# Measurement and expectation values

# Schrödinger's Equation

Time-independent S.E.:

$$\hat{H}\psi_i = E_i \ \psi_i$$

As seen for the $|\chi_n\rangle$ eigenvectors
$\psi_i$ must be orthonormal:

$$\int_{-\infty}^{\infty} \psi_i^* \ \psi_j \, dx = \delta_{ij}$$

(in 1D)

If we have discrete energy values, we can express a general w.f. as a linear combination of eigenstates:

$$\Psi(x,t) = \sum_i c_i \ \psi_i(x) \, e^{-i E_i t/\hbar}$$

# Wavefunction expansion





$$\Psi(x,t) = \sum_i c_i\ \psi_i(x)\, e^{-i E_i t/\hbar}$$

with:

$$c_i = \int_{-\infty}^{\infty} \psi_i^*(x)\ \psi(x,0)\, dx$$

The normalization integral requires:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx = \int_{-\infty}^{\infty} \left[\sum_i c_i^*\ \psi_i^*\ e^{i E_i t/\hbar}\right] \bullet \left[\sum_j c_j\ \psi_j e^{-i E_j t/\hbar}\right] dx = 1$$

# Wavefunction expansion



The normalization integral requires:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx = \int_{-\infty}^{\infty} \left[\sum_i c_i^* \, \psi_i^* \, \mathrm{e}^{\mathrm{i} E_i t/\hbar}\right] \bullet \left[\sum_j c_j \, \psi_j \mathrm{e}^{-\mathrm{i} E_j t/\hbar}\right] dx = 1$$

The only terms that can survive are for i=j and the result for the integration will be $|c_i|^2$

Thus, we will have:

$$\sum_i |c_i|^2 = 1$$

And this sum must be linked to a probability!

# Probability



$$\psi(x,t) = \sum_i c_i \, \psi_i(x) \, e^{-i E_i t/\hbar}$$

with:

$$c_i = \int_{-\infty}^{\infty} \psi_i^*(x) \, \psi(x,0) \, dx$$

$|c_i|^2$ is the probability that a measurement of the energy will yield the eigenvalue $E_i$

# Measurement





In quantum mechanics, when we make a measurement of some quantity such as energy, the system collapses into an eigenstate of the energy, with probability:

$$P_i = |c_i|^2$$

Then the system will stay in the corresponding energy eigenstate (being a stationary state).

# Expectation value





Suppose we measure the energy of our system in such an experiment, but we repeat the experiment many times, and get a statistical distribution of the results.

Considering the probabilities $P_i$ the average value of energy E that we would measure would be:

$$\langle E \rangle = \sum_i E_i P_i = \sum_i E_i |c_i|^2$$

This quantity is called
"Expectation value" of the energy.

# Expectation value harmonic oscillator





Let's check the expectation value of the energy for the coherent state.
The general formula can be adapted

$$\langle E \rangle = \sum_i E_i P_i = \sum_i E_i |c_i|^2$$

Considering the expansion coefficients in the case of coherent state:

$$c_{Nn} = \sqrt{\frac{N^n e^{-N}}{n!}}$$

Thus:

$$\langle E \rangle = \sum_n E_n |c_{Nn}|^2 = \sum_n \left(n + \frac{1}{2}\right) \hbar\omega |c_{Nn}|^2 = \sum_n \mathrm{n}\hbar\omega |c_{Nn}|^2 + \frac{1}{2}\hbar\omega |c_{Nn}|^2$$

# Expectation value harmonic oscillator





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$$= \hbar\omega \underline{\left[\sum_n \mathrm{n}\,|c_{Nn}|^2\right]} + \frac{1}{2}\hbar\omega = \hbar\omega \underline{\left[\sum_n \mathrm{n}\frac{N^n e^{-N}}{n!}\right]} + \frac{1}{2}\hbar\omega = \left(N + \frac{1}{2}\right)\hbar\omega$$

In square brackets we have the probability of getting n times its probability, the summation will then give the average N (which is not necessarily an integer value)

