Black Holes, White Dwarfs, and Neutron Stars
THE PHYSICS OF COMPACT OBJECTS
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12 Black Holes

12.1 Introduction

We have seen in previous chapters that both white dwarfs and neutron stars have a maximum possible mass. What happens to a neutron star that accretes matter and exceeds the mass limit? What is the fate of the collapsing core of a massive star, if the core mass is too large to form a neutron star? The answer, according to general relativity, is that nothing can halt the collapse. As the collapse proceeds, the gravitational field near the object becomes stronger and stronger. Eventually, nothing can escape from the object to the outside world, not even light. A black hole has been born.

A black hole is defined simply as a region of spacetime that cannot communicate with the external universe. The boundary of this region is called the surface of the black hole, or the *event horizon*.[1]

The ultimate fate of collapsing matter, once it has crossed the black hole surface, is not known. Densities for a $1 M_{\odot}$ object are $\sim 10^{17}$ g cm^{-3} as the black hole is formed, and are smaller for larger masses. How can we be sure that some hitherto unknown source of pressure does not become important above such extreme densities and halt the collapse? The answer is that by the time a black hole forms it is already too late to hold back the collapse: matter must move on worldlines *inside* the local light cone, and the spacetime geometry is so distorted that even an "outward" light ray does not escape. In fact, since all forms of energy gravitate in relativity, increasing the pressure energy only *accelerates* the late stages of the collapse.

If we extrapolate Einstein's equations all the way inside a black hole, they ultimately break down: a *singularity* develops. There is as yet no quantum theory of gravitation, and some people believe that the singularity would not occur in

[1] Strictly speaking, the event horizon is a three-dimensional hypersurface in spacetime (a 2-surface existing for some time interval). We shall, however, speak loosely of the event horizon or black hole surface as a 2-surface at some instant of time.

such a theory. It would be replaced by finite, though unbelievably extreme, conditions.

Exercise 12.1 Construct a density by dimensional analysis out of c, G, and $\hbar$. Evaluate numerically this "Planck density" at which quantum gravitational effects would become important.

Answer: $\rho \sim 10^{94}\ \mathrm{g\ cm^{-3}}$.

As long as the singularity is hidden inside the event horizon, it cannot influence the outside world. The singularity is said to be "causally disconnected" from the exterior world. We can continue to use general relativity to describe the observable universe, even though the theory breaks down inside the black hole.

One might expect that the solutions of Einstein's equations describing equilibrium black holes would be extremely complicated. After all, black holes can be formed from stars with varying mass distributions, shapes (multipole moments), magnetic field distributions, angular momentum distributions, and so on. Remarkably, the most general stationary black hole solution is known analytically. It depends on only three parameters: the mass M, angular momentum J, and charge Q of the black hole. All other information about the initial state is radiated away in the form of electromagnetic and gravitational waves during the collapse. The remaining three parameters are the only independent observable quantities that characterize a stationary black hole.[2] This situation is summarized by Wheeler's aphorism, "A black hole has no hair."

The mass of a black hole is observable, for example, by applying Kepler's Third Law for satellites in the Newtonian gravitational field far from the black hole. The charge is observable by the Coulomb force on a test charge far away. The angular momentum is observable by non-Newtonian gravitational effects. For example, a torque-free gyroscope will precess relative to an inertial frame at infinity (Lense–Thirring effect).

12.2 History of the Black Hole Idea

As early as 1795 Laplace (1795) noted that a consequence of Newtonian gravity and Newton's corpuscular theory of light was that light could not escape from an object of sufficiently large mass and small radius. In spite of this early foreshadowing of the possibility of black holes, the idea found few adherents, even after the formulation of general relativity.

In December of 1915 and within a month of the publication of Einstein's series of four papers outlining the theory of general relativity, Karl Schwarzschild (1916) derived his general relativistic solution for the gravitational field surround-

[2]See Carter (1979) for a complete discussion.

ing a spherical mass. Schwarzschild sent his paper to Einstein to transmit to the Berlin Academy. In replying to Schwarzschild, Einstein wrote, "I had not expected that the exact solution to the problem could be formulated. Your analytical treatment of the problem appears to me splendid." Although the significance of the result was apparent to both men, neither they nor anyone else knew at that time that Schwarzschild's solution contained a complete description of the external field of a spherical, electrically neutral, nonrotating black hole. Today we refer to such black holes as *Schwarzschild black holes*, in honor of Schwarzschild's great contribution.

As we described in Chapter 3, Chandrasekhar (1931b) discovered in 1930 the existence of an upper limit to the mass of a completely degenerate configuration. Remarkably, Eddington (1935) realized almost immediately that if Chandrasekhar's analysis was to be accepted, it implied that the formation of black holes would be the inevitable fate of the evolution of massive stars. He thus wrote in January 1935: "The star apparently has to go on radiating and radiating and contracting and contracting until, I suppose, it gets down to a few kilometers radius when gravity becomes strong enough to hold the radiation and the star can at last find peace." But he then went on to declare, "I felt driven to the conclusion that this was almost a reductio ad absurdum of the relativistic degeneracy formula. Various accidents may intervene to save the star, but I want more protection than that. I think that there should be a law of Nature to prevent the star from behaving in this absurd way."

As is clear from his concluding remarks, Eddington never accepted Chandrasekhar's result of the existence of an upper limit to the mass of a cold, degenerate star. This in spite of Eddington's being one of the first to understand and appreciate Einstein's theory of general relativity! (His book *The Mathematical Theory of Relativity* (1922) was the first textbook on general relativity to appear in English.) In fact, Eddington subsequently proceeded to modify the equation of state of a degenerate relativistic gas so that finite equilibrium states would exist for stars of arbitrary mass.[3]

But Eddington was not alone in his misgivings about the inevitability of collapse as the end product of the evolution of a massive star. Landau (1932), in the same paper giving his simple derivation of the mass limit (cf. Section 3.4), acknowledged that for stars exceeding the limit, "there exists in the whole quantum theory no cause preventing the system from collapsing to a point." But rather then follow the sober advice put forth at the beginning of his paper ("It seems reasonable to try to attack the problem of stellar structure by methods of theoretical physics"), Landau, in the end, retreats and declares, "As in reality such masses exist quietly as [normal] stars and do not show any such tendencies,

[3]Chandrasekhar (1980) has recently lamented Eddington's shortsightedness regarding black holes, declaring, "Eddington's supreme authority in those years effectively delayed the development of fruitful ideas along these lines for some thirty years."

we must conclude that all stars heavier than $1.5M_\odot$ certainly possess regions in which the laws of quantum mechanics (and therefore quantum statistics) are violated."

In 1939 Oppenheimer and Snyder (1939) revived the discussion by calculating the collapse of a homogeneous sphere of pressureless gas in general relativity. They found that the sphere eventually becomes cut off from all communication with the rest of the Universe. This was the first rigorous calculation demonstrating the formation of a black hole.

Black holes and the problem of gravitational collapse were generally ignored until the 1960s, even more so than neutron stars. However, in the late 1950s, J. A. Wheeler and his collaborators began a serious investigation of the problem of collapse.[4] Wheeler (1968) coined the name "black hole" in 1968.

In 1963 R. Kerr (1963) discovered an exact family of charge-free solutions to Einstein's vacuum field equations. The charged generalization was subsequently found as a solution to the Einstein–Maxwell field equations by Newman et al. (1965). Only later was the connection of these results to black holes appreciated. We know today that the *Kerr–Newman geometry* described by these solutions provides a unique and complete description of the external gravitational and electromagnetic fields of a stationary black hole.

A number of important properties of black holes were discovered and several powerful theorems concerning black holes were proved during this period. The discovery of quasars in 1963, pulsars in 1968, and compact X-ray sources in 1962 helped motivate this intensive theoretical study of black holes. Observations of the binary X-ray source Cygnus X-1 in the early 1970s (cf. Section 13.5) provided the first plausible evidence that black holes might actually exist in space.

We turn now from history to a discussion of the physics of black holes. We shall begin our treatment with a discussion of the simplest black hole, one with $J = Q = 0$.

12.3 Schwarzschild Black Holes

We repeat here the Schwarzschild solution from Eq. (5.6.8):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \tag{12.3.1}$$

We are using the geometrized units ($c = G = 1$) of Section 5.5.

[4]See Harrison et al. (1965) for an account of these investigations.

A static observer in this gravitational field is one who is at fixed r, θ, ϕ. The lapse of proper time for such an observer is given by Eq. (12.3.1) as

$$d\tau^2 = -ds^2 = \left(1 - \frac{2M}{r}\right) dt^2, \tag{12.3.2}$$

or

$$d\tau = \left(1 - \frac{2M}{r}\right)^{1/2} dt. \tag{12.3.3}$$

This simply shows the familiar gravitational time dilation (redshift) for a clock in the gravitational field compared with a clock at infinity (i.e., $d\tau < dt$). Note that Eq. (12.3.3) breaks down at $r = 2M$, which is the *event horizon* ($\equiv$ *surface of the black hole* $\equiv$ *Schwarzschild radius*). Another name for this is the *static limit*, because static observers cannot exist inside $r = 2M$; they are inexorably drawn into the central singularity, as we shall see later.

A static observer makes measurements with his or her local orthonormal tetrad (Section 5.1). Using carets to denote quantities in the local orthonormal frame, we have from Eq. (12.3.1)

$$\begin{aligned}
\vec{\mathbf{e}}_{\hat{t}} &= \left(1 - \frac{2M}{r}\right)^{-1/2} \vec{\mathbf{e}}_t, \\
\vec{\mathbf{e}}_{\hat{r}} &= \left(1 - \frac{2M}{r}\right)^{1/2} \vec{\mathbf{e}}_r, \\
\vec{\mathbf{e}}_{\hat{\theta}} &= \frac{1}{r} \vec{\mathbf{e}}_\theta, \\
\vec{\mathbf{e}}_{\hat{\phi}} &= \frac{1}{r \sin\theta} \vec{\mathbf{e}}_\phi.
\end{aligned} \tag{12.3.4}$$

This is clearly an orthonormal frame, since[5]

$$\vec{\mathbf{e}}_{\hat{t}} \cdot \vec{\mathbf{e}}_{\hat{t}} = \left(1 - \frac{2M}{r}\right)^{-1} \vec{\mathbf{e}}_t \cdot \vec{\mathbf{e}}_t = \left(1 - \frac{2M}{r}\right)^{-1} g_{tt} = -1, \text{ etc.} \tag{12.3.5}$$

12.4 Test Particle Motion

To explore the Schwarzschild geometry further, let us consider the motion of freely moving test particles. Recall from Eq. (5.2.21) that such particles move

[5] The reader may wish to review the last part of Section 5.2, which discusses the relationship between an orthonormal frame and a general coordinate system.

along geodesics of spacetime, the geodesic equations being derivable from the Lagrangian

$$2L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2, \quad (12.4.1)$$

where $\dot{t} \equiv dt/d\lambda = p^t$ is the t-component of 4-momentum, and so on. Here we have chosen the parameter λ to satisfy $\lambda = \tau/m$ for a particle of mass m.

