Duality and the Envelope Theorem. Notes for Clamses students

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A (maximum or minimum) **value function** is an objective function where the choice variables have been assigned their optimal values. These optimal values of the choice variables are, in turn, functions of the exogenous variables and parameters of the original optimization problem (constrained or unconstrained). Once the optimal values of the choice variables have been substituted into the original objective function, this function indirectly becomes a function of the parameters (through the parameters' influence on the optimal values of the choice variables). Thus, the resulting value function is also referred to as the **indirect objective function**. A typical example in microeconomics is the indirect utility function.

The unconstrained case.

To illustrate, consider the following unconstrained optimization problem with two choice variables *x* and *y*, and one parameter, α. We want to

Maximize (minimize)

$$Z = f(x, y, \alpha) \qquad (1)$$

The first order necessary conditions are

$$f_x(x, y, \alpha) = f_y(x, y, \alpha) = 0 \qquad (2)$$

if second-order conditions are met, these two equations implicitly define the value of *x* and *y* that generate the maximum (minimum) level of *Z*, for any possible α:

$$x^*(\alpha) \text{ and } y^*(\alpha) \qquad (3)$$

If we substitute these solutions into the original objective function, we obtain a new function

$$V(x^*, y^*; \alpha) = f(x^*(\alpha), y^*(\alpha); \alpha) \qquad (4)$$

where this function is the value of *f* when the values of *x* and *y* are those that maximize (minimize) *f(x, y, α)*. Therefore, $V(x^*, y^*; \alpha)$ is the (maximum or minimum) **value function** (or indirect objective function). If we assume that *V* (i.e. *f*) is continuously differentiable around α, we can differentiate equation 4 with respect to α to obtain

$$\frac{\partial}{\partial \alpha} V(x^*, y^*; \alpha) = f_{x*} \frac{\partial x *}{\partial \alpha} + f_{y*} \frac{\partial y *}{\partial \alpha} + \frac{\partial f}{\partial \alpha} \qquad (5)$$

However, from the first order conditions we know $f_{x^*} = f_{y^*} = 0$. Therefore, the first two terms disappear and the result becomes

$$\frac{\partial}{\partial \alpha} V(x^*, y^*; \alpha) = \frac{\partial f}{\partial \alpha} \tag{6}$$

This result says that, at the optimum, as α varies, with $x*$ and $y*$ optimally determined we obtain the same result as if $x*$ and $y*$ were held constant! In other words, the result says that the change in the **maximal (minimal) value** of the function *f (x, y,* α) as the parameter α changes is the change caused by **the direct impact** of the parameter on the function, holding the value of *x* and *y* fixed at their optimal values. The indirect effect, resulting from the change in the optimal value of *x* and *y* caused by a change in the parameter, is zero.
Note that α enters the value function (equation 4) in three places: one direct and two indirect (through x∗ and y∗). Equations 5 and 6 show that, at the optimum, **only the direct effect of α on the objective function matters**. This is the essence of the envelope theorem. The envelope theorem says that only the direct effects of a change in the parameter need be considered, **even though the parameter may enter the maximum (minimum) value function indirectly as part of the solution to the endogenous choice variables**.

An economic example: the profit function and the Hotelling Lemma.

Let us apply the above approach to an economic application, namely the profit function of a competitive firm. Consider the case where a firm uses two inputs: capital, *K*, and labour, *L*. If the (competitive in all markets) prices are *p*, *w* and *r* (for output, *L* and *K*) the profit function is

$$\pi(w, r, p) = pf(L, K) - wL - rK \tag{7}$$

The first order conditions (recall that in perfect competition $\frac{\partial w}{\partial L} = \frac{\partial r}{\partial K} = 0$ by hypothesis) are

$$\begin{aligned} \frac{\partial \pi}{\partial L} &= p\frac{\partial}{\partial L} f(L, K) - w = 0 \\ \frac{\partial \pi}{\partial K} &= p\frac{\partial}{\partial K} f(L, K) - r = 0 \end{aligned} \tag{8}$$

which respectively define the uncompensated factor demand equations

$$\begin{aligned} L &= L^*(w, r, p) \\ K &= K^*(w, r, p) \end{aligned} \tag{9}$$

Substituting the solutions $K*$ and $L*$ into the objective function gives us a "new" function

$$\pi^*(w, r, p) = pf\big(L^*(w, r, p), K^*(w, r, p)\big) - wL^*(w, r, p) - rK^*(w, r, p) \tag{10}$$