# Remember slide: Coherent state

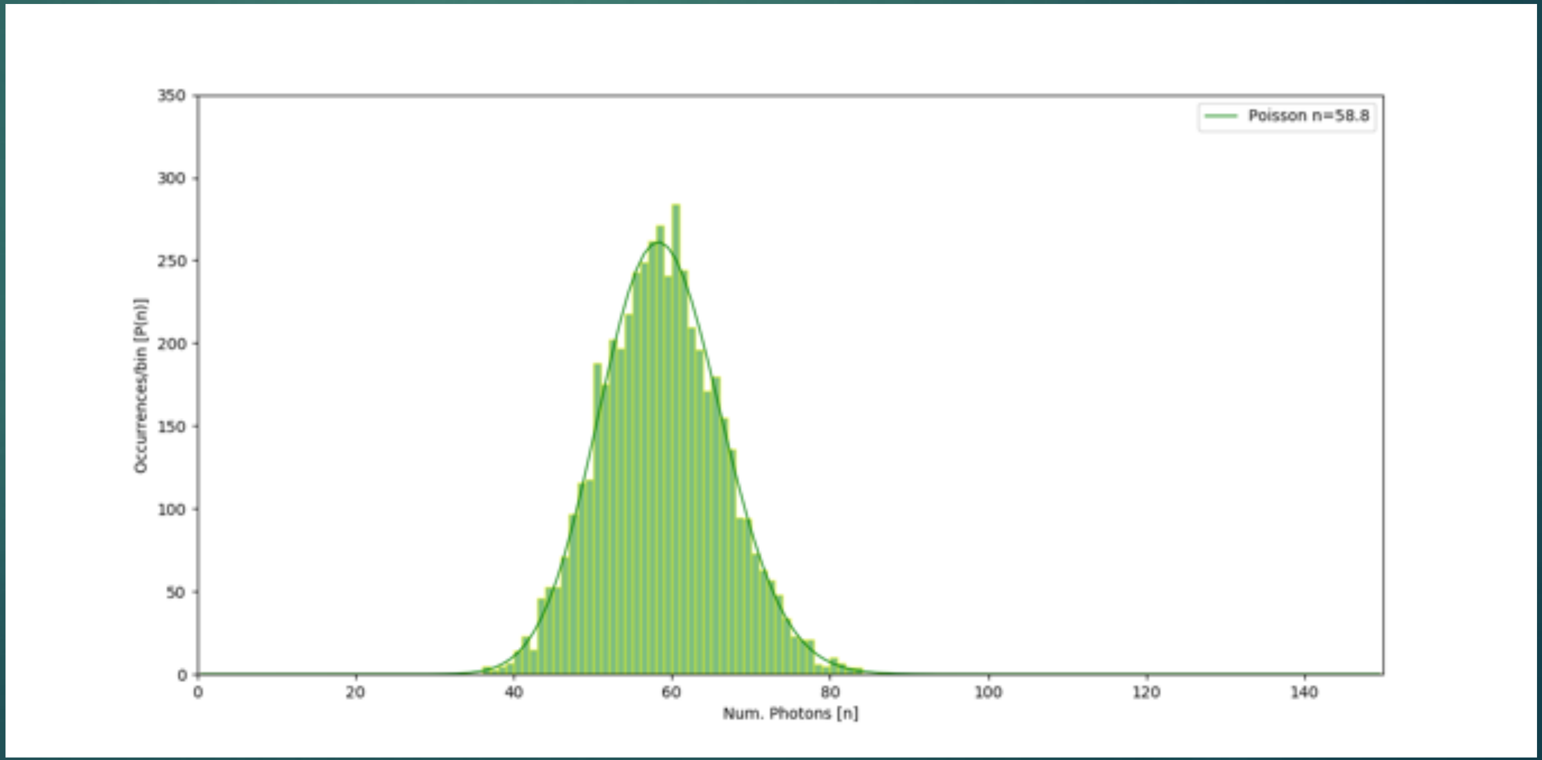


The modulus squared of the expansion coeffiecietnts

$$|C_{Nn}|^2 = \frac{N^n\ e^{-N}}{n!}$$

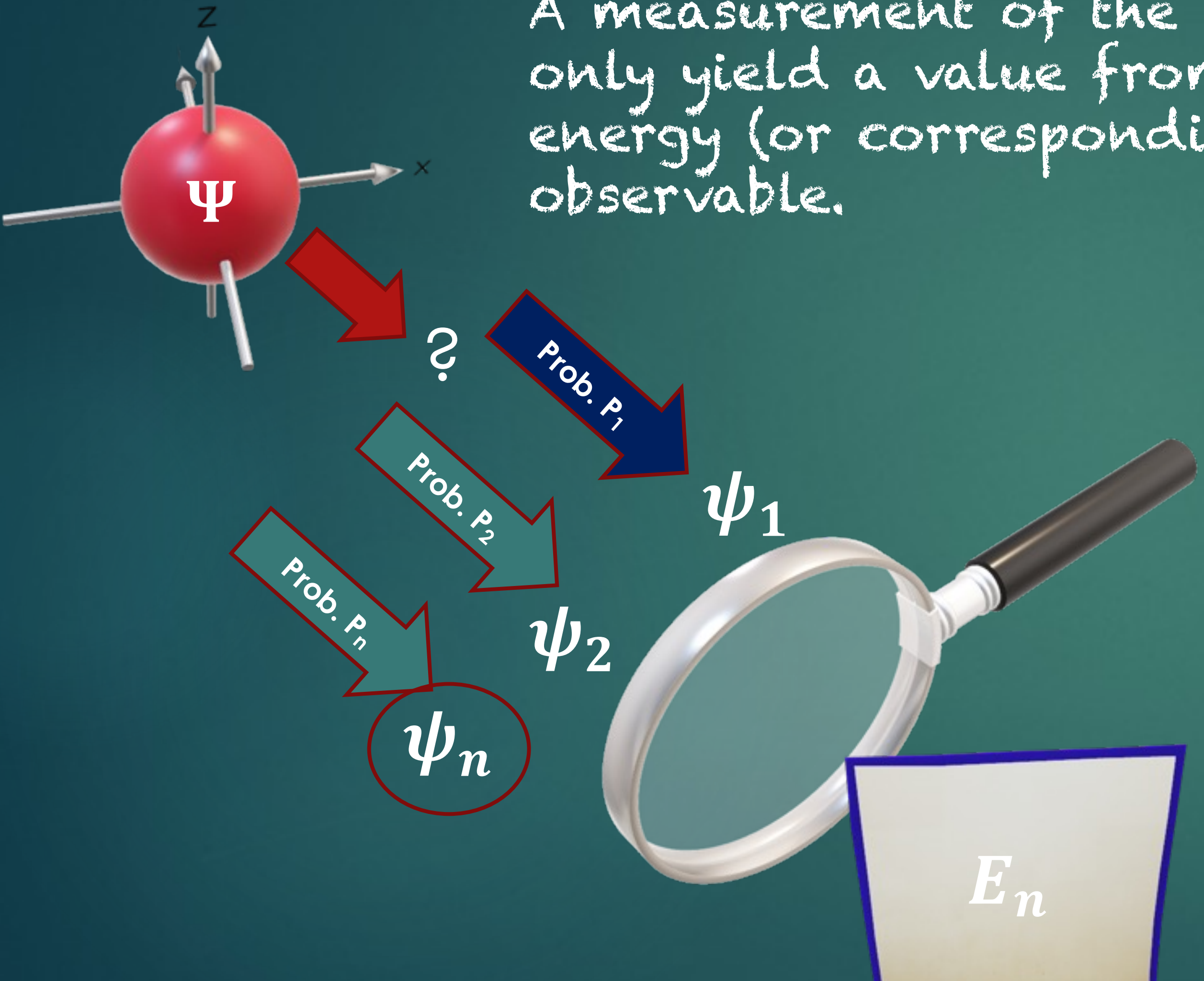
Is the Poisson distribution with mean value N

and standard deviation $\sqrt{N}$



# Statistical interpretation

A measurement of the energy (or any other observable) can only yield a value from the set of the eigenvalues of the energy (or corresponding operator) representing the measured observable.




