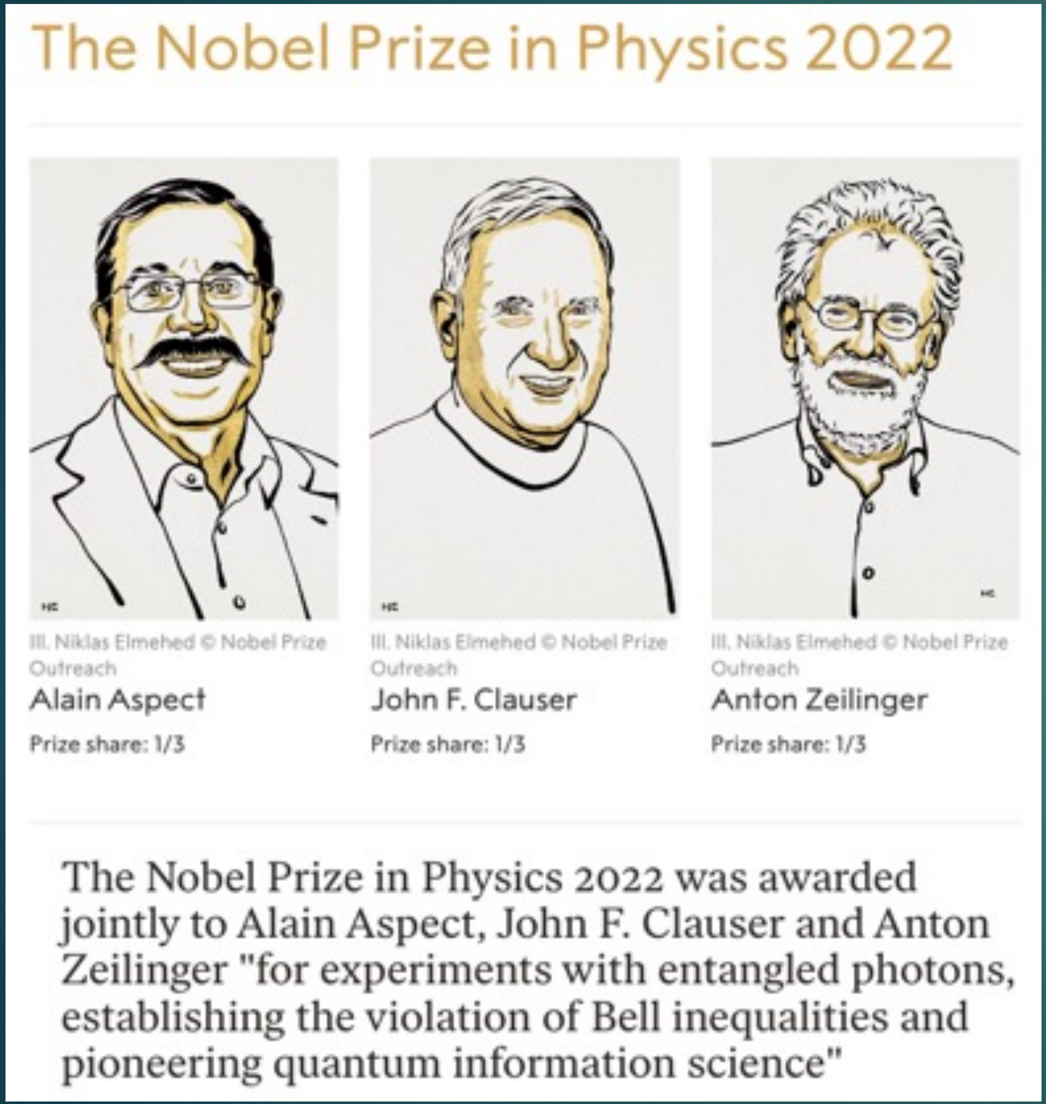
# Before starting…


The Nobel Prize in Physics 2022

Ill. Niklas Elmehed © Nobel Prize Outreach
Alain Aspect
Prize share: 1/3

Ill. Niklas Elmehed © Nobel Prize Outreach
John F. Clauser
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Anton Zeilinger
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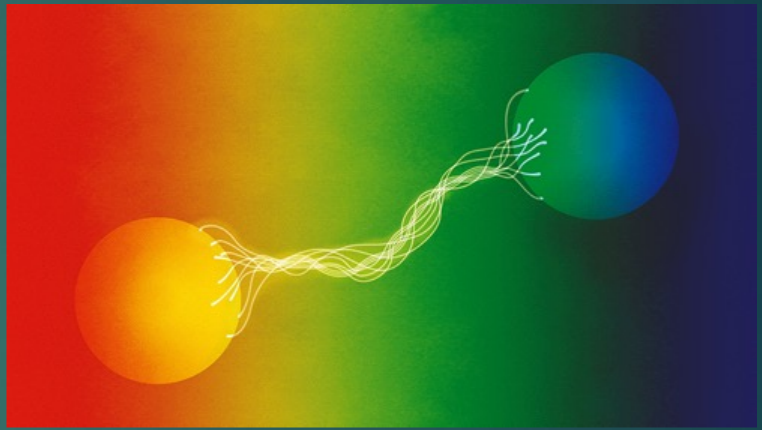