$\pi*(w, r, p)$ is the profit value function (or indirect objective function). The profit value function gives the maximum profit as a function of L^*, K^*, and the exogenous parameters *w*, *r*, and *p*.
To evaluate a change in the maximum profit function from a change in *w*, we differentiate $\pi*(w, r, p)$ with respect to *w* yielding

$$\frac{\partial}{\partial w}\pi^*(w,r,p) = [pf_L(L,K) - w]\frac{\partial L^*}{\partial w} + [pf_K(L,K) - r]\frac{\partial K^*}{\partial w} - L^* \qquad (11)$$

From the first order conditions, the two bracketed terms are equal to zero. Therefore, the resulting equation becomes

$$\frac{\partial}{\partial w}\pi^*(w,r,p) = -L^*(w,r,p)$$

This result says that, at the profit maximizing position, a change in profits with respect to a change in the (parameter) wage is the same whether or not the factors (the original independent variables of the objective function) are held constant (no adjustment to changes in *w*) or allowed to vary (adjustments are allowed) as the factor price changes. In any case the derivative of the profit function with respect to *w* is the negative of the factor demand function $L^*(w, r, p)$.
Following the above procedure, we can also show the additional comparative statics results

$$\frac{\partial}{\partial r}\pi^*(w,r,p) = -K^*(w,r,p)$$

and

$$\frac{\partial}{\partial p}\pi^*(w,r,p) = f\big(L^*(w,r,p), K^*(w,r,p)\big) = \text{Max quantity offered}$$

Under perfectly competitive conditions on all relevant markets, the supply function corresponds to the derivative of the (maximum) profit function. This result is known as Hotelling's Lemma.

Hotelling's Lemma is simply an application of the envelope theorem.

A specific example.

Let $f(L,K) = 3\sqrt[3]{LK}$ with the variables having the usual meaning (or lack of any meaning…) and $0 < L < \infty$ and $0 < K < \infty$. Then

$$\pi = p(3\sqrt[3]{LK}) - wL - rK$$

And from f.o.c.[1]

$$\{L = -\frac{\sqrt{K}p^{3/2}}{w^{3/2}}, L = \frac{\sqrt{K}p^{3/2}}{w^{3/2}}\}$$

$$\{K = -\frac{\sqrt{L}p^{3/2}}{r^{3/2}}, K = \frac{\sqrt{L}p^{3/2}}{r^{3/2}}\}$$

[1] The Hessian of the production function is $\begin{bmatrix} -\frac{2K^2p}{3(KL)^{5/3}} & -\frac{2KLp}{3(KL)^{5/3}} + \frac{p}{(KL)^{2/3}} \\ -\frac{2KLp}{3(KL)^{5/3}} + \frac{p}{(KL)^{2/3}} & -\frac{2L^2p}{3(KL)^{5/3}} \end{bmatrix}$ with $\Delta = \frac{(KL)^{2/3}p^2}{3K^2L^2} = \frac{p^2}{3L^{4/3}T^{4/3}} > 0$.
Hence the Hessian is positive definite for any finite *L*, *K* both positive.