The Euler–Lagrange equations are

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0, \qquad x^\alpha = (t, r, \theta, \phi). \quad (12.4.2)$$

For θ, ϕ, and t these are, respectively,

$$\frac{d}{d\lambda}(r^2\dot{\theta}) = r^2\sin\theta\cos\theta\,\dot{\phi}^2, \quad (12.4.3)$$

$$\frac{d}{d\lambda}(r^2\sin^2\theta\,\dot{\phi}) = 0, \quad (12.4.4)$$

$$\frac{d}{d\lambda}\left[\left(1 - \frac{2M}{r}\right)\dot{t}\right] = 0. \quad (12.4.5)$$

Instead of using the r-equation directly, it is simpler to use the fact that

$$g_{\alpha\beta}p^\alpha p^\beta = -m^2. \quad (12.4.6)$$

In other words, in Eq. (12.4.1) L has the value $-m^2/2$.

Now Eq. (12.4.3) shows that if we orient the coordinate system so that initially the particle is moving in the equatorial plane (i.e., $\theta = \pi/2$, $\dot{\theta} = 0$), then the particle remains in the equatorial plane. This result follows from the uniqueness theorem for solutions of such differential equations, since $\theta = \pi/2$ for all λ satisfies the equation. Physically, the result is obvious from spherical symmetry.

With $\theta = \pi/2$, Eqs. (12.4.4) and (12.4.5) become

$$p_\phi \equiv r^2\dot{\phi} = \text{constant} \equiv l, \quad (12.4.7)$$

$$-p_t \equiv \left(1 - \frac{2M}{r}\right)\dot{t} = \text{constant} \equiv E. \quad (12.4.8)$$

These are simply the constants of the motion corresponding to the ignorable coordinates ϕ and t in Eq. (12.4.1) (cf. Section 5.2). To understand their physical significance, consider a measurement of the particle's energy made by a static observer in the equatorial plane. This locally measured energy is the time component of the 4-momentum as measured in the observer's local orthonormal frame—that is, the projection of the 4-momentum along the time basis vector:

$$E_{\text{local}} \equiv p^{\hat{t}} = -p_{\hat{t}} = -\vec{\mathbf{p}} \cdot \vec{\mathbf{e}}_{\hat{t}} = -\vec{\mathbf{p}} \cdot \left(1 - \frac{2M}{r}\right)^{-1/2} \vec{\mathbf{e}}_t$$

$$= -\left(1 - \frac{2M}{r}\right)^{-1/2} p_t,$$

that is,

$$E = \left(1 - \frac{2M}{r}\right)^{1/2} E_{\text{local}}. \qquad (12.4.9)$$

For $r \to \infty$, $E_{\text{local}} \to E$, so the conserved quantity E is called the "energy-at-infinity." It is related to E_{local} by a redshift factor.

Exercise 12.2 For an alternative derivation of the redshift formula, use the fact that E is constant along the photon's path to show that

$$\frac{\nu_{\text{em}}}{\nu_{\text{rec}}} = \left(1 - \frac{2M}{r_{\text{em}}}\right)^{-1/2} \qquad (12.4.10)$$

for a static emitter at $r = r_{\text{em}}$ and a receiver at $r \to \infty$. Explain why the event horizon for a Schwarzschild black hole is sometimes called the "surface of infinite redshift."

The physical interpretation of l follows from considering the locally measured value of $v^{\hat{\phi}}$, the tangential velocity component:

$$v^{\hat{\phi}} = \frac{p^{\hat{\phi}}}{p^{\hat{t}}} = \frac{p_{\hat{\phi}}}{p^{\hat{t}}} = \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{e}}_{\hat{\phi}}}{E_{\text{local}}} = \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{e}}_{\phi}/r}{E_{\text{local}}} = \frac{p_{\phi}/r}{E_{\text{local}}},$$

and so

$$l = E_{\text{local}} r v^{\hat{\phi}}. \qquad (12.4.11)$$

Comparing with the Newtonian expression $mv^{\hat{\phi}}r$, we see that l is the conserved angular momentum of the particle.

We now consider separately the cases $m \neq 0$ and $m = 0$. For particles of nonzero rest mass, it is convenient to renormalize E and l to quantities expressed per unit mass. Define

$$\tilde{E} = \frac{E}{m}, \qquad \tilde{l} = \frac{l}{m}. \qquad (12.4.12)$$

Then, recalling that $\lambda = \tau/m$, we find from Eqs. (12.4.6)–(12.4.8):

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{l}^2}{r^2}\right), \tag{12.4.13}$$

$$\frac{d\phi}{d\tau} = \frac{\tilde{l}}{r^2}, \tag{12.4.14}$$

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r}. \tag{12.4.15}$$

Equation (12.4.13) can be solved for $r = r(\tau)$ (in general, an elliptic integral); then Eq. (12.4.14) gives $\phi(\tau)$ and Eq. (12.4.15) gives $t(\tau)$.

It is interesting to consider orbits just outside the event horizon. The locally measured value of $v^{\hat{r}}$, the radial velocity component, is given by

$$v^{\hat{r}} = \frac{p^{\hat{r}}}{p^{\hat{t}}} = \frac{p_{\hat{r}}}{p^{\hat{t}}} = \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{e}}_{\hat{r}}}{E_{\text{local}}} = \frac{p_r(1 - 2M/r)^{1/2}}{E_{\text{local}}} = \frac{p^r}{E}, \tag{12.4.16}$$

from Eqs. (12.3.4) and (12.4.9). Recalling $p^r \equiv m\, dr/d\tau$ and Eq. (12.4.13), we get

$$v^{\hat{r}} = \frac{dr}{\tilde{E}\, d\tau} = \left[1 - \frac{1}{\tilde{E}^2}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{l}^2}{r^2}\right)\right]^{1/2}. \tag{12.4.17}$$

So as $r \to 2M$, $v^{\hat{r}} \to 1$ and the particle is observed by a local static observer at r to approach the event horizon along a *radial* geodesic at the *speed of light*, independent of $\tilde{l}$.

Exercise 12.3 Show that the same observer at r finds that the tangential velocity of the particle satisfies

$$v^{\hat{\phi}} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{\tilde{l}}{r\tilde{E}}, \tag{12.4.18}$$

so that $v^{\hat{\phi}} \to 0$ as $r \to 2M$.

Exercise 12.4

(a) Show from Eq. (12.4.17) that a local observer at r finds that the velocity of a radially freely-falling particle released from rest at infinity is given by

$$v^{\hat{r}} = \left(\frac{2m}{r}\right)^{1/2}, \tag{12.4.19}$$

which has precisely the same form as the Newtonian velocity!

(b) Obtain the same result from Eq. (12.4.9), noting that $E_{\text{local}} \equiv \gamma m$.

Exercise 12.5 A particle moves along a geodesic from r and ϕ to $r + dr$ and $\phi + d\phi$ in time dt. A local static observer at (r, ϕ) measures the proper length of the particle's path to have increased by $ds(t, \theta, \phi = \text{const}) = g_{rr}^{1/2}\, dr(= d\hat{r})$ and $ds(t, r, \theta = \text{const}) = g_{\phi\phi}^{1/2}\, d\phi(= d\hat{\phi})$ in the r and ϕ directions, respectively, during this time; the proper time for this motion as measured on the observer's clock lasts $[-ds^2(r, \theta, \phi = \text{const})]^{1/2} = (-g_{00})^{1/2}\, dt(= d\hat{t})$. [Note that $d\hat{t}$ for the *observer* is *not* equal to $d\tau$ appearing, e.g., in Eqs. (12.4.13)–(12.4.15) for the particle!] Use the expressions for these measurements together with Eqs. (12.4.13)–(12.4.15) to rederive Eqs. (12.4.17) and (12.4.18).

The simplest geodesics are those for radial infall, $\phi =$ constant. This occurs if $\tilde{l} = 0$, and Eq. (12.4.13) becomes

$$\frac{dr}{d\tau} = -\left(\tilde{E}^2 - 1 + \frac{2M}{r}\right)^{1/2}. \tag{12.4.20}$$

By considering the limit $r \to \infty$ of Eq. (12.4.20), we see that there are three cases: (i) $\tilde{E} < 1$, particle falls from rest at $r = R$, say; (ii) $\tilde{E} = 1$, particle falls from rest at infinity; (iii) $\tilde{E} > 1$, particle falls with finite inward velocity from infinity, $v \equiv v_\infty$.

Exercise 12.6

(a) Integrate Eq. (12.4.20) for the case $\tilde{E} < 1$, so that $1 - \tilde{E}^2 = 2M/R$, to get ($\tau = 0$ at $r = R$):

$$\tau = \left(\frac{R^3}{8M}\right)^{1/2}\left[2\left(\frac{r}{R} - \frac{r^2}{R^2}\right)^{1/2} + \cos^{-1}\left(\frac{2r}{R} - 1\right)\right]. \tag{12.4.21}$$

(b) Introduce the "cycloid parameter" η by

$$r = \frac{R}{2}(1 + \cos\eta), \tag{12.4.22}$$

and show that

$$\tau = \left(\frac{R^3}{8M}\right)^{1/2}(\eta + \sin\eta). \tag{12.4.23}$$

(c) Integrate Eq. (12.4.15) for t in terms of η to get ($t = 0$ at $r = R$):

$$\frac{t}{2M} = \ln\left|\frac{(R/2M - 1)^{1/2} + \tan(\eta/2)}{(R/2M - 1)^{1/2} - \tan(\eta/2)}\right| + \left(\frac{R}{2M} - 1\right)^{1/2}\left[\eta + \frac{R}{4M}(\eta + \sin\eta)\right]. \tag{12.4.24}$$

Note the following important results for radial infall: from Eq. (12.4.21), the *proper time* to fall from rest at $r = R > 2M$ to $r = 2M$ is *finite*. In fact, the proper time to fall to $r = 0$ is $\pi(R^3/8M)^{1/2}$, also finite. However, from Eqs. (12.4.23) and (12.4.24), the *coordinate time* (proper time for an observer at infinity) to fall to $r = 2M$ is *infinite* [at $r = 2M$, $\tan(\eta/2) = (R/2M - 1)^{1/2}$]. These results are displayed in Figure 12.1.