The Nobel Prize in Physics 2022 was awarded jointly to Alain Aspect, John F. Clauser and Anton Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science"



entangled quantum states



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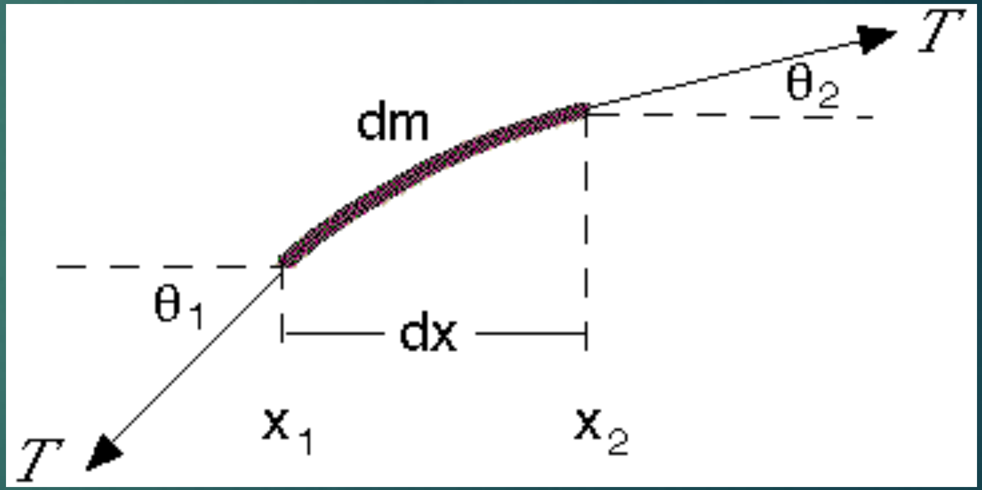
# What we have seen yesterday….

## Harmonic Oscillator


## Wave equation in 1d





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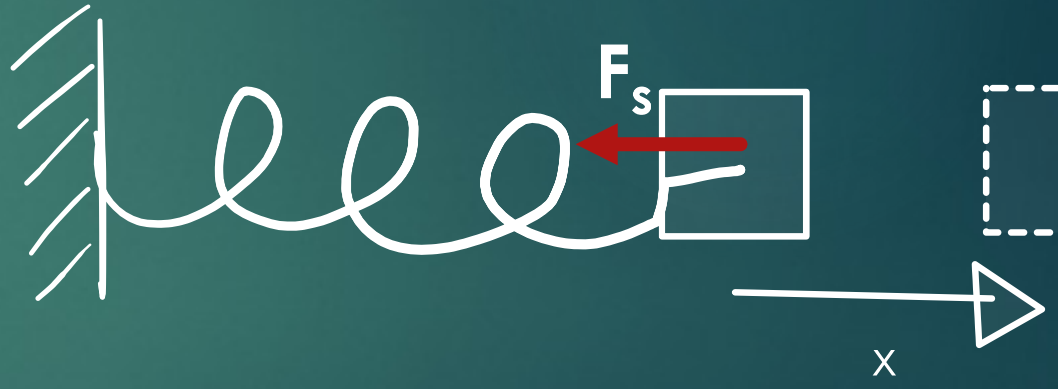
# Harmonic Oscillator