Using positive solutions to the f.o.c., the system

$$L = \frac{\sqrt{K} p^{3/2}}{w^{3/2}}$$
$$K = \frac{\sqrt{L} p^{3/2}}{r^{3/2}}$$

gives 4 solutions

$$\{L = 0\}, \{L = \frac{p^3}{rw^2}\}, \{L = -\frac{(-1)^{1/3} p^3}{rw^2}\}, \{L = \frac{(-1)^{2/3} p^3}{rw^2}\}$$
$$\{K = 0\}, \{K = \frac{p^3}{r^2 w}\}, \{K = -\frac{(-1)^{1/3} p^3}{r^2 w}\}, \{K = \frac{(-1)^{2/3} p^3}{r^2 w}\}$$

We use only **real and strictly positive** values and substitute them into the above profit expression to get, after some trivial manipulation,

$$\pi^*(w, r, p) = \frac{p^3}{rw}$$

Then, by differentiation we obtain the supply function

$$\frac{\partial}{p} \pi^*(w, r, p) = 3\frac{p^2}{rw} = 3\sqrt[3]{L^* K^*} = 3\sqrt[3]{\frac{p^3}{rw^2}\frac{p^3}{r^2 w}} = \text{Max quantity offered by the firm} = S(p, w, r)$$

i.e the output produced when the firm employs the optimal quantity of each factor. Clearly

$$\frac{\partial}{p} S(w, r, p) = \frac{\partial}{p}\left[3\left(\sqrt[3]{\frac{p^6}{r^3 w^3}}\right)\right] = \frac{6p}{rw} > 0$$

Supply is continuous and monotonically increasing in *p (convex in p in this case)*.
The above is the Hoteling Lemma.

The constrained case

Now let us consider the case of constrained optimization. Again, we will have an objective function (call it *U*), two choice variables, (*x* and *y*) and one parameter (*α*) except now we introduce the following constraint:

g(x, y; α) = 0

The derivation of the envelope theorem for the models with one constraint is as follows. The problem becomes: Maximize

$$U = f(x, y; \alpha)$$

subject to

$$g(x, y; \alpha) = 0$$

The Lagrangian for this problem is

$$\Lambda = f(x, y; \alpha) + \lambda g(x, y; \alpha)$$

The first order conditions are

$$\frac{\partial \Lambda}{\partial x} = \frac{\partial \mathrm{f}}{\partial x} + \lambda \frac{\partial \mathrm{g}}{\partial x} = 0$$
$$\frac{\partial \Lambda}{\partial y} = \frac{\partial \mathrm{f}}{\partial y} + \lambda \frac{\partial \mathrm{g}}{\partial y} = 0$$
$$\frac{\partial \Lambda}{\partial \lambda} = g(x, y; \alpha = 0$$

Solving this system of equations gives us

$x = x^*(\alpha)$ $y = y^*(\alpha)$ $\lambda = \lambda^*(\alpha)$

Substituting the solutions **into the objective function,** we get

$$U^* = f(x^*(\alpha), y^*(\alpha); \alpha) = V(\alpha)$$

where *V(α)* is the indirect objective function, or maximum value function. This is the maximum value of *f* for any α and *x* and *y* that satisfy the constraint. Microeconomists call it **indirect utility function**.
Let us evaluate how *V(α)* changes as α changes. First, we differentiate *V* with respect to α

$$\frac{\partial}{\partial \alpha} V(\alpha) = \frac{\partial}{\partial x^*} f(x^*(\alpha), y^*(\alpha); \alpha) \frac{\partial x^*}{\partial \alpha} + \frac{\partial}{\partial y^*} f(x^*(\alpha), y^*(\alpha); \alpha) \frac{\partial y^*}{\partial \alpha} + \frac{\partial f}{\partial \alpha} \quad (12)$$

In this case, the above equation **will not** simplify to the above equation 6 (unconstrained)

$$\frac{\partial}{\alpha} V(\alpha) = \frac{\partial f}{\partial \alpha}$$

since

$$\frac{\partial}{\partial x^*} f(x^*(\alpha), y^*(\alpha); \alpha) \neq 0, \ \frac{\partial}{\partial y^*} f(x^*(\alpha), y^*(\alpha); \alpha) \neq 0.$$

Yet, if we substitute the solutions to *x* and *y* into the constraint (producing an identity) we get

$$g(x^*(\alpha), y^*(\alpha); \alpha) = 0 \quad (13)$$

and differentiating with respect to α yields

$$\frac{\partial g}{\partial x^*} \frac{\partial x^*}{\partial \alpha} + \frac{\partial g}{\partial y^*} \frac{\partial y^*}{\partial \alpha} + \frac{\partial g}{\partial \alpha} \equiv 0 \quad (14)$$