Exercise 12.7

(a) Find $\tau(r)$ and $t(r)$ for radial infall when $\tilde{E} = 1$.

(b) Find $\tau(r)$, $r(\eta)$, $\tau(\eta)$, and $t(\eta)$ when $\tilde{E} > 1$. You can get these from Eqs. (12.4.21)–(12.4.24) by defining R such that $2M/R = \tilde{E}^2 - 1$ and changing the sign of R in these equations. Show that $2M/R = v_\infty^2/(1 - v_\infty^2)$.

Answer: See Lightman et al. (1975), p. 407.

Turn now to nonradial motion. The elliptic integrals resulting from Eqs. (12.4.13)–(12.4.15) are not particularly informative, but we can get a general picture of the orbits by considering an "effective potential,"

$$V(r) \equiv \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{l}^2}{r^2}\right). \tag{12.4.25}$$

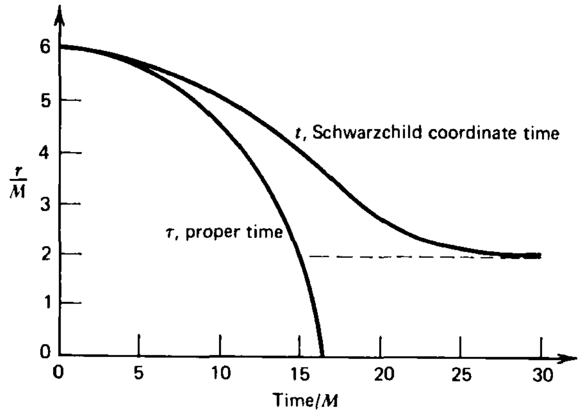

Figure 12.1 Fall from rest toward a Schwarzschild black hole as described (a) by a comoving observer (proper time τ) and (b) by a distant observer (Schwarzschild coordinate time t). In the one description, the point $r = 0$ is attained, and quickly [see Eq. (12.4.23)]. In the other description, $r = 0$ is never reached and even $r = 2M$ is attained only asymptotically [Eq. (12.4.24)]. [From *Gravitation* by Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, W. H. Freeman and Company. Copyright © 1973.]

Equation (12.4.13) then becomes

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - V(r). \tag{12.4.26}$$

For a fixed value of $\tilde{l}$, V is depicted schematically in Figure 12.2. Shown on the diagram are three horizontal lines corresponding to different values of $\tilde{E}^2$. From Eq. (12.4.26) we see that the distance from the horizontal line to V gives $(dr/d\tau)^2$. Consider orbit 1, the horizontal line labeled 1 corresponding to a particle coming in from infinity with energy $\tilde{E}^2$. When the particle reaches the value of r corresponding to point A, $dr/d\tau$ passes through zero and changes sign—the particle returns to infinity. Such an orbit is *unbound*, and A is called a *turning point*. Orbit 2 is a *capture* orbit; the particle plunges into the black hole. Orbit 3 is a *bound* orbit, with two turning points A_1 and A_2. The point B corresponds to a *stable circular* orbit. If the particle is slightly perturbed away from B, the orbit remains close to B. The point C is an *unstable circular* orbit; a particle placed in such an orbit will, upon experiencing the slightest inward radial perturbation, fall toward the black hole and be captured. If it is perturbed outward, it flies off to infinity. Orbits like 1 and 3 exist in the Newtonian case for motion in a central gravitational field; capture orbits are unique to general relativity.

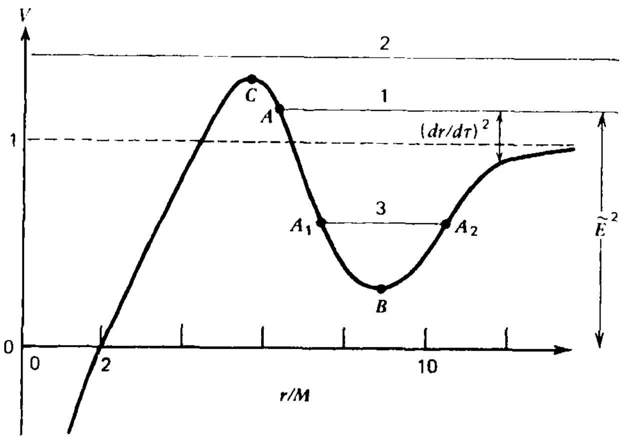

Figure 12.2 Sketch of the effective potential profile for a particle with *nonzero* rest mass orbiting a Schwarzschild black hole of mass M. The three horizontal lines labeled by different values of $\tilde{E}^2$ correspond to an (1) unbound, (2) capture, and (3) bound orbit, respectively. See text for details.

Exercise 12.8 Show that Eq. (12.4.26) reduces to the familiar Newtonian expression for particle motion in a central gravitational field when $2M/r \ll 1$.

Exercise 12.9

(a) Show that $\partial V/\partial r = 0$ when

$$Mr^2 - \tilde{l}^2 r + 3M\tilde{l}^2 = 0, \tag{12.4.27}$$

and hence that there are no maxima or minima of V for $\tilde{l} < 2\sqrt{3}\,M$.

(b) Show that $V_{\max} = 1$ for $\tilde{l} = 4M$.

The variation of V with $\tilde{l}$ is shown in Figure 12.3.

Circular orbits occur when $\partial V/\partial r = 0$ and $dr/d\tau = 0$. Equations (12.4.26) and (12.4.27) give

$$\tilde{l}^2 = \frac{Mr^2}{r - 3M}, \tag{12.4.28}$$

$$\tilde{E}^2 = \frac{(r - 2M)^2}{r(r - 3M)}. \tag{12.4.29}$$

Thus circular orbits exist down to $r = 3M$, the limiting case corresponding to a photon orbit ($\tilde{E} = E/m \to \infty$). The circular orbits are stable if V is concave up; that is, $\partial^2 V/\partial r^2 > 0$ and unstable if $\partial^2 V/\partial r^2 < 0$ (Why?).

Exercise 12.10 Show the circular Schwarzschild orbits are stable if $r > 6M$, unstable if $r < 6M$.

Exercise 12.11

(a) Show that in Newtonian theory, a distant nonrelativistic test particle can only be captured by a star of mass M and radius R if

$$\tilde{l} < \tilde{l}_{\text{crit}} \simeq (2MR)^{1/2}.$$

(b) Taking into account general relativity, can particles with much larger values of angular momentum be captured by neutron stars? by white dwarfs?

The binding energy per unit mass of a particle in the last stable circular orbit at $r = 6M$ is, from Eq. (12.4.29),

$$\tilde{E}_{\text{binding}} = \frac{m - E}{m} = 1 - \left(\frac{8}{9}\right)^{1/2} = 5.72\%. \tag{12.4.30}$$

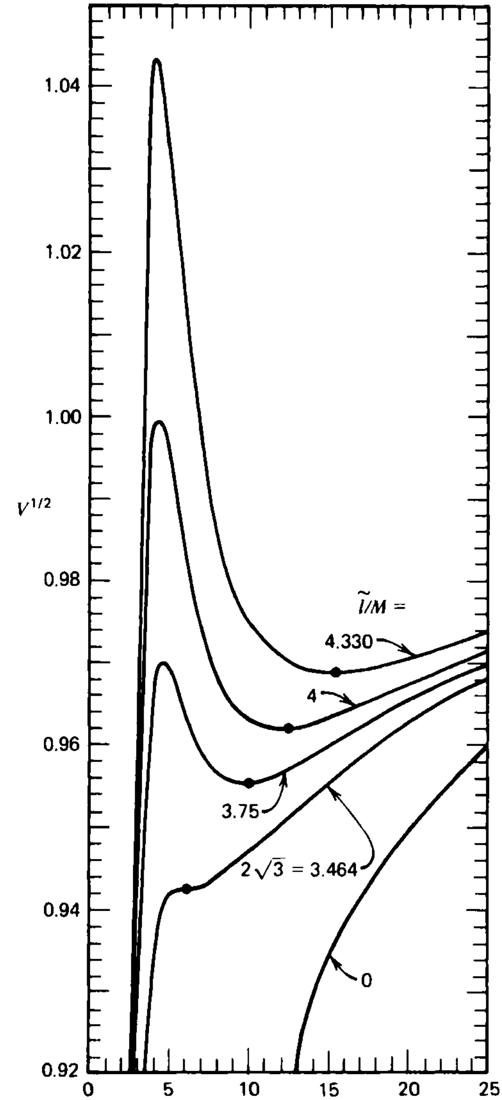

Figure 12.3 The effective potential profile for *nonzero* rest-mass particles of various angular momenta $\tilde{l}$ orbiting a Schwarzschild black hole of mass M. The dots at local minima locate radii of stable circular orbits. Such orbits exist only for $\tilde{l} > 2\sqrt{3}\,M$. [From *Gravitation* by Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, W. H. Freeman and Company. Copyright © 1973.]

This is the fraction of rest-mass energy released when, say, a particle originally at rest at infinity spirals slowly toward a black hole to the innermost stable circular orbit, and then plunges into the black hole. Thus, the conversion of rest mass to other forms of energy is potentially much more efficient for accretion onto a black hole than for nuclear burning, which releases a maximum of only 0.9% of the rest mass (H $\rightarrow$ Fe). This high efficiency will be important in our discussion of accretion disks around black holes (Section 14.5). It is the basis for invoking black holes as the energy source in numerous models seeking to explain astronomical observations of huge energy output from compact regions (e.g., Cygnus X-1; quasars; double radio galaxies, etc.).