So: $$\frac{d^2x}{dt^2} = -\frac{k}{M}x = -\omega^2 x$$

We define $$\omega^2 = \frac{k}{M}$$




We have solutions like

$$x \propto \sin \omega t$$

With $\omega = \sqrt{k/M}$ angular freq $\omega = 2\pi f$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# Simple harmonic oscillator

It is a physical system described by the equation:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x$$

We have solutions like

$x \propto \sin \omega t$ With $\omega = \sqrt{k/M}$ angular freq $\omega = 2\pi f$

Many examples:

- Spring systems (and not just the simple classical one)
- Electrical resonator
- Acoustic resonators
- Linear oscillators (in general)


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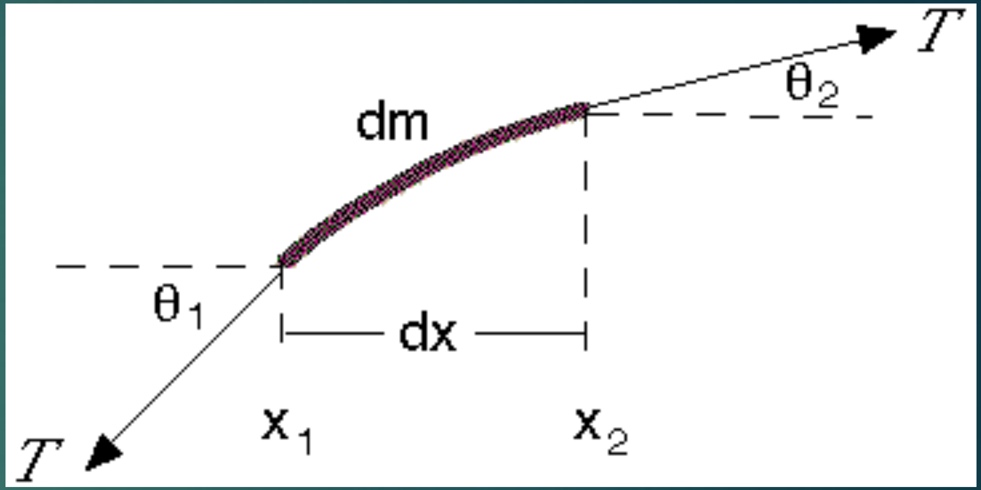
# Classical wave equation

$$F_y = T\left(\left(\frac{\delta y}{\delta x}\right)_2 - \left(\frac{\delta y}{\delta x}\right)_1\right) = \rho\, \mathrm{dx}\, \frac{\delta^2 y}{\delta t^2}$$

Rearranging it:

$$\frac{\delta^2 y}{\delta t^2} = \frac{T}{\rho} \frac{\left(\frac{\delta y}{\delta x}\right)_2 - \left(\frac{\delta y}{\delta x}\right)_1}{dx}$$




Because the last term expresses the change in first derivative between $x_1$ and $x_2$, it correspond to the second derivative…we get:

$$\frac{\delta^2 y}{\delta t^2} = \frac{T}{\rho} \frac{\delta^2 y}{\delta x^2}$$



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# Wave equation solutions

Often, we work with waves travelling at the light velocity c, thus:

$$\frac{\delta^2 y}{\delta t^2} - c^2 \frac{\delta^2 y}{\delta x^2} = 0 \qquad \text{with} \qquad f(x \pm ct)$$

Often, we're interested in waves that are oscillating at one specific angular frequency ω

$$T(t) = \mathrm{e}^{\mathrm{i}\omega t}, \mathrm{e}^{-\mathrm{i}\omega t}, \cos(\pm\omega t), \sin(\pm\omega t)$$