$E_1, E_2, E_n$ Eigenvalues

$\psi_1, \psi_2, \psi_n$ Eigenvectors

$$\Psi(x,t) = C_1\psi_1 + C_2\psi_2 + C_3\psi_3$$

# Statistical interpretation

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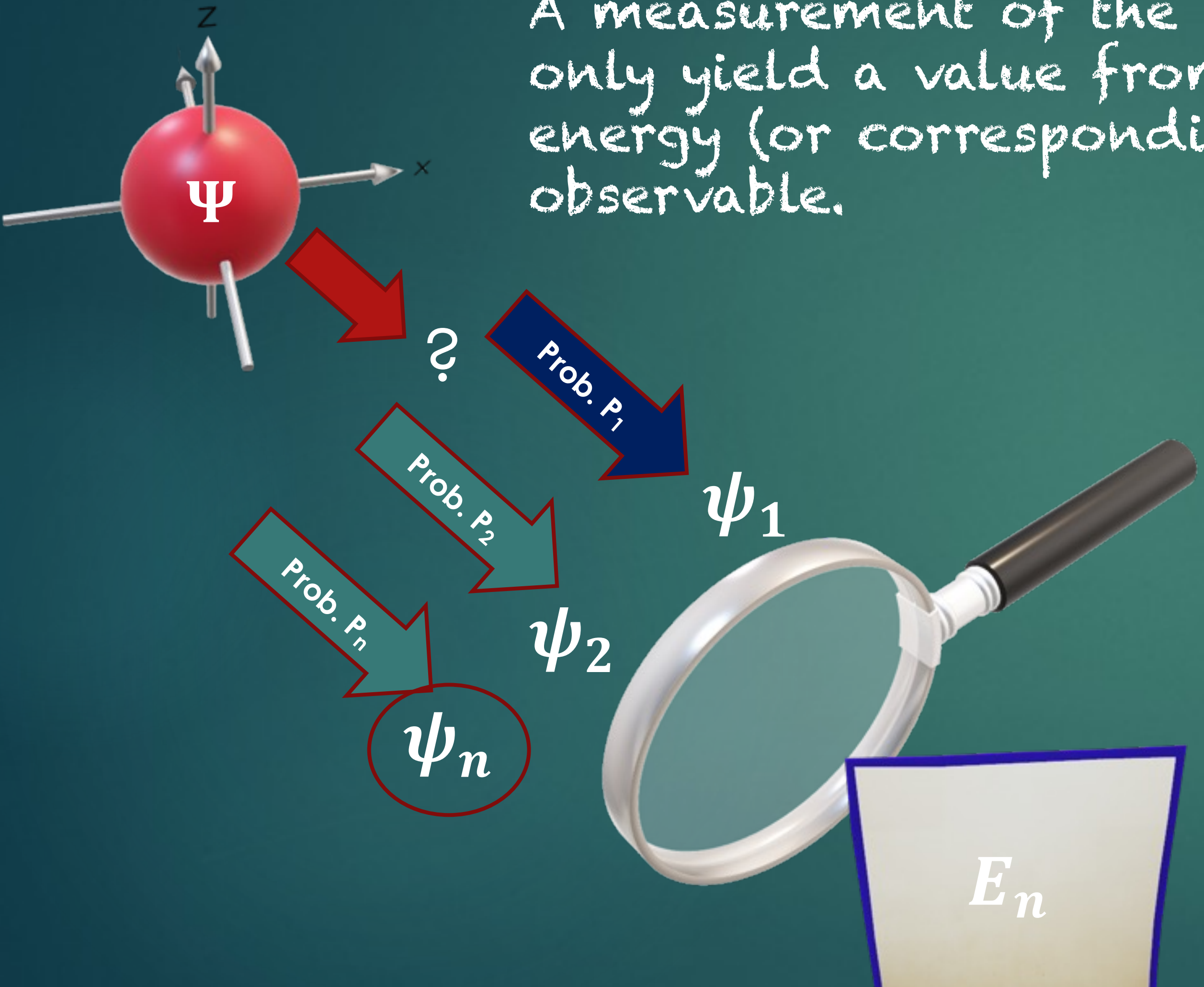
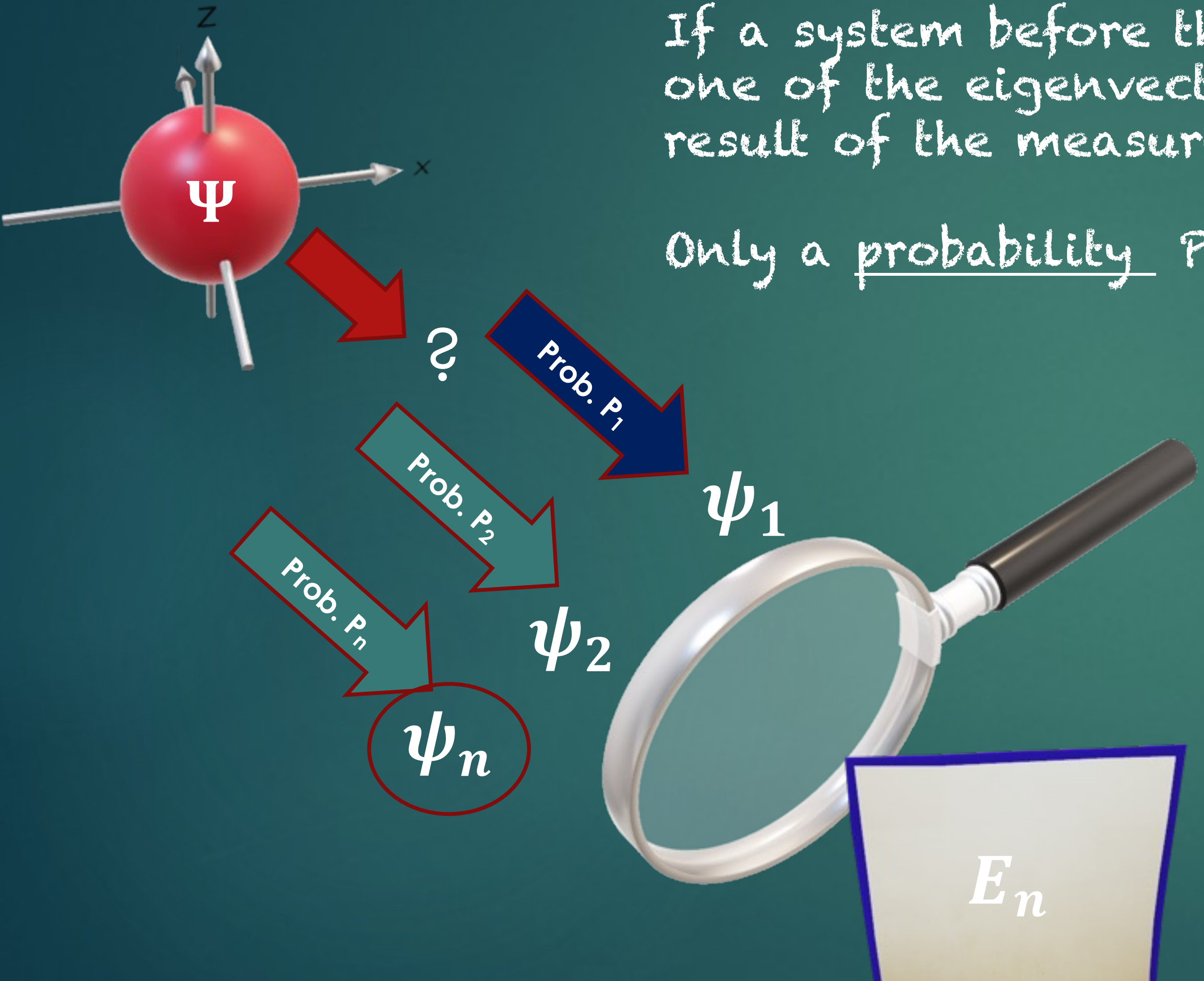