Exercise 12.12

(a) Use Eq. (12.4.18) to show that the velocity of a particle in the innermost stable circular orbit as measured by a local static observer is $v^{\hat{\phi}} = \frac{1}{2}$ $(c = 1)$.

(b) Suppose the particle in part (a) is emitting monochromatic light at frequency ν_{em} in its rest frame. Show that the frequency received at infinity varies periodically between

$$\frac{\sqrt{2}}{3}\nu_{\text{em}} < \nu_\infty < \sqrt{2}\,\nu_{\text{em}}.$$

Hint: Write $\nu_\infty/\nu_{\text{em}} = (\nu_\infty/\nu_{\text{stat}})(\nu_{\text{stat}}/\nu_{\text{em}})$, where ν_{stat} is the frequency measured by the local static observer and is related to ν_{em} by the special relativistic Doppler formula.

(c) Compute the orbital period for the particle as measured by the local static observer and by the observer at infinity.

Hint: Since $d\hat{\phi} = r\,d\phi$, the proper circumference of the orbit is simply $2\pi r$.

Answer: $T_{\text{stat}} = 24\pi M$, $T_\infty = T_{\text{stat}}/(2/3)^{1/2} = 4.5 \times 10^{-4}$ s $(M/M_\odot)$

Exercise 12.13

(a) Show that the angular velocity as measured from infinity, $\Omega \equiv d\phi/dt$, has the same form in the Schwarzschild geometry as for circular orbits in Newtonian gravity—namely,

$$\Omega = \left(\frac{M}{r^3}\right)^{1/2}. \tag{12.4.31}$$

(b) Use this result to confirm the value of T_∞ found in Exercise 12.12.

In our later discussion of accretion onto black holes, we will need to know the capture cross section for particles falling in from infinity. This is simply

$$\sigma_{\text{capt}} = \pi b_{\text{max}}^2, \tag{12.4.32}$$

where b_{max} is the maximum impact parameter of a particle that is captured. To express b in terms of $\tilde{E}$ and $\tilde{l}$, consider the definition of the impact parameter (cf. Fig. 12.4)

$$b = \lim_{r \to \infty} r \sin \phi. \tag{12.4.33}$$

Now for $r \to \infty$, Eqs. (12.4.13) and (12.4.14) give

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 \simeq \frac{\tilde{E}^2 - 1}{\tilde{l}^2}. \tag{12.4.34}$$

Substituting $r \simeq b/\phi$, we identify

$$\frac{1}{b^2} = \frac{\tilde{E} - 1}{\tilde{l}^2}, \tag{12.4.35}$$

or in terms of the velocity at infinity, $\tilde{E} = (1 - v_\infty^2)^{-1/2}$,

$$\begin{aligned} \tilde{l} &= b v_\infty \left(1 - v_\infty^2\right)^{-1/2} \\ &\to b v_\infty \quad \text{for } v_\infty \ll 1. \end{aligned} \tag{12.4.36}$$

Consider now a nonrelativistic particle moving towards the black hole ($\tilde{E} \simeq 1$, $v_\infty \ll 1$). From Exercise 12.9, we know that it is captured if $\tilde{l} < 4M$. Thus

$$b_{\max} = \frac{4M}{v_\infty}, \tag{12.4.37}$$

which gives a capture cross section

$$\sigma_{\text{capt}} = \frac{4\pi (2M)^2}{v_\infty^2}. \tag{12.4.38}$$

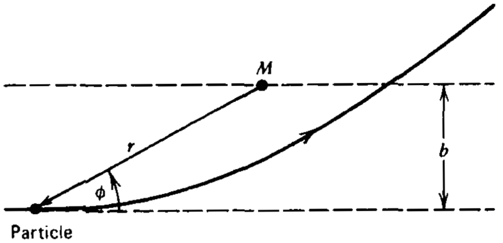

Figure 12.4 Impact parameter b for a particle with trajectory $r = r(\phi)$ about mass M.

This value should be compared with the geometrical capture cross section of a particle by a sphere of radius R in Newtonian theory:

$$\sigma_{\text{Newt}} = \pi R^2\left(1 + \frac{2M}{v_\infty^2 R}\right). \tag{12.4.39}$$

A black hole thus captures nonrelativistic particles like a Newtonian sphere of radius $R = 8M$.

12.5 Massless Particle Orbits in the Schwarzschild Geometry

For $m = 0$ (e.g., a photon), Eqs. (12.4.6)–(12.4.8) become

$$\frac{dt}{d\lambda} = \frac{E}{1 - 2M/r}, \tag{12.5.1}$$

$$\frac{d\phi}{d\lambda} = \frac{l}{r^2}, \tag{12.5.2}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right). \tag{12.5.3}$$

Now by the Equivalence Principle, we know that the particle's worldline should be independent of its energy. We can see this by introducing a new parameter

$$\lambda_{\text{new}} = l\lambda. \tag{12.5.4}$$

Writing

$$b \equiv \frac{l}{E} \tag{12.5.5}$$

and dropping the subscript "new," we find

$$\frac{dt}{d\lambda} = \frac{1}{b(1 - 2M/r)}, \tag{12.5.6}$$

$$\frac{d\phi}{d\lambda} = \frac{1}{r^2}, \tag{12.5.7}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right). \tag{12.5.8}$$

The worldline depends only on the parameter b, which is the particle's *impact parameter*, and not on l or E separately. Taking the limit $m \to 0$ of Eq. (12.4.35), we see that b of Eq. (12.5.5) is the same quantity defined in the previous section for massive particles.

We can understand photon orbits by means of an effective potential

$$V_{\text{phot}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right), \tag{12.5.9}$$

so that Eq. (12.5.8) becomes

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - V_{\text{phot}}(r). \tag{12.5.10}$$

Clearly the distance from a horizontal line of height $1/b^2$ to V_{phot} gives $(dr/d\lambda)^2$. The quantity V_{phot} has a maximum of $1/(27M^2)$ at $r = 3M$; it is displayed in Figure 12.5. We see that the critical impact parameter separating capture from scattering orbits is given by $1/b^2 = 1/(27M^2)$, or

$$b_c = 3\sqrt{3}\,M. \tag{12.5.11}$$

The capture cross section for photons from infinity is thus

$$\sigma_{\text{phot}} = \pi b_c^2 = 27\pi M^2. \tag{12.5.12}$$

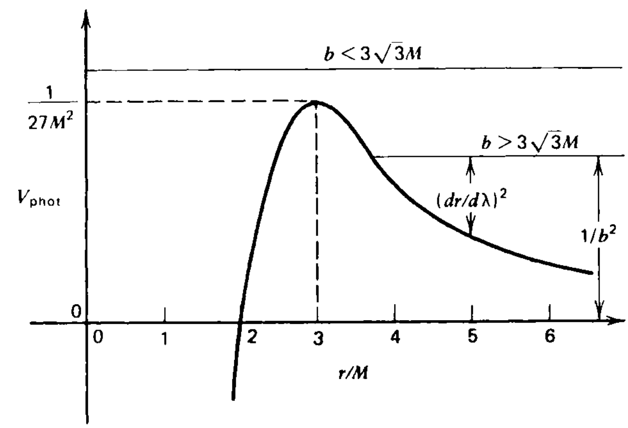

Figure 12.5 Sketch of the effective potential profile for a particle with *zero* rest mass orbiting a Schwarzschild black hole of mass M. If the particle falls from $r = \infty$ with impact parameter $b > 3\sqrt{3}\,M$ it is scattered back out to $r = \infty$. If, however, $b < 3\sqrt{3}\,M$ the particle is captured by the black hole.

To calculate the observed emission from gas near a black hole we must know those propagation directions, as measured by a static observer, for which a photon emitted at radius r can escape to infinity. Referring to Figure 12.5, we see that a photon at $r \geqslant 3M$ escapes only if (i) $v^{\hat{r}} > 0$, or (ii) $v^{\hat{r}} < 0$ and $b > 3\sqrt{3}\,M$. In terms of the angle ψ between the propagation direction and the radial direction (see Figure 12.6), we have since $|\mathbf{v}| = 1$,

$$v^{\hat{\phi}} = \sin\psi, \qquad v^{\hat{r}} = \cos\psi. \tag{12.5.13}$$

But Eqs. (12.4.12) and (12.4.18) give, with $b = l/E$,

$$v^{\hat{\phi}} = \frac{b}{r}\left(1 - \frac{2M}{r}\right)^{1/2}. \tag{12.5.14}$$

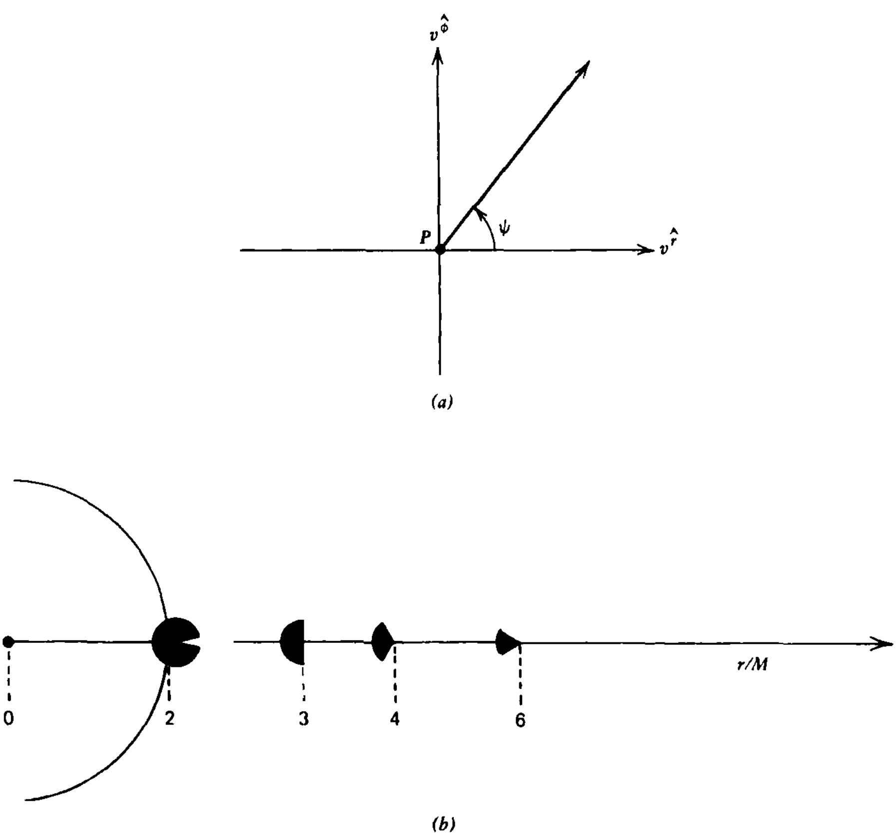

Figure 12.6 (a) The angle ψ between the propagation direction of a photon and the radial direction at a given point P. (b) Gravitational capture of radiation by a Schwarzschild black hole. Rays emitted from each point into the interior of the *shaded* conical cavity are captured. The indicated capture cavities are those measured in the orthonormal frame of a local static observer.