We can write a solution of the wave equation as some product of the variation in x times the variation in t:

$$\phi(x,t) = X(x)T(t)$$



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# Wave equation solutions

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We can write a solution of the wave equation as some product of the variation in x times the variation in t:

$$\phi(x,t) = X(x)T(t)$$

And we will have

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -\omega^2\phi$$

That we can put back into the wave equation:

$$\frac{\delta^2\phi}{\delta t^2} - c^2\frac{\delta^2\phi}{\delta x^2} = 0$$

Obtaining:

$$c^2\frac{\delta^2\phi}{\delta x^2} + \omega^2\phi = 0$$


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# Wave equation solutions

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$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -\omega^2\phi$$

Thus the remaining spatial part equation is:

$$\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} + k^2 X(x) = 0$$ with $$k^2 = \frac{\omega^2}{c^2}$$


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# Wave equation solutions

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Helmholtz wave equation

This is working if we have only one frequency (monochromatic wave).
A simple version of the general expression for a sine wave travelling in the positive x direction is:

$$\phi = A \sin(kx - \omega t)$$


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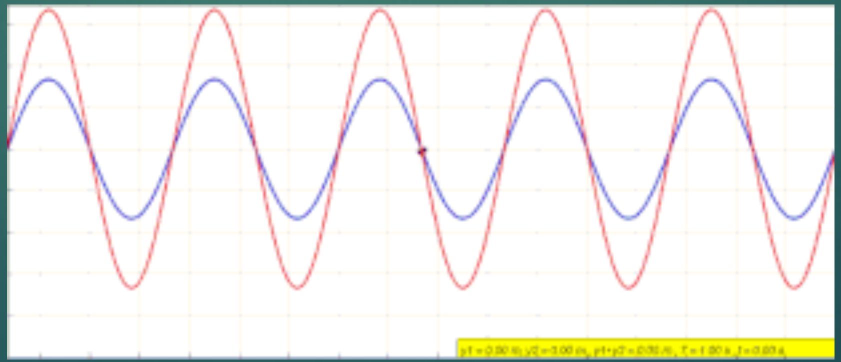
# Standing waves

A combination of waves travelling one to the right and one to the left is:

$$\phi(\mathrm{x},\mathrm{t}) = \sin(kx - \omega t) + \sin(kx + \omega t) \equiv 2\cos(\omega t)\sin(kx)$$

This gives a standing waves, because it is always the same shape in space.

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# Standing waves

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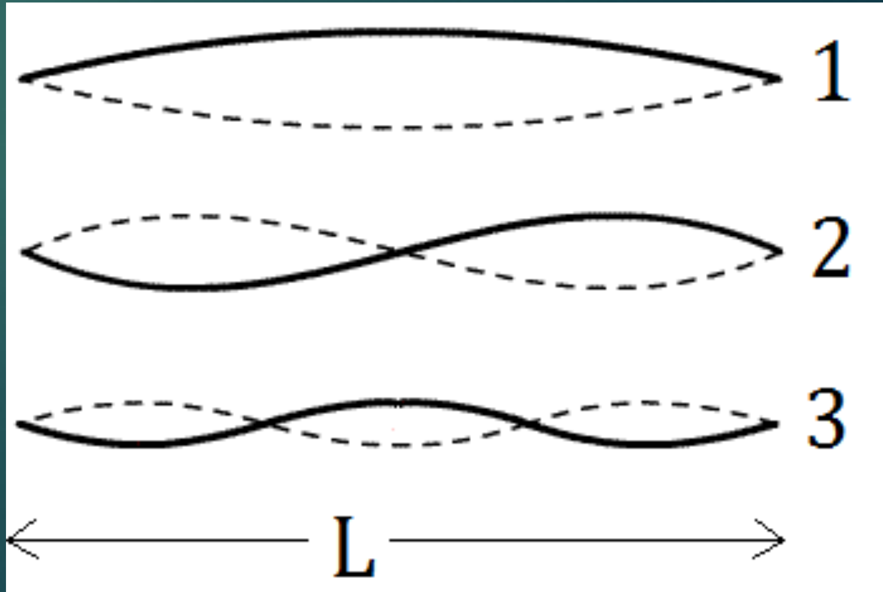
$$\phi(\mathrm{x},\mathrm{t}) = \sin(kx - \omega t) + \sin(kx + \omega t)$$
$$\equiv 2\cos(\omega t)\sin(kx)$$