$$\Psi(x,t) = C_1\psi_1 + C_2\psi_2 + C_3\psi_3$$

OBSERVABLE: any quantity whose numerical value can be experimentally measured.
Actually, the list of Observables for a quantum system is not very different than that characterizing classical mechanics: coordinates, momentums, energies, angular momentums, etc

# Statistical interpretation




If a system before the measurement is not in a state described by one of the eigenvectors, but it is in a superposition of states, the result of the measurement cannot be predicted a priori.

Only a probability $P_n$ of a particular outcome can be known

$$P_n = |c_n|^2$$

$$c_n = \int_{-\infty}^{\infty} \psi_n^*(x)\ \Psi(x, t_0)\, dx$$

**Ψ** Is the state of the system before the mesurement

$$\Psi(x,t) = C_1\psi_1 + C_2\psi_2 + C_3\psi_3$$

# The Stern Gerlach experiment

Note: Inhomogeneous magnetic field

We will get a net force along z

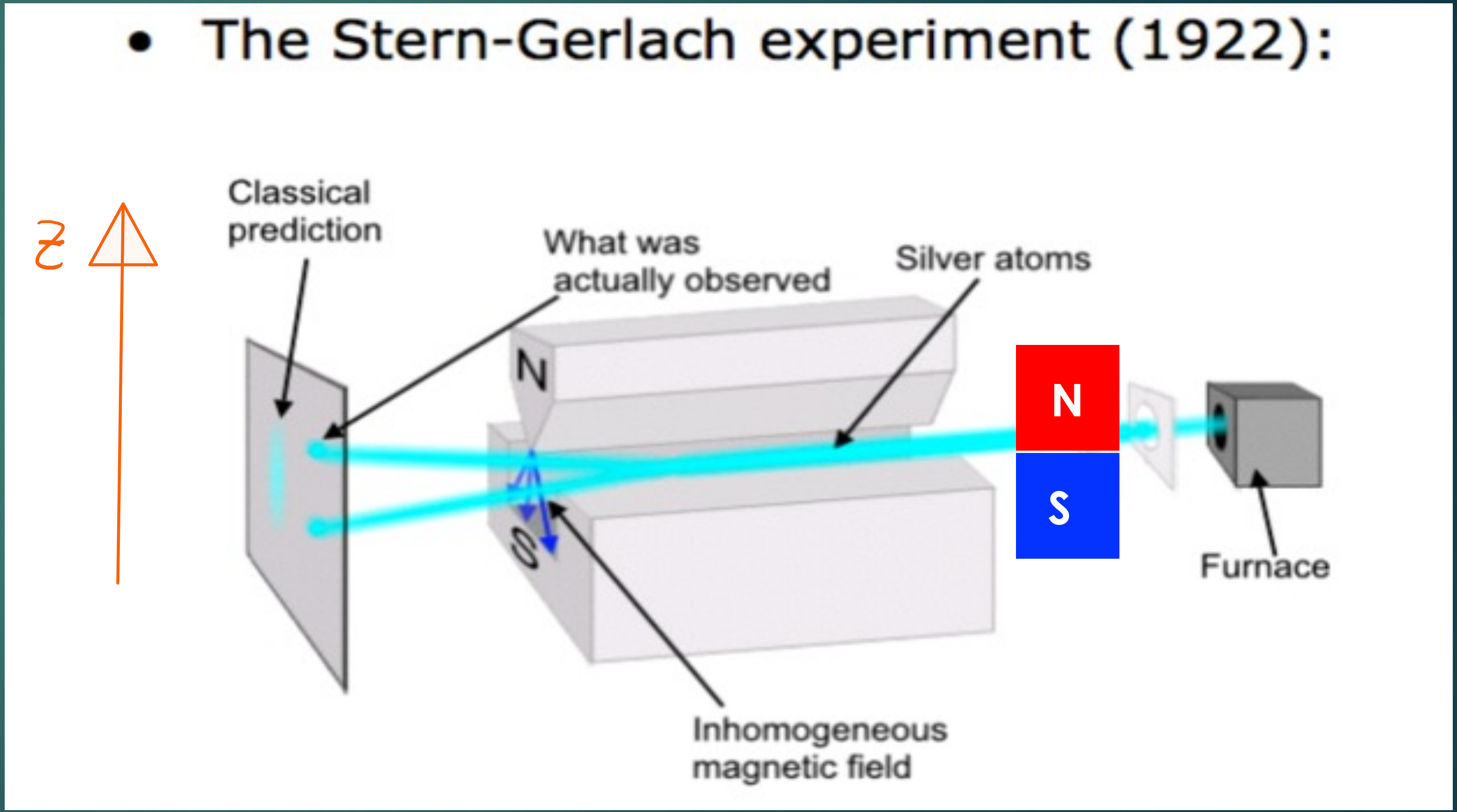
When we make a measurement, we collapse the state of the system

$$\Psi(x,t) = C_1\psi_{1/2} + C_2\psi_{-1/2}$$

into one of the eigen states:

$\psi_{1/2}$ or $\psi_{-1/2}$

For
Spin up ($m_s$=1/2)
Spin Down ($m_s$= =1/2)