Thus an inward-moving photon escapes the black hole if

$$\sin\psi > \frac{3\sqrt{3}\,M}{r}\left(1 - \frac{2M}{r}\right)^{1/2}. \qquad (12.5.15)$$

At $r = 6M$, escape requires $\psi < 135°$; at $r = 3M$, $\psi < 90°$ so that all inward-moving photons are captured (i.e., 50% of the radiation from a stationary, isotropic emitter at $r = 3M$ is captured).

Exercise 12.14 Show that an outward-directed photon emitted between $r = 2M$ and $r = 3M$ escapes if

$$\sin\psi < \frac{3\sqrt{3}\,M}{r}\left(1 - \frac{2M}{r}\right)^{1/2}.$$

Only the outward-directed radial photons escape as the source approaches $r = 2M$. See Figure 12.6 for a diagram of these effects.

12.6 Nonsingularity of the Schwarzschild Radius

The metric (12.3.1) appears singular at $r = 2M$; the coefficient of dt^2 goes to zero, while the coefficient of dr^2 becomes infinite. However, we cannot immediately conclude that this behavior represents a true physical singularity. Indeed, the coefficient of $d\phi^2$ becomes zero at $\theta = 0$, but we know this is simply because the polar coordinate system itself is singular there. The coordinate singularity at $\theta = 0$ can be removed by choosing another coordinate system (e.g., stereographic coordinates on the 2-sphere).

We already have a clue that the Schwarzschild radius $r = 2M$ is only a *coordinate singularity*. Recall that a radially infalling particle does not notice anything strange about $r = 2M$; there is nothing special about $r(\tau)$ at this point. However, the coordinate time t becomes infinite as $r \to 2M$. This strongly suggests the presence of a coordinate singularity rather than a physical singularity.

There are many different coordinate transformations that can be used to show explicitly that $r = 2M$ is not a physical singularity. We shall exhibit one—the *Kruskal*[6] *coordinate system*. It is defined by the transformation

$$u = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\frac{t}{4M}, \qquad (12.6.1)$$

$$v = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\frac{t}{4M}. \qquad (12.6.2)$$

[6] Kruskal (1960); Szekeres (1960).

The inverse transformation is

$$\left(\frac{r}{2M} - 1\right)e^{r/2M} = u^2 - v^2, \tag{12.6.3}$$

$$\tanh\frac{t}{4M} = \frac{v}{u}. \tag{12.6.4}$$

The metric (12.3.1) takes the form

$$ds^2 = \frac{32M^3}{r}e^{-r/2M}(-dv^2 + du^2) + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2, \tag{12.6.5}$$

where r is defined implicitly in terms of u and v by Eq. (12.6.3). Clearly the metric (12.6.5) is nonsingular at $r = 2M$. However, $r = 0$ is still a singularity. One can show that it is a real *physical singularity* of the metric, with infinite gravitational field strengths there.

Note that $r = 0$ is, from Eq. (12.6.3), at $v^2 - u^2 = 1$, or $v = \pm(1 + u^2)^{1/2}$. There are two singularities! Note also that $r \geqslant 2M$ is the region $u^2 \geqslant v^2$—that is, $u \geqslant |v|$ or $u \leqslant -|v|$. Two regions correspond to $r \geqslant 2M$!

The original Schwarzschild coordinate system covers only part of the spacetime manifold. Kruskal coordinates give an analytic continuation of the same solution of the field equations to cover the whole spacetime manifold. This situation is depicted in the *Kruskal diagram*, Figure 12.7. Kruskal coordinates have the nice property that radial light rays travel on 45° straight lines [see Eq. (12.6.5) with $ds^2 = 0$]. A Kruskal diagram is a spacetime diagram, with the time coordinate v plotted vertically and the spatial coordinate u plotted horizontally. It can be read like a special relativistic spacetime diagram, because the light cones are at 45° at every point, and particle worldlines must lie inside the light cones.

Region I is "our universe," the original region $r > 2M$. Region II is the "interior of the black hole," the region $r < 2M$. Regions III and IV are the "other universe": region III is asymptotically flat, with $r > 2M$, while region IV corresponds to $r < 2M$.

If one checks the signs of u and v in the various quadrants, one finds that the relationship between Kruskal and Schwarzschild coordinates in the various regions is [cf. Eqs. (12.6.1) and (12.6.2)]

$$u = \pm\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M}\cosh\frac{t}{4M}, \tag{12.6.6}$$

$$v = \pm\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M}\sinh\frac{t}{4M}, \quad (r \geqslant 2M), \tag{12.6.7}$$

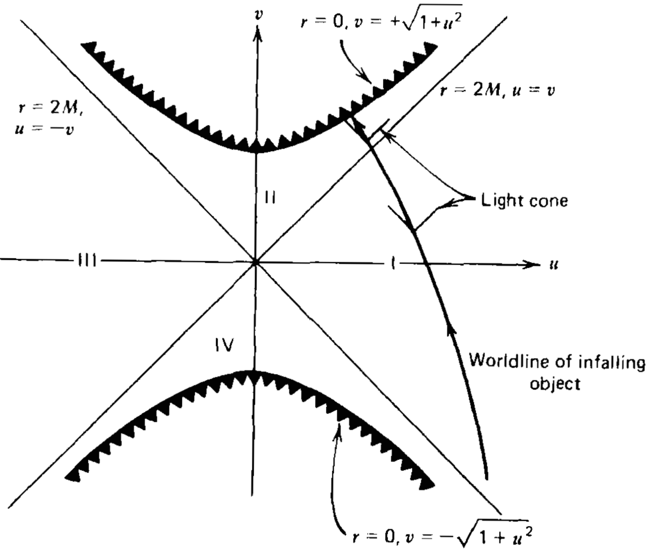

Figure 12.7 Kruskal diagram of the Schwarzschild metric.

and

$$u = \pm\left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh\frac{t}{4M}, \tag{12.6.8}$$

$$v = \pm\left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh\frac{t}{4M}, \quad (r \leqslant 2M). \tag{12.6.9}$$

Here the upper sign refers to "our universe," the lower to the "other universe." Equation (12.6.3) holds everywhere, while the right-hand side of Eq. (12.6.4) becomes u/v for $r \leqslant 2M$. Equation (12.6.4) shows that lines of constant t are straight lines. These relationships are shown in Figure 12.8.

The singularity at the top of the Kruskal diagram at $r = 0$ occurs inside of the black hole. Clearly any timelike worldline at $r \leqslant 2M$ (i.e., in region II) *must* strike the singularity. The singularity at the bottom of the diagram represents a "white hole," from which anything can come spewing out.

It is important to realize that the full analytically continued Schwarzschild metric is merely a mathematical solution of Einstein's equations. For a black hole formed by gravitational collapse, part of spacetime must contain the collapsing matter. We know from Birkhoff's theorem (Section 5.6) that outside the collapsing star the geometry is still described by the Schwarzschild metric. Thus the worldline of a point on the surface of the star will be the boundary of the physically meaningful part of the Kruskal diagram (see Fig. 12.9). The "white hole" and the "other universe" are not present in real black holes.

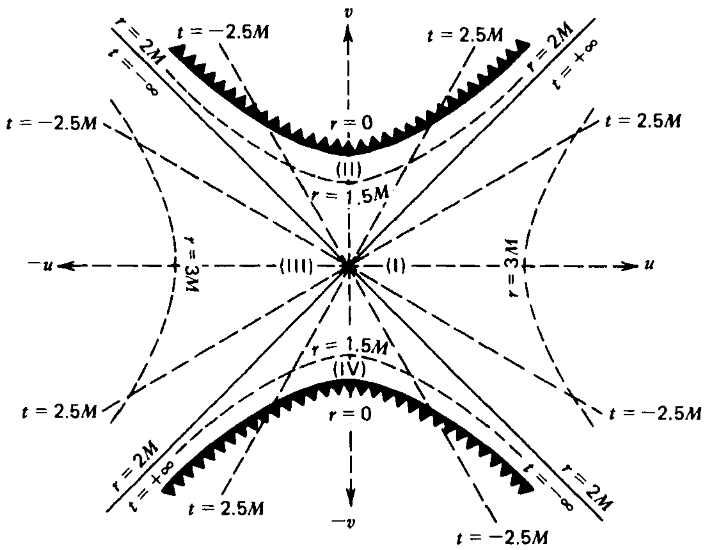

Figure 12.8 Kruskal diagram of the Schwarzschild metric showing the relation between Schwarzschild (t, r) and Kruskal (v, u) coordinates.

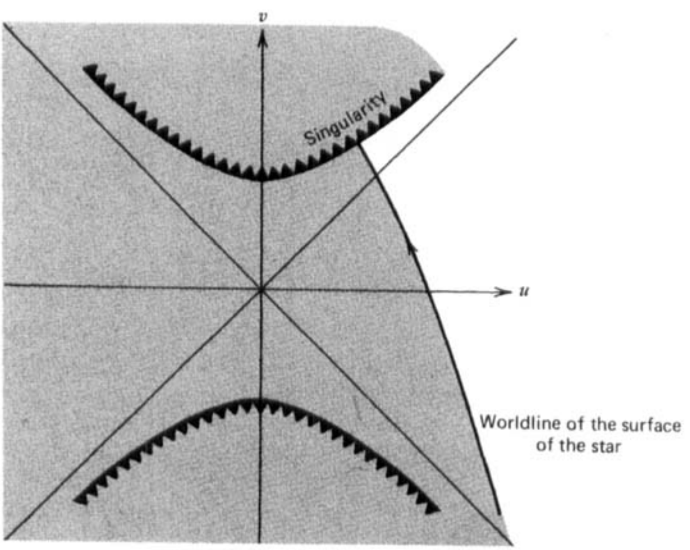

Figure 12.9 Kruskal diagram for gravitational collapse. Only the *unshaded* portion to the right of the surface of the star is physically meaningful. The remainder of the diagram must be replaced by the spacetime geometry of the interior of the star.