If we impose the boundary conditions because of fixes ends:

$$k = \frac{n\pi}{L} \qquad \omega = \frac{n\pi c}{L}$$



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# Standing waves

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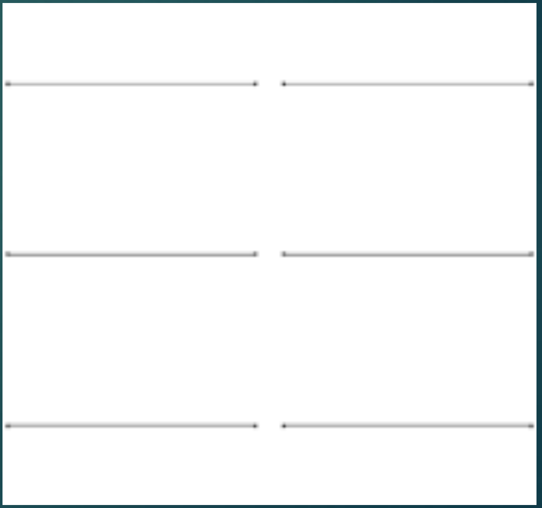

$$k = \frac{n\pi}{L} \qquad \omega = \frac{n\pi c}{L}$$


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# Wave equation in 3D

Generalizing to 3D, the wave equation is:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\delta^2\phi}{\delta t^2} - c^2 = 0$$

With:

$$\nabla^2 \equiv \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$$

Or:

$$\nabla^2 \equiv \nabla \cdot \nabla = \left[\frac{\delta V}{\delta x}\mathbf{i} + \frac{\delta V}{\delta y}\mathbf{j} + \frac{\delta V}{\delta z}\mathbf{k}\right] \cdot \left[\frac{\delta V}{\delta x}\mathbf{i} + \frac{\delta V}{\delta y}\mathbf{j} + \frac{\delta V}{\delta z}\mathbf{k}\right]$$

Plane waves are solutions of the 3d wave equation…

# Wavefunction (mathematical description)





We can imagine that a wave is propagating as a "disturbance of the medium..."

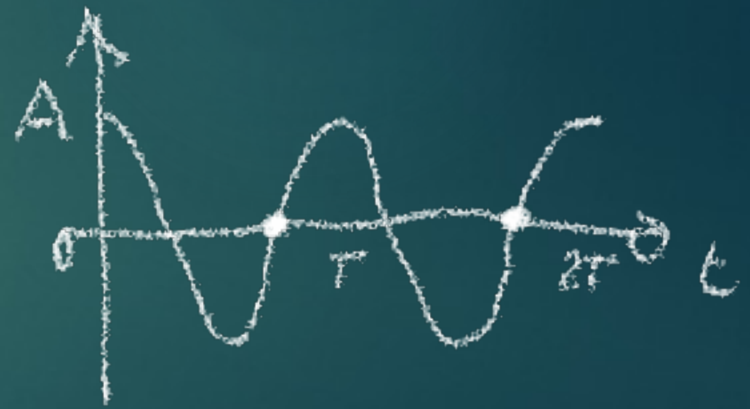
water wave air wave earth wave

# Wavefunction (mathematical description)

We can imagine that a wave is propagating as a "disturbance of the medium..."

Mathematically, in one dimension, a wave is generally represented in terms of a wavefunction:

$$\psi(x,t) = A\cos(k\,x - \omega\,t + \varphi)$$




This represents a wave of amplitude $A$, wavenumber $k$, angular frequency $\omega$, and phase angle $\varphi$, propagating in the positive $x$-direction...

