup N_2)$
3	$(P_1 \cap N_2^c, N_1 \cup N_2)$
4	$(N_1^c \cap P_2, N_1 \cup N_2)$
5	$(P_1 \cap P_2, N_1 \cup N_2)$
6	$(N_1^c \cap P_2, N_1 \cup P_2^c)$
7	$(P_1 \cap P_2, N_1 \cup P_2^c)$
8	$(P_1 \cap P_2, P_1^c \cup P_2^c)$
9	$(P_1 \cap P_2, P_1^c \cup N_2)$
10	$(N_1^c \cap P_2, (P_1^c \cap P_2^c) \cup N_1 \cup N_2)$
11	$(P_1 \cap P_2, (P_1^c \cap P_2^c) \cup N_1 \cup N_2)$
12	$(P_1 \cap N_2^c, P_1^c \cup N_2)$
13	$(P_1 \cap N_2^c, (P_1^c \cap P_2^c) \cup N_1 \cup N_2)$
14	$((P_1 \cup P_2) \cap N_1^c \cap N_2^c, (P_1^c \cap P_2^c) \cup U_1 \cup U_2)$

Nested pairs - implications

n	$(L_1, U_1) \Rightarrow (L_2, U_2)$	
1	$(U_1 \to L_2, U_1 \to L_2)$	Sette
2	$(U_1 \to L_2, (L_1 \to L_2) \cap (U_1 \to U_2))$	Sobociński
3	$(U_1 \to L_2, L_1 \to L_2)$	
4	$(U_1 \to L_2, U_1 \to U_2)$	Jaśkowski
5	$(U_1 \to L_2, L_1 \to U_2)$	Kleene
6	$(U_1 \to U_2, U_1 \to U_2)$	
7	$(U_1 \to U_2, L_1 \to U_2)$	
8	$(L_1 \to U_2, L_1 \to U_2)$	Bochvar
9	$(L_1 \to L_2, L_1 \to U_2)$	Nelson
10	$((L_1 \to L_2) \cap (U_1 \to U_2),\ U_1 \to U_2)$	Gödel
11	$((L_1 \to L_2) \cap (U_1 \to U_2),\ L_1 \to U_2)$	Łukasiewicz
12	$(L_1 \to L_2,\ L_1 \to L_2)$	
13	$((L_1 \to L_2) \cap (U_1 \to U_2),\ L_1 \to L_2)$	
14	$((L_1 \to L_2) \cap (U_1 \to U_2), (L_1 \to L_2) \cap (U_1 \to U_2))$	Gaines-Rescher

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Order relations 1/2

Pointwise Ordering

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Pointwise Ordering

Order on V	Order on $O(X)$	Symbol	Type
$0 \leq \frac{1}{2} \leq 1$	$P_1 \subseteq P_2,\ N_2 \subseteq N_1$	$\leq_t$	Total
$\frac{1}{2} \leq 1 \leq 0$	$N_1 \subseteq N_2,\ Bnd_2 \subseteq Bnd_1$	$\leq_N$	Total
$\frac{1}{2} \leq 0 \leq 1$	$P_1 \subseteq P_2,\ Bnd_2 \subseteq Bnd_1$	$\leq_P$	Total
$\frac{1}{2} \leq 1,\ \frac{1}{2} \leq 0$	$P_1 \subseteq P_2,\ N_1 \subseteq N_2$	$\leq_I$	Partial
$0 \leq \frac{1}{2},\ 0 \leq 1$	$P_1 \subseteq P_2,\ Bnd_1 \subseteq Bnd_2$	$\leq_{PB}$	Partial
$1 \leq \frac{1}{2},\ 1 \leq 0$	$N_1 \subseteq N_2,\ Bnd_1 \subseteq Bnd_2$	$\leq_{NB}$	Partial

- $\leq_t$ truth ordering: O_2 is “more true” than O_1
- $\leq_I$ knowledge ordering: O_2 is more informative than the orthopair O_2 (boundary is smaller)