# Operators



$$\Psi(r,t) = \psi_i(r)\, e^{-i E_i t/\hbar}$$

$\psi_i$ satisfies the time-independent S.E.:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathrm{r})\right]\psi_i(r) = E_i\ \psi_i(r)$$

which can be also written as:

$$\hat{H}\psi_i = E_i\ \psi_i$$

With:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathrm{r},t)$$

# Operators and expectation values



t.i.S.E.:

$$\hat{H}\psi(r) = E\,\psi(r)$$

With: $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathrm{r}, t)$

But we wrote the t.d.S.E. as :

$$i\hbar\frac{\delta\Psi(\mathrm{r}, t)}{dt} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathrm{r}, t)\right]\Psi(\mathrm{r}, t)$$

Thus, we can rewrite it:

$$\hat{H}\;\Psi(\mathrm{r}, t) = i\hbar\frac{d\,\Psi(\mathrm{r}, t)}{dt}$$

# Operators





$$\hat{H}\psi(r) = E\,\psi(r)$$

$\hat{H}$ is nor a number or a function, but it is an operator.

It is an entity that when applied to a function, transforms that function into another.

Conceptually, it is just like a derivative operator $\frac{d}{dx}$

The "hat" is indicating that H is not a function or a constant, but an operator.

# Operators



$$\hat{H}\psi(r) = E\,\psi(r)$$

$\hat{H}$ is nor a number or a function, but it is an operator.

The $\hat{H}$ operator is called Hamiltonian
and like the classical Hamiltonian
In non-relativistic quantum mechanics
The Hamiltonian operator is related to the total energy of the system

# Operators and expectation values



We can use $\hat{H}$ to write another expression of the expectation value of the energy:

$$\langle E \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\,\hat{H}\,\Psi(x,t)\mathrm{d}x$$

In fact, if we introduce into the integral

$$\Psi(x,t) = \sum_i c_i\, \psi_i(x)\, \mathrm{e}^{-\mathrm{i}\,E_i\,t/\hbar}$$

We will get:

$$\langle E \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\,\hat{H}\,\Psi(x,t)\mathrm{d}x = \sum_i E_i |c_i|^2$$

Thus, the expectation value!

# Operators and expectation values





We can use $\hat{H}$ to write another expression of the expectation value of the energy:

$$\langle E\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\,\hat{H}\,\Psi(x,t)\mathrm{dx}$$

In general

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\,f(x)\,\Psi(x,t)\mathrm{dx}$$

for any operator which is a function of x

Thus, to get the expectation value of an observable, we can use the corresponding operator, solving the integral above but not necessarily the S.E.

# Quantum States

- We are dealing with "mechanical states"

  The term "state" in physics is used with different meanings, i.e.:

  States of matter (liquid, solid, etc.)

  Thermodynamic states (e.g. the state for a gas is identified by its P,V,T)

- The exact meaning of "mechanical states" depends upon the framework in which we are working:

  classical or quantum physics?

# Mechanical States in classical phys.



- In classical physics:
  a mechanical state is described by specifying its
  **position** and velocity (or **momentum** p)

- If I know the initial coordinates and velocities of any object in the universe, we can determine (or predict) its future position with any accuracy limited only by the accuracy of the available instrument (experimental and computational).

- The evolution of classical states is described by the laws of classical physics

# Quantum States

- OBSERVABLE: any quantity whose numerical value can be experimentally measured.
  Actually, the list of Observables for a quantum system is not very different than that characterizing classical mechanics: coordinates, momentums, energies, angular momentums, etc
- But in QM not all observables can be measured within the same set of experiments: e.g. Heisenberg's uncertainty principle

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

# Quantum States

- But in QM not all observables can be measured within the same set of experiments: e.g. Heisenberg's uncertainty principle
- Still, one can consider observables that can be measured with certainty, called **mutually consistent**.
- The largest set of such observables is called a complete set of **mutually consistent observables**.

# Quantum States: Dirac notation

- But in QM not all observables can be measured within the same set of experiments: e.g. Heisenberg's uncertainty principle

- Still, one can consider observables that can be measured with certainty, called **mutually consistent**.

- The largest set of such observables is called a complete set of **mutually consistent observables**.

$$\{q^{(i)}\}$$

$q^{(i)}$: i'th observable

# Quantum States: Dirac notation

- The largest set of such observables is called a complete set of **mutually consistent observables:** $\{\mathrm{q}^{(\mathrm{i})}\}$
- $\{q_k^{(i)}\}$ represent a *k*'th value of *i*'th observable
- According to the Dirac's notation the quantum state can be represented by:

$$\left| q_k^{(1)}, q_m^{(2)} \cdots q_p^{(N)} \right\rangle$$

# Quantum States: Dirac notation

$$\left|q_k^{(1)}, q_m^{(2)} \cdots q_p^{(N)}\right\rangle$$

Observables (example)

$\mathrm{q}^{(1)}: x - position,\ \mathrm{q}^{(2)}: y - position,\ \mathrm{q}^{(3)}: z - position$

Results of measurements

$q_k^{(1)}$ =3 Å, $q_m^{(2)}$ =1 Å, $q_n^{(3)}$ =2 Å

Note that the $\mathrm{q}^{(\mathrm{i})}$ observables can change continiusly (see the example above) or can have a discrete spectrum.

# Quantum States: Dirac notation

$$\left|q_k^{(1)}, q_m^{(2)} \cdots q_p^{(N)}\right\rangle$$

Observables (example)

$\mathrm{q}^{(1)}: x - position$, $\mathrm{q}^{(2)}: y - position$, $\mathrm{q}^{(3)}: z - position$

Results of measurements

$q_k^{(1)}$ =3 Å, $q_m^{(2)}$ =1 Å, $q_p^{(3)}$ =2 Å $|0,0,0\rangle$

$q_q^{(1)}$ =-3 Å, $q_r^{(2)}$ =-1 Å, $q_s^{(3)}$ =-2 Å $|1,1,1\rangle$

Assuming a discrete spectrum and in the hypotesys that the system can be in just two states.

# Quantum States: Dirac notation



- But it is not always that simple: there are states in QM in which the **observables may have not definite values**, because we cannot predict with absolute certainty the output of the measurement. (remember the complementary principle: it is the measurement itself to “force” the observable to have a determined value)

- In general, repeating measurements on the same system (provided it is returned back to its initial state) **returns different outcomes after each measurement**.