# Plane-waves (mathematical description)

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$$\psi(x,t) = A\cos(k\,x - \omega\,t + \varphi)$$

This type of wave is conventionally termed a one-dimensional plane-wave because the wave maxima, which are located at:

$$\mathrm{k\,x - \omega\,t + \varphi = j\,2\pi}$$

consist of a series of parallel planes, normal to the $x$-axis, that are equally spaced a distance $\lambda=2\pi/k$ apart, and propagate along the positive $x$-axis at the velocity $v=\omega/k$

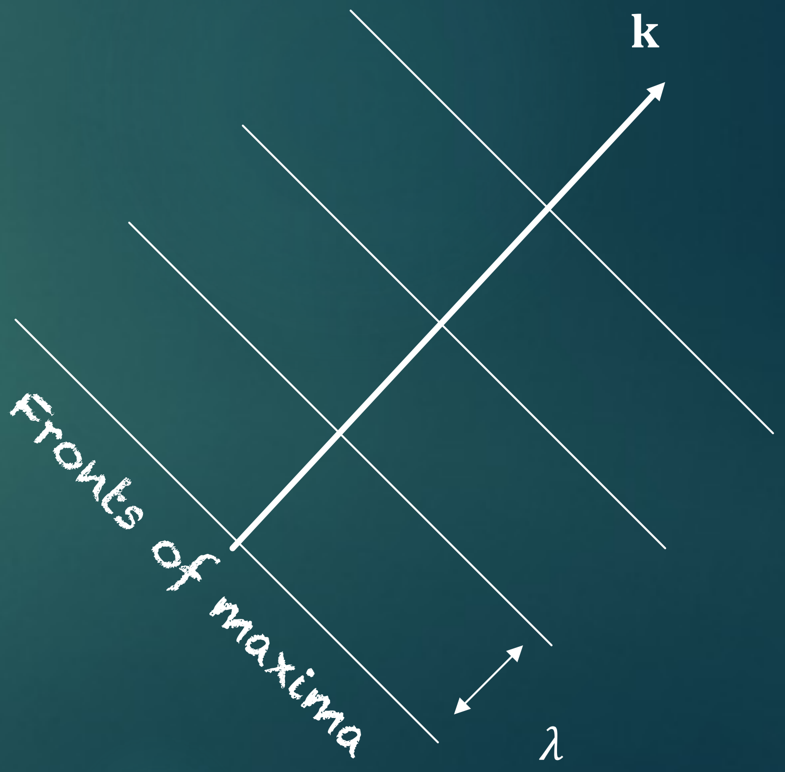
# Plane wave (mathematical description)



In 3D $$\psi(r,t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi)$$

where the constant vector $\mathbf{k} = (k_x, k_y, k_z)$
is called the wavevector

In this case the maxima are located at
$$\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi = j\,2\pi,$$




Thus,
the direction of the wavevector **k** specifies
the wave propagation direction,
its magnitude determines the wavenumber,
$k$, and the wavelength $\lambda=2\pi/k$

