(Hints on) Decision Under Uncertainty

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Introduction

- Decision theoretic tools rely on mathematical models for representing agent's preferences
  - "prefer 200€ instead of 100€"
- Real world problems require also models for
  - Representing risk
  - Representing uncertainties
- Expected Utility, Prospect Theory, Belief Expected Utility, …

Risk/Uncertainty

- Risk: “a condition that occurs with known probability as the result of selecting an alternative”*
  - Example: decide on stock trading, according to the probability of raise/decline of interest rates
  - Probability theory
- Uncertainty: “the probability as the result of selecting an alternative is not known” or is ambigous
  - Interest rates are not known or known with imprecision (e.g.,“fairly low” increase)
  - Beyond probability theory

*K. Takemura, Behavioral Decision Theory, Springer, 2014

Expected Utility

- A set of decisions D, with a preference order $d_1 \preceq d_2$
- A set of possibile consequences X, x(d) is the consequence of decision d
  - Lottery: a set of probabilities associated to a set of consequences
  - The preference is reflected on consequences $x(d_1) \preceq x(d_2) \quad \text{iff} \quad d_1 \preceq d_2$
  - Example: roll a dice. L1 = Win 100€ if even, 10€ if odd
  - Rigged dice, even p =0.01, odd p=0.99
- Utility function $u : X \mapsto \mathbb{R}$
  - $x \preceq y \quad \text{iff} \quad u(x) \preceq u(y)$

EU - axiomatic approach

- Eu used since the 18th century (Cramer, Bernoulli)
- Axiomatic approach in the 20th century
  - Von Neumann - Morgenstern, 1944, assumes the existence of an "objective" probability distribution on consequences X
  - Savage, 1954, probability derived from the rationality of the agent —> "subjective probability"
- Axioms represent rationality
  - Agent is rational iff it follows Expected Utility

Von Neumann-Morgenstern

- Axiom 1: $\preceq$ is reflexive, transitive and complete
  - Given two possible consequences of a decision, an agent is always capable of determining the preferred one and if $P \preceq Q$ and $Q \preceq R$ then $P \preceq R$
- Axiom 2: $P \prec Q \prec R$
  - If R is preferred to Q, small perturbations do not change this preference
  - $\alpha R + (1 - \alpha)P$ with probability α we have R, otherwise P
  - there exists α, β such that $\alpha R + (1 - \alpha)P < Q < \beta R + (1 - \beta)P$

Von Neumann-Morgenstern

- Axiom 3: independence
  - $P \preceq Q$ iff $\alpha P + (1-\alpha)R \preceq \alpha Q + (1-\alpha)R$
  - With probability $(1-\alpha)$: $R \preceq R$
  - With probability α: $P \preceq Q$

- Are we rational (in the sense of axioms 1-3)?
- Do we use probability to reach a decision?

Allais Paradox

- Which lottery do you prefer?
  - L1 = Win 1M €, probability 1
  - L2 = Win 1M €, probability 0.89; 5M € prob. 0.1; 0€, prob. 0.01
- Which lottery do you prefer?
  - L1' = win 1M€, p=0.11; 0€, p=0.89
  - L2' = win 5M€, p=0.10; 0€, p=0.90
- It can be shown that according to independence Axiom 3
  - If $L2 \prec L1$ then $L2' \prec L1'$

Ellsberg paradox

- An urn with 1/3 of red balls, the other balls are black or yellow
- Select an alternative of winning according to drawn a ball from a urn
- CASE 1
  - Alternative A: win 1M€ if the ball is red, 0€ otherwise
  - Alternative B: win 1M€ if the ball is black, 0€ otherwise

Ellsberg paradox

- CASE 2
  - Alternative C: win 1M€ if the ball is red or yellow, 0€ otherwise
  - Alternative D: win 1M€ if the ball is black or yellow, 0€ otherwise
- However, in EU and assuming the additivity of probability:
  - If $B \prec A$ then $D \prec C$

Rank Dependent Utility

- Prospect Theory and successively Rank Dependent Utility (RDU) take into account the perception of probabilities by the agent
- A generalization of Expected Utility
  - Allais paradox can be explained
- “The axiomatic foundation of RDU are quite complicated”*
- Cannot cope with Ellsberg’s paradox
- A particular case of Choquet Expect Utility: capacities instead of probabilities

* C. Gonzales, P. Perny, “Decision Under Uncertainty”, 2020

Choquet/Belief Expected Utility

- Modify axioms 1-3 using capacities/belief functions instead of probabilities
- Agent is rational iff it follows Choquet/Belief Expected Utility
- Belief function of Ellsberg's urn:
  - $f(\emptyset) = f(\{yellow\}) = f(\{black\}) = 0$
  - $f(\{red\}) = 1/3$
  - $f(\{black, yellow\}) = 2/3$

Ellberg's urn - rivisited

- Assume the utility:
  - $u(\{0\}) = 0$, $u(\{1M\}) = 1$, $u(\{0{,}1M\}) = \alpha$
  - If $\alpha < 1/2$, the common agent's preferences are respected
- Alternative A: win 1M€ if the ball is red, 0€ otherwise
  - BEU(A) = $\frac{2}{3} \cdot u(\{0\}) + \frac{1}{3} \cdot u(\{1M\}) = \frac{1}{3}$
- Alternative B: win 1M€ if the ball is black, 0€ otherwise
  - BEU(B) = $\frac{1}{3} \cdot u(\{0\}) + \frac{2}{3} \cdot u(\{0{,}1M\}) = \frac{2}{3}\alpha$

Qualitative decision making

- Capacities —> Qualitative Capacities
- Utility —> Qualitative Utility

- Possibility distributions π on an ordered set L to represent lotteries
- Pessimisitic utility function: to which extent it is *sure* to get a consequence having a “good” utility value
- Optimistic utility function: to which extent it is *possible* to get a consequence having a “good” utility value

Going further

- Sequential decision models: multiple decisions taken one after the other
  - Decision trees
- Multi-criteria decision making (MCDM): several criteria has to be taken into account
  - Possibility to use possibility theory, rough sets, etc…
- Behavioural decision theory: how people make decisions
  - A descriptive (vs normative) decision theory
  - At the crossroads of mathematics, psychology, economy