- Still, states with uncertain outcomes of a measurement **can be described as a linear superposition of the simple states.**

# Linear vector space

$$|SUP\rangle = a_1|q_1\rangle + a_2|q_2\rangle$$

All quantum states can be represented by special objects belonging to a certain "space" and have some properties.

1. There is a null object $|0\rangle$ such that $\boldsymbol{|q\rangle + |0\rangle = |q\rangle}$ and $\boldsymbol{0 \cdot |q\rangle = |0\rangle}$

2. Distributive property between these objects
   $\boldsymbol{a(|q_1\rangle + |q_2\rangle) = a|q_1\rangle + a|q_2\rangle}$

3. Associative properties w.r.t. the complex numbers and their multiplication
   $$\boldsymbol{a_1|q_i\rangle + a_2|q_i\rangle = (a_1 + a_2)|q_i\rangle}$$
   $\boldsymbol{a_1(a_2\,|q_i\rangle) = (a_1\,a_2)\,|q_i\rangle}$

# Linear vector space

We know from basic physics courses that examples of objects satisfying all the properties mentioned so far are **three-dimensional vectors** (e.g. displacement and velocity vectors).

- More in general, abstract objects satisfying these properties are called **vectors** belonging to **linear space vector** and are generally represent as:

$$|\alpha\rangle\ ,\ |\beta\rangle\ ,\ |\gamma\rangle$$

- We will use what you (may) know from linear algebra course to work with these abstract objects.

# Linearly independent vectors

- A **linear combination** of the vectors $|\alpha\rangle$ , $|\beta\rangle$, $|\gamma\rangle$ , ...:

  $$a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle + \cdots$$

- A vector $|\gamma\rangle$ is **<u>linearly independent</u>** of the set $|\alpha\rangle$ , $|\beta\rangle$, $|\gamma\rangle$ , ... if it cannot be written as a linear combination of them. (example: **x**, **y**, **z**, unit vectors in three dimensions)

- A set of vectors is **<u>linearly independent</u>** if each one is linearly independent of all the rest

- A set of vectors is said to **<u>span</u>** the space if every vector can be written as a linear combination of the members of this set. In other words, this set of linearly independent vectors is also **<u>complete</u>** because adding any other distinct vector to the set makes it linearly dependent.

- A set of linearly independent vectors that spans the space is called a **<u>basis</u>** and the number of vectors in any basis is called the **dimension of the space**.

# Linearly independent vectors





- A set of vectors is said to span the space if every vector can be written as a linear combination of the members of this set. In other words, this set of linearly independent vectors is also complete because adding any other distinct vector to the set makes it linearly dependent.

# Linear vector space





- With respect to a prescribed basis $|\boldsymbol{e_1}\rangle$ , $|\boldsymbol{e_2}\rangle$ , $|\boldsymbol{e_3}\rangle$ ,…, $|\boldsymbol{e_n}\rangle$

We can state:

$$|\alpha\rangle \; = a_1|\boldsymbol{e_1}\rangle \; + a_2|\boldsymbol{e_2}\rangle \; + a_3|\boldsymbol{e_3}\rangle \; + \ldots + \, a_n\,|\boldsymbol{e_n}\rangle$$

$$|\alpha\rangle \leftrightarrow \; (a_1, a_2, a_3, \ldots, a_n)$$

$$|\alpha\rangle + |\beta\rangle \; \leftrightarrow \; (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + bn)$$

$$\mathrm{c}|\alpha\rangle \leftrightarrow \; (ca_1, ca_2, ca_3, \ldots, \mathrm{c}\; a_n)$$

$$|0\rangle \leftrightarrow \; (0, 0, 0, \ldots, 0)$$

$$-\,|\alpha\rangle \leftrightarrow \; (-a_1, -a_2, -a_3, \ldots, -a_n)$$

# Inner products

- The dot product, that we know in three-dimensional space vector, generalize in n-dimensional space vector as **inner product (in the Dirac notation bra-ket)**

$$\langle \alpha | \beta \rangle$$

- **With the following properties:**

1. $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^*$

2. $\langle \alpha | \alpha \rangle \geq \emptyset \quad \sqrt{\langle \alpha | \alpha \rangle} = \|\alpha\| \geq 0$ $\quad\|\alpha\|$ **NORM of** $|\alpha\rangle$

3. $\langle \alpha \quad (b|\beta\rangle + c\,|\gamma\rangle) = b\langle \alpha | \beta \rangle + \text{c}\,\langle \alpha | \gamma \rangle$

# Orhonormal set

- Normalized vector:

$$||\alpha|| = 1$$

- Orthogonal vectors:

$$\langle\alpha|\beta\rangle = 0$$

Normalization:

$$\frac{|\alpha\rangle}{||\alpha||}$$

- Collection of mutually orthogonal normalized vectors is called **ORTHONORMAL SET**

$$\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$$

# Inner products

- **Bra-ket**

$$\langle \alpha | \beta \rangle$$

**In term of vectors:**

$$|\alpha\rangle = a_1|\boldsymbol{e_1}\rangle + a_2|\boldsymbol{e_2}\rangle + a_3|\boldsymbol{e_3}\rangle + \ldots + a_n|\boldsymbol{e_n}\rangle$$

$$|\beta\rangle = b_1|\boldsymbol{e_1}\rangle + b_2|\boldsymbol{e_2}\rangle + b_3|\boldsymbol{e_3}\rangle + \ldots + b_n|\boldsymbol{e_n}\rangle$$

$$|\alpha\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ \\ a_n \end{bmatrix}$$ and $$|\beta\rangle = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \\ bn \end{bmatrix}$$

# Inner products

- **Bra-ket**

$$\langle \alpha | \beta \rangle = [a_1^* \quad a_2^* \ldots \quad a_n^*] \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \\ bn \end{bmatrix} = \sum_{i=1}^{N} a_i^* b_i$$

**Complex Conjugate:**

$$a = x + iy$$

$$a^* = x - iy$$

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# Inner products

- **Bra-ket**

$$\langle \alpha | \beta \rangle = [a_1^* \quad a_2^* \ldots \quad a_n^*] \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \\ bn \end{bmatrix} = \sum_{i=1}^{N} a_i^* b_i$$