# Wavefunction (complex function)

$$\psi(r,t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi)$$

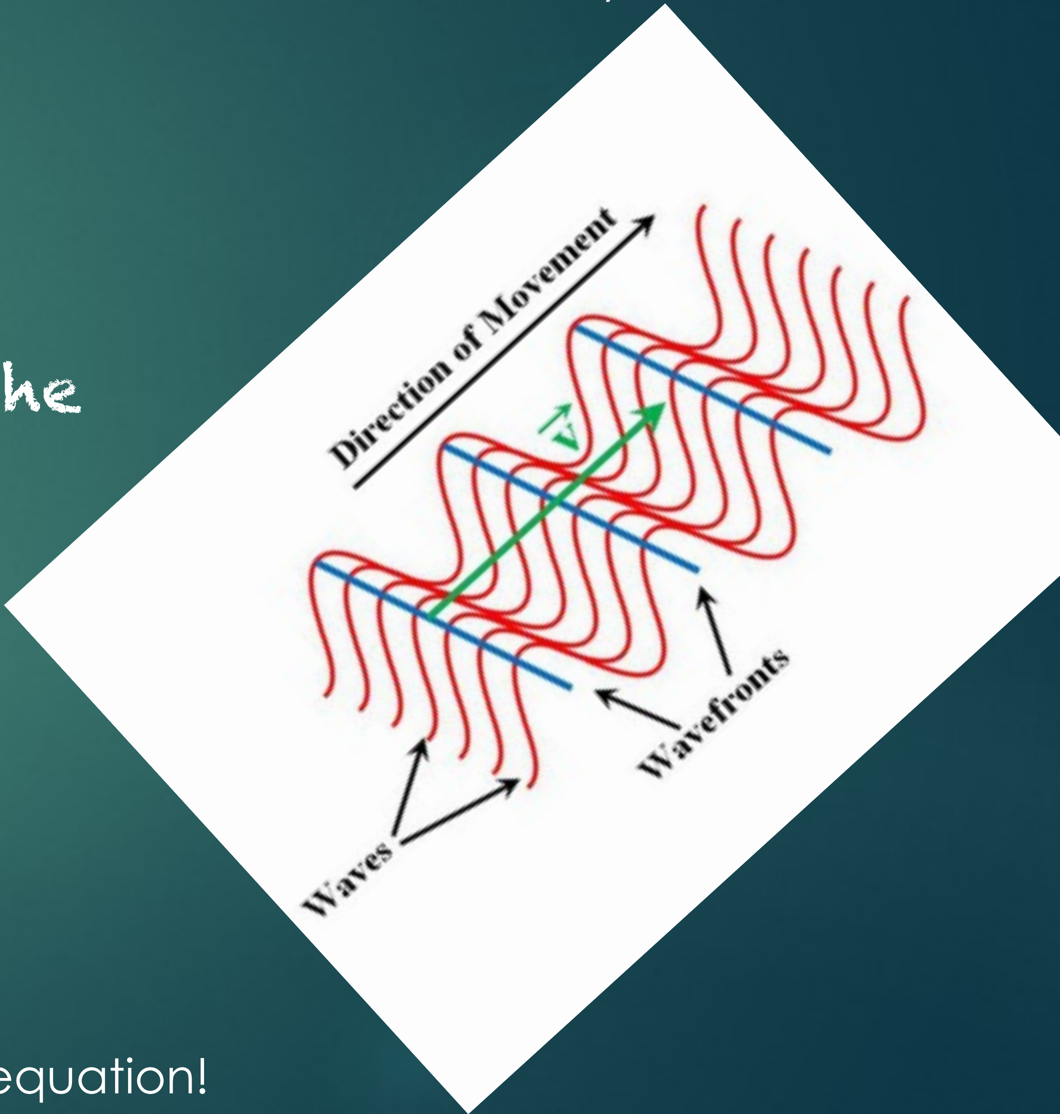
*usually omitted!

By exploiting Euler's Th.

$$e^{i\phi} \equiv \cos\phi + i\sin\phi$$

One can rewrite the wavefunction as the real part* of a complex number:

$$\psi(r,t) = \psi_0\, e^{i(k\cdot r - \omega t)}$$




with $\psi_0$ a complex constant
and $\lambda = 2\pi/k$

$$\psi_0 = A\,e^{i\varphi}$$

Exercise: verify that this wfc is a solution of the 3D wave equation!

# Wavefunction (Fourier Transform)

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One can rewrite the wavefunction as the real part of a complex number:

$$\psi(r,t) = \psi_0\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

Can I write $\psi(r,t)$ as a linear combination of plane-waves of different wavenumbers?

There is a useful mathematical theorem, known as Fourier's theorem , which states that if:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(k)\, e^{ikx}\, dk$$

Then

$$\bar{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

Where $\bar{f}(k)$ is the *Fourier transform* of the function $f(x)$.
We can use Fourier's theorem to find the linear combination of plane-waves forming and producing $\psi(x)$

# Wavefunction (Fourier transform)

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One can rewrite the wavefunction as the real part of a complex number:

$$\psi(\boldsymbol{r},t) = \psi_0\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

Can I write $\psi(r,t)$ as a linear combination of combination of plane-waves of different wavenumbers?

We can use Fourier's theorem to find the k-space $\bar{\psi}(k)$ function that generates any given x-space wavefunction $\psi(x)$ at a given time:

$$\psi(x,t) = \int_{-\infty}^{\infty} \bar{\psi}(k)\, e^{i(kx-\omega t)}\, dk$$