The Kruskal diagram makes clear the two key properties of black holes: once an object crosses $r = 2M$, it must strike the singularity at $r = 0$; and once inside $r = 2M$, it cannot send signals back out to infinity. However, there is no local test for this horizon. An observer does not notice anything significantly different in crossing from $r = 2M + \varepsilon$ to $r = 2M - \varepsilon$.

12.7 Kerr Black Holes

The most general stationary black hole metric, with parameters M, J, and Q, is called the Kerr–Newman metric.[7] Special cases are the Kerr metric ($Q = 0$), the Reissner–Nordstrom metric ($J = 0$), and the Schwarzschild metric ($J = 0$, $Q = 0$).

It is usually true that a charged astrophysical object is rapidly neutralized by the surrounding plasma. We shall accordingly simplify our discussion by assuming that charged black holes are not likely to be important astrophysically. All astrophysical objects rotate however, and so we expect black holes formed by gravitational collapse to be rotating in general. Remarkably, when all the radiation of various kinds produced by the collapse has been radiated away, the gravitational field settles down asymptotically to the Kerr metric.

This solution of Einstein's equations, discovered by Kerr in 1963, was not at first recognized to be a black hole solution. Its properties are more transparent in Boyer–Lindquist (1967) coordinates, where

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr\sin^2\theta}{\Sigma} dt\, d\phi + \frac{\Sigma}{\Delta} dr^2$$
$$+ \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2. \qquad (12.7.1)$$

Here the black hole is rotating in the ϕ direction, and

$$a \equiv \frac{J}{M}, \qquad \Delta \equiv r^2 - 2Mr + a^2, \qquad \Sigma \equiv r^2 + a^2\cos^2\theta. \qquad (12.7.2)$$

The metric is stationary (independent of t) and axisymmetric about the polar axis (independent of ϕ). Note that a, the angular momentum per unit mass, is measured in cm when expressed in units with $c = G = 1$. Setting $a = 0$ in Eq. (12.7.1) gives the Schwarzschild metric.

Exercise 12.15 The angular momentum of the sun (assuming uniform rotation) is $J = 1.63 \times 10^{48}$ g cm^2 s^{-1}. What is a/M for the sun?

Answer: 0.185.

[7] See, for example, Misner, Thorne, and Wheeler (1973).

The horizon occurs where the metric function Δ vanishes. This occurs first at the larger root of the quadratic equation $\Delta = 0$,

$$r_+ = M + (M^2 - a^2)^{1/2}. \tag{12.7.3}$$

Note that a must be less than M for a black hole to exist. If a exceeded M, one would have a gravitational field with a "naked" singularity (i.e., one not "clothed" by an event horizon). A major unsolved problem in general relativity is Penrose's Cosmic Censorship Conjecture, that gravitational collapse from well-behaved initial conditions never gives rise to a naked singularity. Certainly no mechanism is known to take a Kerr black hole with $a < M$ and spin it up so that a becomes greater than M (see Section 12.8). A black hole with $a \equiv M$ is called a *maximally rotating* black hole.

Static observers were useful for understanding the Schwarzschild metric. We can generalize to rotating black holes by introducing *stationary* observers, who are at fixed r and θ, but rotate with a constant angular velocity

$$\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t}. \tag{12.7.4}$$

The condition $\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} = -1$ (i.e., that the observers follow a timelike worldline) is

$$-1 = (u^t)^2\left[g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\right], \tag{12.7.5}$$

where Eq. (12.7.4) has been used to eliminate u^ϕ. The quantity in square brackets in Eq. (12.7.5) must therefore be negative. Since $g_{\phi\phi}$ in Eq. (12.7.1) is positive, this is true only if Ω lies between the roots of the quadratic equation obtained by setting the bracketed expression equal to zero. Thus

$$\Omega_{\min} < \Omega < \Omega_{\max}, \tag{12.7.6}$$

where

$$\Omega_{\substack{\max \\ \min}} = \frac{-g_{t\phi} \pm \left(g_{t\phi}^2 - g_{tt}g_{\phi\phi}\right)^{1/2}}{g_{\phi\phi}}. \tag{12.7.7}$$

Exercise 12.16 Discuss the restriction (12.7.6) in the weak-field limit.

Answer: $-c/(r\sin\theta) < \Omega < c/(r\sin\theta)$; stationary observers must rotate about the z-axis with $v < c$.

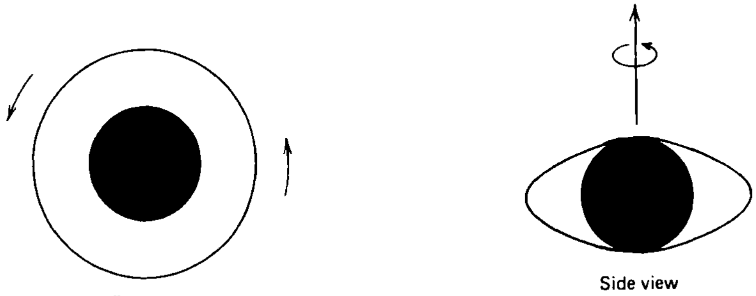

Figure 12.10 Ergosphere of a Kerr black hole: the region between the static limit [the flattened outer surface $r = M + (M^2 - a^2\cos^2\theta)^{1/2}$] and the event horizon [inner sphere $r = M + (M^2 - a^2)^{1/2}$].

Note that $\Omega_{\min} = 0$ when $g_{tt} = 0$; that is, when $r^2 - 2Mr + a^2\cos^2\theta = 0$. This occurs at

$$r_0 = M + (M^2 - a^2\cos^2\theta)^{1/2}. \tag{12.7.8}$$

Observers between r_+ and r_0 must have $\Omega > 0$; no static observers ($\Omega = 0$) exist within $r_+ < r < r_0$. The surface $r = r_0$ is therefore called the static limit. It is also called the "boundary of the ergosphere," for reasons that will become clearer later.

Note that the horizon r_+ and the static limit r_0 are distinct for $a \neq 0$. This is depicted in Figure 12.10. From Eq. (12.7.7), we see that stationary observers fail to exist when $g_{t\phi}^2 - g_{tt}g_{\phi\phi} < 0$—that is, when $\Delta < 0$; this can occur only when $r < r_+$. This is the generalization of the fact that static observers do not exist inside the horizon for a Schwarzschild black hole.

The Kerr metric can be analytically continued inside $r = r_+$ in a manner similar to the Kruskal continuation of the Schwarzschild metric. However, this interior solution is not physically meaningful for two reasons. First, part of it must be replaced by the interior of the collapsing object that formed the black hole. More important, there is no Birkhoff's theorem for rotating collapse. The Kerr metric is not the exterior metric *during* the collapse; it is only the *asymptotic* form of the metric when all the dynamics has ceased. Its mathematical continuation inside r_+ is essentially irrelevant. We shall therefore restrict our discussion to the region $r \geqslant r_+$, adopting the viewpoint that anything entering inside that region becomes causally disconnected from the rest of the universe.

A complete description of the geodesics of a Kerr black hole is quite complicated because of the absence of spherical symmetry. There is, however, a "hidden" symmetry that can be exploited to solve for the geodesics analytically.[8]

[8]Carter (1968); see also Misner, Thorne, and Wheeler (1973).

To get some insight into the effects of rotation on the geodesics, we shall restrict ourselves to considering test particle motion in the equatorial plane. This can be done in a straightforward way without invoking the hidden symmetry.

Setting $\theta = \pi/2$ in Eq. (12.7.1), we obtain the Lagrangian

$$2L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \frac{4aM}{r}\dot{t}\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi}^2, \tag{12.7.9}$$

where $\dot{t} = dt/d\lambda$, and so on. Corresponding to the ignorable coordinates t and ϕ, we obtain two first integrals:

$$p_t \equiv \frac{\partial L}{\partial \dot{t}} = \text{constant} = -E, \tag{12.7.10}$$

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \text{constant} \equiv l. \tag{12.7.11}$$

Evaluating the derivatives from Eq. (12.7.9) and solving the two equations for $\dot{t}$ and $\dot{\phi}$, we obtain

$$\dot{t} = \frac{(r^3 + a^2 r + 2Ma^2)E - 2aMl}{r\Delta}, \tag{12.7.12}$$

$$\dot{\phi} = \frac{(r - 2M)l + 2aME}{r\Delta}. \tag{12.7.13}$$

We get a third integral of the motion as usual be setting $g_{\alpha\beta}p^\alpha p^\beta = -m^2$—that is, $L = -m^2/2$. Substituting Eqs. (12.7.12) and (12.7.13), we obtain after some simplification

$$r^3\left(\frac{dr}{d\lambda}\right)^2 = R(E, l, r), \tag{12.7.14}$$

where

$$R \equiv E^2(r^3 + a^2 r + 2Ma^2) - 4aMEl - (r - 2M)l^2 - m^2 r\Delta. \tag{12.7.15}$$

We can regard R as an effective potential for radial motion in the equatorial plane.[9] For example, circular orbits occur where $dr/d\lambda$ remains zero (perpetual

[9] Some authors define the effective potential as that value of E which makes $R = 0$, as we did in the Schwarzschild case, Eq. (12.4.26).

turning point). This requires

$$R = 0, \qquad \frac{\partial R}{\partial r} = 0. \tag{12.7.16}$$

After considerable algebra, Eqs. (12.7.16) can be solved for E and l to give

$$\tilde{E} = \frac{r^2 - 2Mr \pm a\sqrt{Mr}}{r(r^2 - 3Mr \pm 2a\sqrt{Mr})^{1/2}}, \tag{12.7.17}$$

$$\tilde{l} = \pm \frac{\sqrt{Mr}\,(r^2 \mp 2a\sqrt{Mr} + a^2)}{r(r^2 - 3Mr \pm 2a\sqrt{Mr})^{1/2}}. \tag{12.7.18}$$

Here the upper sign refers to *corotating* or direct orbits (i.e., orbital angular momentum of particle parallel to black hole spin angular momentum), the lower sign to *counterrotating* orbits. These formulas generalize Eqs. (12.4.28)–(12.4.29) for the Schwarzschild metric.