**If we choose an orthonormal base:**

$$\langle e_i | e_j \rangle = \delta_{ij} \Rightarrow \langle e_n | e_n \rangle = 1$$

$$\langle \alpha | \beta \rangle = a_1^* b_1 + a_2^* b_2 + \cdots + a_n^* b_n$$

# Inner products

**Thus in term of vectors, ket and bra- :**

$$|\alpha\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \\ \\ a_n \end{bmatrix}$$

$$\langle\alpha| = [a_1^* \; a_2^* \ldots a_n^*]$$

$$|\alpha\rangle^* = \begin{bmatrix} a_1^* \\ a_2^* \\ \\ \\ a_n^* \end{bmatrix}$$

$$\left(|\alpha\rangle^*\right)^T = |\alpha\rangle^\dagger = \langle\boldsymbol{\alpha}|$$

**Hermitian conjugate or Hermitian transpose or adjoint**

# Inner products

- **Bra-ket**

$$\langle\alpha|\beta\rangle = [a_1^* \quad a_2^* \dots \quad a_n^*] \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ \\ bn \end{bmatrix} = \sum_{i=1}^{N} a_i^* b_i$$

**If $\beta = \alpha$**

$$\langle\alpha|\alpha\rangle = \sum_{i=1}^{n} a_n^* a_n = \sum_i^n |a_i|^2 \geq 0$$

**and the component $a_1$ can be found:**

$$a_i = \langle e_1|\alpha\rangle$$

# Hilbert Space

- **Bra-ket**

$$\langle \alpha | \beta \rangle = \sum_{i=1}^{N} a_i^* \cdot b_i$$

- **Norm**

$$\langle \alpha | \alpha \rangle = \sum_{i}^{M} |a_i|^2 \geq 0$$

- **A linear vector space with a defined inner product and a norm is an Hilbert space**

# Functions of coninuous variable

Consider a class of complex functions $\psi(x)$ with $x \in [-\infty, \infty]$ and:

- $\int_{-\infty}^{\infty} |\psi(x)|^2 \,\mathrm{d}x < \infty$ square integrable functions with
- $\|\psi\| = \sqrt{\int_{-\infty}^{\infty} |\psi(x)|^2 \,\mathrm{d}x}$ defining the norm

- The linear combinations of such functions also belong to the same class (forming a linear vector space)
- The inner product of $\psi(x)\varphi(x)$ is defined as:

$$(\psi, \varphi) = \int_{-\infty}^{\infty} \psi^*(x)\varphi(x)\,\mathrm{d}x$$

**Thus, these functions do form a Hilbert vector space**

- **This is also called the "overlap integral" of $\psi$ and $\varphi$**

# Functions of coninuous variable

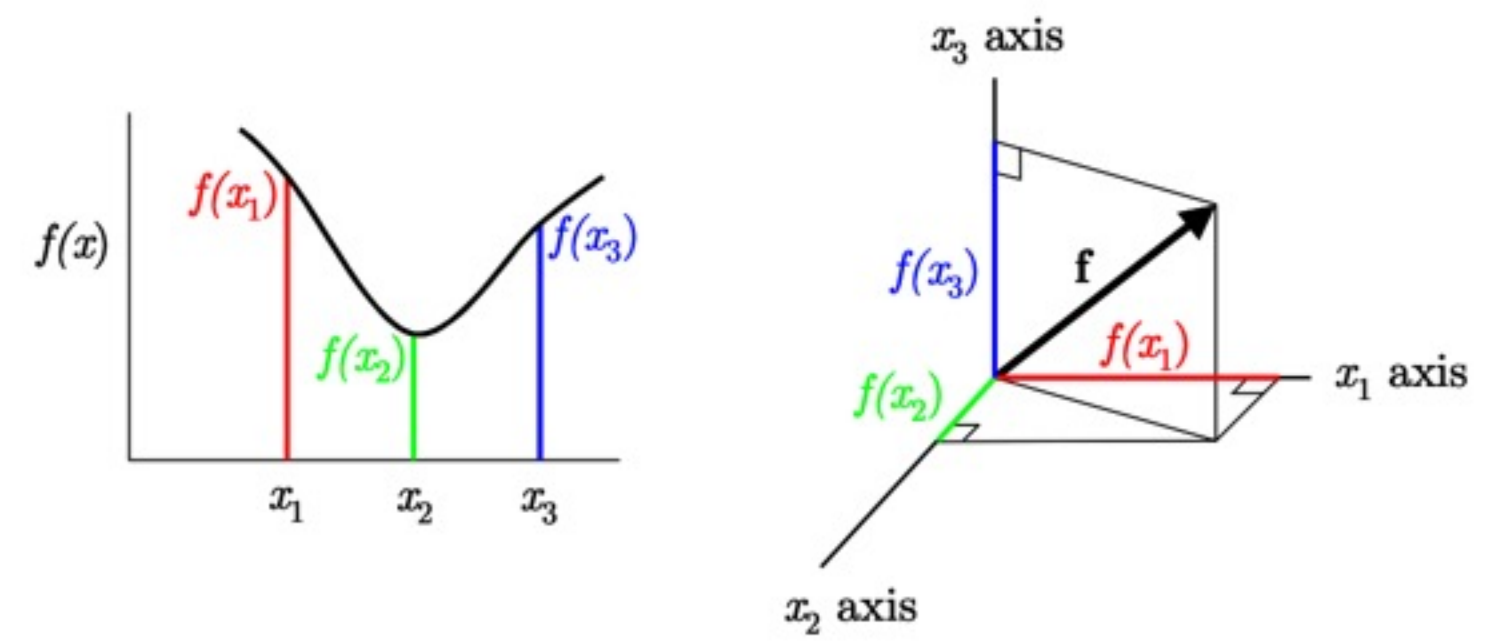
We can make a link between the bra and ket vectors and these functions:

$$|\alpha\rangle \equiv \psi(x) \quad \text{and} \quad \langle\alpha| \equiv \psi^*(x)$$

- We have shown two very different concrete realizations of abstract Hilbert space: column vectors and square-integrable functions with two different operational definitions of the inner product. Despite their differences they have the same defining properties.