Aggregation operations from order relations

From the three total order we derive three lattice structures:

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- Kleene meet and join from *truth ordering* (usual min/max)

$$(P_1, N_1) \sqcap_t (P_2, N_2) := (P_1 \cap P_2, N_1 \cup N_2)$$
$$(P_1, N_1) \sqcup_t (P_2, N_2) := (P_1 \cup P_2, N_1 \cap N_2)$$

- Weak Kleene meet and join (not in the table of 14 conjunctions)

$$(P_1, N_1) \sqcap_P (P_2, N_2) := (P_1 \cap P_2, (N_1 \cap N_2) \cup [(N_1 \cap P_2) \cup (N_2 \cap P_1)])$$
$$(P_1, N_1) \sqcap_N (P_2, N_2) := ((P_1 \cap P_2) \cup [(P_1 \cap N_2) \cup (P_2 \cap N_1)], N_1 \cap N_2))$$

- Sobocinski meet and join

$$(P_1, N_1) \sqcup_N (P_2, N_2) := (P_1 \backslash N_2 \cup P_2 \backslash N_1, N_1 \cup N_2)$$
$$(P_1, N_1) \sqcup_P (P_2, N_2) := (P_1 \cup P_2, N_1 \backslash P_2 \cup N_2 \backslash P_1)$$

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$$(P_1, N_1) \sqcap_I (P_2, N_2) := (P_1 \cap P_2, N_1 \cap N_2)$$

Conjunction and disjunction from $\sqcap_I$

Meet and join with respect to *information ordering* are

- The pessimistic combination operator

$$(P_1, N_1) \sqcap_I (P_2, N_2) := (P_1 \cap P_2, N_1 \cap N_2)$$

- the optimistic combination operator

$$(P_1, N_1) \sqcup_I (P_2, N_2) := (P_1 \cup P_2, N_1 \cup N_2)$$

(it makes sense whenever the two orthopairs are consistent: $P_1 \cap N_2 = \emptyset$ and $P_2 \cap N_1 = \emptyset$)

Difference

Several ways to define a difference. For instance

- $O_1 \ominus O_2 := (P_1 \setminus N_2, N_1 \setminus P_2)$
  The consensus (agreement) operation can then be defined
  $O_1 \odot O_2 = (O_1 \ominus O_2) \sqcup_I (O_2 \ominus O_1)$

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- Example
  $O_1 = (\{x_1, x_2\}, \{x_3, x_4\})$: x_1, x_2 are true x_3, x_4 are false
  $O_2 = (\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\})$: x_1, x_3, x_5 are true x_2, x_4, x_6 are false
  $O_1 \odot O_2 = (\{x_1, x_5\}, \{x_4, x_6\})$

Use of operations: some example

- If two orthopairs represent two agents opinion on the same fact, then
  - we can reach an agreement between them using the operator $\odot$;
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- Sobocinski operations are standard conjunction and disjunction operations on conditional events;
- If we want to aggregate two shadowed sets, then a first choice is to use Kleene lattice operations, that corresponds to min and max on fuzzy sets;
- In case of (three-way) decision theory, where the regions of the orthopair represent accept and reject, operations can be used to aggregate two different decisions on the same subject

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Rough Sets and three-valued logics

- $\mathcal{R}(X) := \{(l(A), u(A)) : A \subseteq X\}$ is a subset of all nested pairs, and equivalently, of all orthopairs
  $\rightarrow$ three-valued connectives can be inherited through orthopairs

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- Problem

> Given $(l(A), u(A)) \odot (l(B), u(B))$ does there exist an operation $\cdot$ on 2^X such that
> $(l(A \cdot B), u(A \cdot B))$?