Exercise 12.17 Show that Kepler's Third Law takes the form

$$\Omega = \pm \frac{M^{1/2}}{r^{3/2} \pm aM^{1/2}} \tag{12.7.19}$$

for circular equatorial orbits in the Kerr metric. Here $\Omega \equiv d\phi/dt = \dot{\phi}/\dot{t}$.

Circular orbits exist from $r = \infty$ all the way down to the limiting circular photon orbit, when the denominator of Eq. (12.7.17) vanishes. Solving the resulting cubic equation in $r^{1/2}$, we find for the photon orbit[10]

$$r_{\rm ph} = 2M\{1 + \cos[\tfrac{2}{3}\cos^{-1}(\mp a/M)]\}. \tag{12.7.20}$$

For $a = 0$, $r_{\rm ph} = 3M$, while for $a = M$, $r_{\rm ph} = M$ (direct) or $4M$(retrograde).

For $r > r_{\rm ph}$, not all circular orbits are bound. An unbound circular orbit has $E/m > 1$. Given an infinitesimal outward perturbation, a particle in such an orbit will escape to infinity on an asymptotically hyperbolic trajectory. Bound circular orbits exist for $r > r_{\rm mb}$, where $r_{\rm mb}$ is the radius of the marginally bound circular orbit with $E/m = 1$:

$$r_{\rm mb} = 2M \mp a + 2M^{1/2}(M \mp a)^{1/2}. \tag{12.7.21}$$

Note also that $r_{\rm mb}$ is the minimum periastron of all parabolic ($E/m = 1$) orbits.

[10] Bardeen et al. (1972).

In astrophysical problems, particle infall from infinity is very nearby parabolic, since $v_\infty \ll c$. Any parabolic trajectory which penetrates to $r < r_{\rm mb}$ must plunge directly into the black hole. For $a = 0$, $r_{\rm mb} = 4M$; for $a = M$, $r_{\rm mb} = M$(direct) or $5.83M$(retrograde).

Even the bound circular orbits are not all stable. Stability requires that

$$\frac{\partial^2 R}{\partial r^2} \leqslant 0. \tag{12.7.22}$$

From Eq. (12.7.15), we get

$$1 - (\tilde{E})^2 \geqslant \frac{2}{3}\frac{M}{r}. \tag{12.7.23}$$

Substituting Eq. (12.7.17), we get a quartic equation in $r^{1/2}$ for the limiting case of equality. The solution for $r_{\rm ms}$, the radius of the marginally stable circular orbit, is given by Bardeen et al. (1972):

$$\begin{aligned} r_{\rm ms} &= M\{3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}\}, \\ Z_1 &\equiv 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/3}\left[\left(1 + \frac{a}{M}\right)^{1/3} + \left(1 - \frac{a}{M}\right)^{1/3}\right], \\ Z_2 &\equiv \left(3\frac{a^2}{M^2} + Z_1^2\right)^{1/2}. \end{aligned} \tag{12.7.24}$$

For $a = 0$, $r_{\rm ms} = 6M$; for $a = M$, $r_{\rm ms} = M$(direct) or $9M$(retrograde). A quantity of great interest for the potential efficiency of a black hole accretion disk as an energy source is the binding energy of the marginally stable circular orbit. If we eliminate r from Eq. (12.7.17) using Eq. (12.7.23), we find

$$\frac{a}{M} = \mp\frac{4\sqrt{2}\,(1 - \tilde{E}^2)^{1/2} - 2\tilde{E}}{3\sqrt{3}\,(1 - \tilde{E}^2)}. \tag{12.7.25}$$

The quantity $\tilde{E}$ *decreases* from $\sqrt{8/9}$ $(a = 0)$ to $\sqrt{1/3}$ $(a = M)$ for *direct* orbits, while it *increases* from $\sqrt{8/9}$ to $\sqrt{25/27}$ for *retrograde* orbits. The maximum binding energy $1 - \tilde{E}$ for a maximally rotating black hole is $1 - 1/\sqrt{3}$, or 42.3% of the rest-mass energy! This is the amount of energy that is released by matter spiralling in toward the black hole through a succession of almost circular equatorial orbits. Negligible energy is released during the final plunge from $r_{\rm ms}$ into the black hole.

Note that the Boyer–Lindquist coordinate system collapses $r_{\rm ms}$, $r_{\rm mb}$, $r_{\rm ph}$, and r_+ into $r = M$ as $a \rightarrow M$. This is simply an artifact of the coordinates as discussed by Bardeen et al. (1972): the radii actually correspond to distinct spacetime regions.

An exceedingly interesting property of rotating black holes is that there exist negative energy test particle trajectories. If we solve Eq. (12.7.14) for E, we find

$$E = \frac{2aMl + (l^2r^2\Delta + m^2r\Delta + r^3\dot{r}^2)^{1/2}}{r^3 + a^2r + 2Ma^2}. \tag{12.7.26}$$

(The sign of the square root is determined by letting $r \to \infty$.) To have $E < 0$, we require a retrograde orbit ($l < 0$), with

$$l^2r^2\Delta + m^2r\Delta + r^3\dot{r}^2 < 4a^2M^2l^2. \tag{12.7.27}$$

The boundary of the region of negative energy orbits is found by making the left-hand side of inequality (12.7.27) as small as possible. Thus let $m \to 0$ (highly relativistic particle) and $\dot{r} \to 0$. We then find that the boundary is at $r = 2M = r_0$ ($\theta = \pi/2$). One can in fact show that the static limit r_0 is the boundary of the region containing negative energy orbits for *all* values of θ. A particle can only be injected into such an orbit *inside* the static limit; it then plunges into the black hole.

Penrose (1969) exploited this property of Kerr black holes in a remarkable thought experiment to demonstrate that rotating black holes are potentially vast storehouses of energy. Imagine sending a particle in from infinity with an energy E_{in}. The trajectory is carefully chosen so that it penetrates inside the static limit. The particle is then "instructed" (or preprogrammed) to split into two. One piece goes into a negative energy trajectory and down the black hole, with energy $E_{\text{down}} < 0$. The other comes back out to infinity with energy E_{out}. Energy conservation gives

$$E_{\text{in}} = E_{\text{out}} + E_{\text{down}}, \qquad \text{i.e.,} \quad E_{\text{out}} > E_{\text{in}}! \tag{12.7.28}$$

Even though some rest-mass has been lost down the hole, there is a net gain of energy at infinity. This energy is extracted from the rotational energy of the hole, which slows down slightly when the retrograde negative energy particle is captured.

The region $r_+ < r < r_0$, where energy extraction is possible, is called the *ergosphere*, from the Greek word for work.

Unfortunately, the original Penrose process is not likely to be important astrophysically. Bardeen et al. (1972) have shown that the breakup of the two particles inside the ergosphere must happen with a relative velocity of at least $\frac{1}{2}c$; it is hard to imagine astrophysical processes that produce such large relative velocities.

Energy amplification also occurs when waves (electromagnetic or gravitational) of suitable frequency are scattered by a rotating black hole. Part of the wave is absorbed, but the part that scatters can, under the right conditions, have more energy than the incident wave. Whether such "superradiant scattering" is important astrophysically or not is an open question. Superradiant scattering has been invoked to design a "black hole bomb," and for advanced civilizations to solve their energy crisis.[11] Incidentally, it has been shown that rotating black holes are dynamically stable objects, in the sense that they cannot spontaneously explode in a burst of energy.[12]

Exercise 12.18 Consider a particle with $\tilde{l} = 0$ released from rest far from a Kerr black hole. Show that the particle "corotates with the geometry" as it spirals toward the hole along a conical surface of constant θ. In other words, show that the particle acquires an angular velocity $d\phi/dt = \omega(r, \theta)$ as viewed from infinity, where

$$\omega(r, \theta) = \frac{2aMr}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}.$$

Note: Observers at fixed r and θ with zero angular momentum also "corotate with the geometry" with angular velocity $\omega(r, \theta)$. Such observers define the so-called "locally nonrotating frame" (LNRF) (see Bardeen et al., 1972); according to such observers, the released particle described above appears to move *radially* locally.

The procedure for determining the emission angles leading to photon capture and escape from a radiating source near a Kerr hole has been outlined by Bardeen (1973). Capture must be accounted for whenever one wishes to determine the actual radiation observed at infinity that originates from a local source near the hole. For a rotating hole, photons emitted with $v^{\phi} > 0$ in the LNRF preferentially escape to infinity. In general applications, the calculation of the escape angles must be performed numerically.[13]

12.8 The Area Theorem and Black Hole Evaporation

Hawking[14] proved a remarkable theorem about black holes: in any interaction, the surface area of a black hole can never decrease. If several black holes are present, it is the sum of the surface areas that can never decrease.

We can compute the surface area of a Kerr black hole quite easily from the metric (12.7.1). Setting t = constant, $r = r_+ =$ constant, and using Eq. (12.7.3), we

[11] Press and Teukolsky (1972).

[12] Press and Teukolsky (1973), Teukolsky and Press (1974); but see Section 12.8.

[13] See Cunningham and Bardeen (1972) and Shapiro (1974).

[14] See Hawking and Ellis (1973).

find for the metric on the surface

$$ds^2 = \left(r_+^2 + a^2\cos^2\theta\right) d\theta^2 + \frac{(2Mr_+)^2}{r_+^2 + a^2\cos^2\theta}\sin^2\theta\, d\phi^2. \tag{12.8.1}$$

The area of the horizon is

$$\begin{aligned} A &= \int\!\!\int \sqrt{g}\, d\theta\, d\phi \\ &= \int\!\!\int 2Mr_+ \sin\theta\, d\theta\, d\phi \\ &= 8\pi M\left[M + (M^2 - a^2)^{1/2}\right], \end{aligned} \tag{12.8.2}$$

where g is the determinant of the metric coefficients appearing in Eq. (12.8.1). Note that for $a = 0$, $A = 4\pi(2M)^2$, as we would expect.