# Functions of coninuous variable

We can see the function and, more generally, the quantum mechanical state as a vector in a space.




$$\mathbf{f} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

# Functions of coninuous variable





We can see the ket as a function:

$$|\boldsymbol{f}(\boldsymbol{x})\rangle = \begin{bmatrix} f(x_1)\sqrt{\delta x} \\ f(x_2)\sqrt{\delta x} \\ f(x_3)\sqrt{\delta x} \\ \vdots \end{bmatrix}$$

**more strictly, in the limit of** $\sqrt{\delta x} \rightarrow 0$

# Functions of coninuous variable

The integral of the modulus squared of the f function:

$$\int |f(x)|^2 dx = \begin{bmatrix} f^*(x_1)\sqrt{\delta x} & f^*(x_1)\sqrt{\delta x} \ldots & f^*(x_n)\sqrt{\delta x} \end{bmatrix} \begin{bmatrix} f(x_1)\sqrt{\delta x} \\ f(x_2)\sqrt{\delta x} \\ \vdots \\ f(x_n)\sqrt{\delta x} \end{bmatrix}$$

$$\equiv \langle \boldsymbol{f(x)} | \boldsymbol{f(x)} \rangle$$

# Observable with Continuous spectrum





**Starting with :** $|\alpha\rangle \equiv \psi(x)$ and $\langle\alpha| \equiv \psi^*(x)$

$$\|\alpha\|^2 = \int \psi^*(x)\,\psi(x)\,dx = \int |\psi(x)|^2\,dx = 1$$

to have normalized func.

} in discrete spectrum

$|\psi(x)|^2$ **In the Probability Density**

**Wavefunction** $\psi(x)$ **Is not the state** $|\alpha\rangle$ **and has no physical meaning**

# Functions of coninuous variable

We can make a link between the bra and ket vectors and these functions:

$$|\boldsymbol{\alpha}\rangle \equiv \boldsymbol{\psi}(\boldsymbol{x}) \quad \text{and} \quad \langle\boldsymbol{\alpha}| \equiv \boldsymbol{\psi}^*(\boldsymbol{x})$$

- We have shown two very different concrete realizations of abstract Hilbert space: column vectors and square-integrable functions with two different operational definitions of the inner product. Despite their differences they have the same defining properties.

- **It has been introduced as a definition of the state $|\alpha\rangle$ but we have seen that $\psi(x)$ has no physical meaning**

# Superposition principle and probabilities

$$|SUP\rangle = a_1|q_1\rangle + a_2|q_2\rangle$$

- Let's use a more convenient symbol for $|SUP\rangle$

$|\alpha\rangle = a_1|q_1\rangle + a_2|q_2\rangle$ multipl. by $\langle q_1|$ both sides

$\langle q_1|\alpha\rangle = \langle q_1|\left(a_1|q_1\rangle\right) + \langle q_1|\left(a_2|q_2\rangle\right) = a_1 \underbrace{\langle q_1|q_1\rangle}_{=1} + a_2 \underbrace{\langle q_1|q_2\rangle}_{=0} =$ orthonorm.

$\Rightarrow a_1 = \langle q_1|\alpha\rangle$

similarly $a_2 = \langle q_2|\alpha\rangle$ $\Rightarrow$

$P(q_1) = |\langle q_1|\alpha\rangle|^2$ or $P(q_i) = |\langle q_i|\alpha\rangle|^2$

$P(q_2) = |\langle q_2|\alpha\rangle|^2$

# Superposition principle and probabilities

- Probability that the measurement of observable q on a state $|\alpha\rangle$ will give $q_i$

$$P(q_i) = |\langle q_i|\alpha\rangle|^2$$

If we calculate the norm of $|\alpha\rangle = a_1|q_1\rangle + a_2|q_2\rangle$

$$\|\alpha\|^2 = \langle\alpha|\alpha\rangle = a_1^*\langle q_1|\left(a_1|q_1\rangle + a_2|q_2\rangle\right) + a_2^*\langle q_2|\left(a_1|q_1\rangle + a_2|q_2\rangle\right) =$$

$$\underbrace{\langle q_1|q_1\rangle}_{1} \quad \underbrace{\langle q_1|q_2\rangle}_{0} \quad \underbrace{\langle q_2|q_1\rangle}_{0} \quad \underbrace{\langle q_2|q_2\rangle}_{1}$$

$$\|\alpha\|^2 = |a_1|^2 + |a_2|^2 \equiv P_1 + P_2$$

# Superposition principle and probabilities

- Probability that the measurement of observable q on a state $|\boldsymbol{\alpha}\rangle$ will give $q_i$

$$p(q_i) = |\langle q_i|\alpha\rangle|^2$$

$$||\alpha||^2 = |a_1|^2 + |a_2|^2 \equiv p_1 + p_2$$

- If the norm of the state $|\boldsymbol{\alpha}\rangle$ is 1 also the sum of probability of measuring the different values is 1 in agreement with what expected for probabilities
- **That's why we need to have normalized vectors**

# Superposition principle and probabilities

Thus in general:

$$|\alpha\rangle \quad = a_1|q_1\rangle + a_2|q_2\rangle + a_3|q_3\rangle + \cdots = \sum_{i=1}^{N} a_i\,|q_i\rangle$$

and:

$$a_{\mathrm{i}} = \langle q_i|\alpha\rangle$$

$$p(q_i) = |\langle q_i|\alpha\rangle|^2$$

# Superposition principle and probabilities

Thus, in general:

$$|\alpha\rangle = a_1|q_1\rangle + a_2|q_2\rangle + a_3|q_3\rangle + \cdots = \sum_{i=1}^{N} a_i\,|q_i\rangle$$

and:

$$a_\mathrm{i} = \langle q_i|\alpha\rangle$$

Remembering that:

$$\Psi(x) = \sum_n c_n\,\psi_n(x)$$

# Superposition principle and probabilities





Thus :

$$|\Psi\rangle = \sum_n c_n |\psi_n\rangle$$

and:

$c_n = \langle\psi_n|\Psi\rangle$

# Superposition principle and probabilities

$$|\Psi\rangle = \sum_n c_n \, |\psi_n\rangle$$

and: $c_n = \langle\psi_n|\Psi\rangle$

Thus :

$$|\Psi\rangle = \sum_n c_n \, |\psi_n\rangle = \sum_n |\psi_n\rangle c_n =$$

$$= \sum_n |\psi_n\rangle\langle\psi_n|\Psi\rangle$$

$C_n$ can be moved in the product