If we multiply equation 14 by λ and combine the result with equation 12 we get after rearranging terms,

$$\frac{\partial}{\partial\alpha}V(\alpha)=\left[\frac{\partial f}{\partial x^*}+\lambda\frac{\partial g}{\partial x^*}\right]\frac{\partial x^*}{\partial\alpha}+\left[\frac{\partial f}{\partial y^*}+\lambda\frac{\partial g}{\partial y^*}\right]\frac{\partial y^*}{\partial\alpha}+\frac{\partial f}{\partial\alpha}+\lambda\frac{\partial g}{\partial\alpha}=\frac{\partial\Lambda}{\partial\alpha} \qquad (15)$$

Or

$$\frac{\partial}{\partial\alpha}V(\alpha)=\frac{\partial\Lambda}{\partial\alpha}=\frac{\partial f}{\partial\alpha}+\lambda\frac{\partial g}{\partial\alpha}$$

where $\frac{\partial\Lambda}{\partial\alpha}$ is the partial derivative of the Lagrangian function with respect to α, holding all other variables constant. In this case, the Langrangian functions serves as the objective function in deriving the indirect objective function. Notice that if, like in most microeconomic cases, $\frac{\partial f}{\partial\alpha}=0$ and $\frac{\partial g}{\partial\alpha}=1$ then

$$\frac{\partial}{\partial\alpha}V(\alpha)=\frac{\partial\Lambda}{\partial\alpha}=\lambda$$

This is the textbook case of max direct utility subject to a usually linear in parameters budget constraint, i.e., the case when the Utility function (corresponding to the above *f*) does not depend upon *R* (the amount that can be used to buy the goods which plays the role of α) and *R* (alias, α) enters linearly the consumer's budget constraint *g*. For most microeconomic problems the Envelope Theorem reduces to this simple result: $\frac{\partial\Lambda}{\partial\alpha}=\lambda$ where $\alpha = R$. In this case microeconomists say that an infinitesimal variation of *R* changes the max utility by the amount λ, or that the Lagrangian multiplier measures the effect on max Utility of the infinitesimal change of *R* and call this nonsense the "marginal utility of Income". In the Gravelle and Rees textbook this result is exploited to discuss the Roy's identity (pp. 52 – 55).

While the result in equation 15 nicely parallels the unconstrained case, it is important to note that some of the comparative static results depend critically on whether the parameters enter only the objective function or whether they enter only the constraints or enter both. If a parameter enters only in the objective function then the comparative static results are the same as for unconstrained case. However, if the parameter enters the constraint, the relation between second derivatives coming from eq. 6

$V_{\alpha\alpha} \geq f_{\alpha\alpha}$

will no longer hold.

Example (the parameter enters the constraint only)

Max $U=10X^{0.4}Y^{0.6}$ subject to $xX+yY=R$ where *x* e *y* are the prices of *X* and *Y* respectively. Then the Marshallian demands and λ are

$$X=\frac{0.4R}{x}, Y=\frac{0.6R}{y}, \lambda=\frac{5.1017}{x^{2/5}y^{3/5}}$$

The indirect objective function is the maximum value function (indirect utility):

$$V(x,y;R)=10\left(\frac{0.4R}{x}\right)^{0.4}\left(\frac{0.6R}{y}\right)^{0.6}=\frac{5.1017}{x^{0.4}y^{0.6}}R$$

Then, since the demand functions are continuous and differential in *R*, so is *V*:

$$\frac{\partial V(x, y; R)}{\partial R} = \frac{5.1017}{x^{0.4}y^{0.6}} = \lambda$$

The above is an example the so-called marginal utility of Income.
For both consumption and production, the interpretation of the optimal λ can be found in the Notes and in Gravelle-Rees (page 682 – 684).

You may also consult https://mjo.osborne.economics.utoronto.ca/index.php/tutorial/index/1/toc