Exercise 12.19 Use Hawking's area theorem to find the minimum mass M_2 of a Schwarzschild black hole that results from the collision of two Kerr black holes of equal mass M and opposite angular momentum parameter a. Show that if $|a| \to M$, 50% of the rest mass is allowed to be radiated away. Show that no other combinations of masses and angular momenta lead to higher possible efficiencies. Show that if $a = 0$, the maximum efficiency is 29%.

Note: The *actual* amount of radiation generated by such a collision is amenable to numerical computation. The result is not yet known for the general case, but for $a = 0$ it is $\sim$ 0.1% (Smarr, 1979a). See also Chapter 16.

We can use the area theorem to show that one cannot make a naked singularity by adding particles to a maximally rotating black hole in an effort to spin it up. From Eq. (12.8.2), we find that $\delta A > 0$ implies

$$\left[2M(M^2 - a^2)^{1/2} + 2M^2 - a^2\right]\delta M > Ma\,\delta a. \tag{12.8.3}$$

As $a \to M$, this becomes

$$M\,\delta M > a\,\delta a. \tag{12.8.4}$$

Thus M^2 always remains greater than a^2, and the horizon is not destroyed [cf. Eq. (12.7.3)]. The capture cross section for particles that increase a/M goes to zero as $a \to M$.

The law of increase of area looks very much like the second law of thermodynamics for the increase of entropy. Bekenstein (1973) tried to develop a thermodynamics of black hole interactions. However, in classical general relativity there is no equilibrium state involving black holes. If a black hole is placed in a radiation bath, it continually absorbs radiation without ever coming to equilibrium.

This situation was changed by Hawking's remarkable discovery[15] that when quantum effects are taken into account, black holes radiate with a thermal spectrum. The expectation number of particles of a given species emitted in a mode with frequency ω is

$$\langle N\rangle = \frac{\Gamma}{\exp(\hbar\omega/kT) \mp 1}, \tag{12.8.5}$$

where Γ is the absorption coefficient for that mode incident on the hole. The absorption coefficient Γ is a slowly varying function of ω depending on the kind of particle emitted and is close to unity for wavelengths much less than M; we shall simply take it to be unity. The temperature of the black hole is inversely proportional to its mass:

$$T = \frac{\hbar}{8\pi kM} \simeq 10^{-7}\,\mathrm{K}\left(\frac{M_\odot}{M}\right). \tag{12.8.6}$$

Note that the "Planck mass" and "Planck radius" are

$$\hbar^{1/2} = 2.2 \times 10^{-5}\,\mathrm{g} = 1.6 \times 10^{-33}\,\mathrm{cm} \tag{12.8.7}$$

in units with $c = G = 1$.

Exercise 12.20 Verify the numerical relations in Eqs. (12.8.6) and (12.8.7).

We are giving Hawking's formulas for the case of a Schwarzschild black hole. They can be easily generalized to include charge and rotation. Dimensionally, T is obtained by setting a thermal wavelength[16] $\hbar c/kT$ equal to the Schwarzschild radius. Because of the thermal nature of the spectrum, mainly massless particles are produced (photons, neutrinos, and gravitons). To create a significant amount of particles of mass m, one requires $kT \sim mc^2$; that is, the Schwarzschild radius must be of order the Compton wavelength $\lambda_c \sim \hbar/mc$ of the particle.

We can now compute the entropy of a black hole: since the area is (restoring the c's and G's)

$$A = 4\pi\left(\frac{2GM}{c^2}\right)^2, \tag{12.8.8}$$

[15] Hawking (1974), (1975).

[16] A thermal wavelength is roughly the average separation between photons in equilibrium at temperature T.

we have

$$d(Mc^2) = \frac{c^6}{G^2}\frac{1}{32\pi M}dA \equiv T\,dS \tag{12.8.9}$$

provided we identify the entropy S with

$$S = \frac{kc^3}{G\hbar}\frac{1}{4}A. \tag{12.8.10}$$

The ratio of a macroscopic quantity (A) to a microscopic quantity ($\hbar$) ensures that black holes have large entropy. This is in accord with the "no hair" ideas about many different internal states of a black hole corresponding to the same external gravitational field, and that information is lost from the outside world once black holes form.[17]

Note that during black hole "evaporation" (emission of thermal quanta), M decreases by energy conservation and thus so does A (and S). This violates Hawking's area theorem. However, one of the postulates of the area theorem is that matter obeys the "strong" energy condition, which requires that a local observer always measures positive energy densities and that there are no spacelike energy fluxes. Black hole evaporation can be understood as pair creation in the gravitational field of the black hole, one member of the pair going down the black hole and the other coming out to infinity. In pair creation the pair of particles materializes with a spacelike separation—effectively there is a spacelike energy flux.

The area theorem of classical general relativity gets replaced by a generalized second law of thermodynamics: in any interactions, the *sum* of the entropies of all black holes plus the entropy of matter outside black holes never decreases. Thus black holes do not simply have laws analogous to those of thermodynamics; they actually fit very naturally into an extended framework of thermodynamics.

We can get a qualitative understanding of the Hawking process by first considering the case of pair production in a strong electric field E (cf. Fig. 12.11). In quantum mechanics, the vacuum is continuously undergoing fluctuations, where a pair of "virtual" particles is created and then annihilated. The electric field tends to separate the charges. If the field is strong enough, the particles tunnel through the quantum barrier and materialize as real particles. The critical field strength is achieved when the work done in separating them by a Compton wavelength equals the energy necessary to create the particles:

$$eE\lambda_c \sim 2mc^2. \tag{12.8.11}$$

[17] Compare Bekenstein (1975).

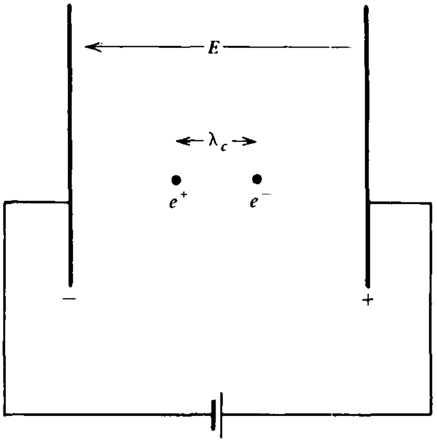

Figure 12.11 Pair production in a strong electric field.

In the black hole case, the tidal gravitational force across a distance λ_c is of order

$$\frac{GmM}{r^3}\lambda_c. \tag{12.8.12}$$

The work done is the product of this with λ_c. Setting $r \sim GM/c^2$, since the maximum field strength is near the horizon, and equating the work to $2mc^2$ gives

$$\lambda_c \sim \frac{GM}{c^2}. \tag{12.8.13}$$

Thus particles are created when their Compton wavelength is of order the Schwarzschild radius, as we mentioned earlier. [The argument has to be modified for massless particles, which have no barrier to penetrate. Their production rate is controlled by the phase space available. If we equate the mean separation of blackbody photons, $\hbar c/kT$, to the only length scale associated with the black hole, GM/c^2, we get a dimensional estimate of the black hole temperature, Eq. (12.8.6).]

The rate of energy loss from an evaporating black hole is given by the blackbody formula

$$\frac{dE}{dt} \sim \text{area} \times T^4 \sim M^2 \times M^{-4} \sim M^{-2}. \tag{12.8.14}$$

The associated timescale is

$$\tau \sim \frac{E}{dE/dt} \sim M^3. \tag{12.8.15}$$

To get the dimensions correct (with $c = G = 1$), we must restore a factor of $\hbar$:

$$\tau \sim \frac{M^3}{\hbar} \sim 10^{10}\,\text{yr}\left(\frac{M}{10^{15}\,\text{g}}\right)^3. \tag{12.8.16}$$

For solar mass black holes, Hawking evaporation is completely unimportant, as is clear from Eqs. (12.8.6) and (12.8.16). Only when $M \lesssim 10^{15}$ g is the timescale shorter than the age of the Universe. Presumably such "mini" black holes could only have been formed from density fluctuations during the Big Bang. Their Schwarzschild radius is about a fermi. If a spectrum of mini black holes were in fact formed in the early Universe, those with $M \ll 10^{15}$ g would long since have exploded. Those with $M \sim 10^{15}$ g would just now be exploding, with

$$\frac{dE}{dt} \sim 10^{20}\,\text{erg s}^{-1}\left(\frac{10^{15}\,\text{g}}{M}\right)^2 \tag{12.8.17}$$

[cf. Eq. (12.8.14)], producing quanta with energy

$$\hbar\omega \sim 100\,\text{MeV}\left(\frac{10^{15}\,\text{g}}{M}\right) \tag{12.8.18}$$

[cf. Eq. (12.8.6)]. Rees (1977a) and Blandford (1977) have discussed the possibility of detecting such events. Page (1976) has computed detailed predictions of the energy spectrum emitted by 10^{15} g black holes.

The observed energy density of γ-rays at around 100 MeV is[18] $\sim 10^{-38}$ g cm^{-3}. The calculations of Page suggest that about 10% of the energy of an exploding black hole is emitted in the form of photons (as opposed to neutrinos, gravitons or particles with mass). Thus the density of 10^{15} g black holes must be less than 10^{-37} g cm^{-3}, which is about 10^{-8} times the critical density to close the universe.[19]

Exercise 12.21

(a) Compute the entropy of a $1 M_\odot$ black hole in units of k, Boltzmann's constant.

Answer: $S = 1.0 \times 10^{77}\,k$.

(b) Estimate the entropy of the sun. Assume it consists of completely ionized hydrogen, with a mean density of 1 g cm^{-3} and a mean temperature of 10^6 K.

Answer: $S \sim 2 \times 10^{58}\,k$.

(c) Estimate the entropy of a $1 M_\odot$ iron white dwarf and a $1 M_\odot$ neutron star. Take the mean temperature to be 10^8 K, and the mean densities to be 10^6 g cm^{-3} and 10^{14} g cm^{-3}, respectively. Note that the expression (11.8.1) for C_v for a degenerate ideal gas is also equal to S, since $C_v = T\,dS/dT$. [Bekenstein (1975) has discussed the very large entropy of black holes from an information-theoretic viewpoint.]

[18] Fichtel et al. (1975).

[19] See also Chapline (1975); Page and Hawking (1976); Carr (1976).