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- Answer: yes... with interpretation problems

> All the 14 implications and 14 conjunctions defined on orthopairs are closed on $\mathcal{R}(X)$

Conjunctions on rough sets

$$r(A) *_1 r(B) = r(u(A) \cap u(B))$$
$$r(A) *_2 r(B) = r([A \cap u(B)] \cup [u(A) \cap B])$$
$$r(A) *_3 r(B) = r(A \cap u(B))$$
$$r(A) *_4 r(B) = r(u(A) \cap B)$$
$$r(A) *_6 r(B) = r(u(A) \cap l(B))$$
$$r(A) *_7 r(B) = r(A \cap l(B))$$
$$r(A) *_8 r(B) = r(l(A) \cap l(B))$$

A ∩ B

$$r(A) *_9 r(B) = r(l(A) \cap B)$$
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Implications on rough sets

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$$r(A) \Rightarrow_1 r(B) = r(u^c(A) \cup l(B))$$
$$r(A) \Rightarrow_2 r(B) = r([A^c \cup l(B)] \cap [l(A^c) \cup B])$$
$$r(A) \Rightarrow_3 r(B) = r(A^c \cup l(B))$$
$$r(A) \Rightarrow_4 r(B) = r(l(A^c) \cup B)$$
$$r(A) \Rightarrow_5 r(B) = r((A^c \cup B) \cap ((A^c \cup l(B)) \cup (l(A^c \cup B)^c)))$$
$$r(A) \Rightarrow_6 r(B) = r(u^c(A) \cup u(B))$$
$$r(A) \Rightarrow_7 r(B) = r(A^c \cup u(B)) \qquad \mathbf{A^c \cup B}$$
$$r(A) \Rightarrow_8 r(B) = r(l^c(A) \cup u(B))$$
$$r(A) \Rightarrow_9 r(B) = r(l^c(A) \cup l(B)))$$
$$r(A) \Rightarrow_{10} r(B) = r([l^c(A) \cap u(B)] \cup B \cup u^c(A))$$
$$r(A) \Rightarrow_{11} r(B) = r([l^c(A) \cap u(B)] \cup B \cup A^c)$$
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$$r(A) \Rightarrow_{14} r(B) = r([(l^c(A) \cup (B)) \cap (u^c(A) \cup u(B))])$$

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$$r(A) \sqcup r(B) = (L(A) \cup L(B), U(A) \cup U(B))$$

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In general $C \neq A \cap B$ and $D \neq A \cup B$

The lattice operation case (2)

Bonikowski proposal (1992)

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Gehrke, Walker proposal (1992)
Banerjee, Chakraborty proposal (1996)

- $C' = (A \cap B) \cup ((A \cap U(B)) \cap (U(A \cap B)^c))$

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The lattice operation case (2)

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Dual situation for D (union)

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$C' = \{a\}$ $r(C') = r(\{a, b, d\}) \sqcap r(\{c, e\}) = (\emptyset, \{a, e\})$

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- The solution is not unique even inside the same partition
- Two languages
  - The language of sets (extension)
  - The language of attributes (intension) or more generally of the granulation

  We can operate on the language of attributes but then we are not able to interpret the results on sets

Outline

Orthopartitions

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An orthopartition is a set $\mathcal{O} = \{O_1, ..., O_n\}$ of orthopairs such that the following axioms hold:

(Ax O1) $\forall O_i, O_j \in \mathcal{O}$ O_i, O_j are disjoint

(Ax O2) $\bigcup_i (P_i \cup Bnd_i) = U$; *(coverage requirement)*

(Ax O3) $\forall x \in U$ $(x \in Bnd_i) \rightarrow (x \in Bnd_j)$, $i \neq j$ (an object cannot belong to only 1 boundary)

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Example $U = \{1, 2, \ldots, 10\}$, the collection $\{O_1, O_2, O_3\}$ is an orthopartition of U where: $O_1 = (\{1, 2\}, \{9, 10\})$, $O_2 = (\{9\}, \{1, 2\})$, $O_3 = (\emptyset, \{1, 2, 9\})$
(O_1, O_2) is not an orthopartition

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