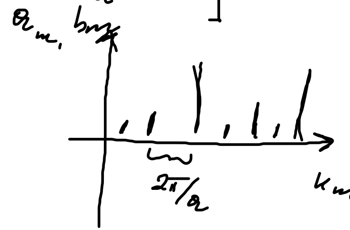
Serie di Fourier

$$f(x) \in \mathcal{L}_2(\mathbb{R}) \qquad \left(-\frac{a}{2}, \frac{a}{2}\right) \qquad a > 0$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} \left[a_m \cos\left(\frac{2\pi m}{a} x\right) + b_m \sin\left(\frac{2\pi m}{a} x\right) \right]$$



$$a_m = \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \cos\left(\frac{2\pi m x}{a}\right) dx$$

$$b_m = \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \sin\left(\frac{2\pi m x}{a}\right) dx$$

λ : lunghezza d'onda

$$k_m = \frac{2\pi m}{a} \qquad m = 1, 2, 3 \ldots \qquad \lambda_m = \frac{2\pi}{k_m} = \frac{\cancel{2\pi}}{\cancel{2\pi} m} a = \frac{a}{m}$$

$$\Delta k = k_{m+1} - k_m = \frac{2\pi}{a}$$

$$\cos\left(\frac{2\pi m x}{a}\right) = Re\left(e^{i\frac{2\pi m x}{a}}\right)$$

$$\sin\left(\frac{2\pi m x}{a}\right) = Im\left(e^{i\frac{2\pi m x}{a}}\right)$$

$$z_m \overset{def}{=} e^{\frac{i 2\pi m x}{a}}$$

complesso coniugato

$$Re(z_m) = \frac{z_m + z_m^*}{2} \qquad Im(z_m) = \frac{z_m - z_m^*}{2i}$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} a_m \frac{z_m + z_m^*}{2} + b_m \frac{z_m - z_m^*}{2i}$$

$$z_{-m} = e^{-i\frac{2\pi m x}{a}} = z_m^*$$

$$b_0 = 0$$

$$a_{-m} = a_m \qquad b_{-m} = -b_{+m}$$

$$= \frac{a_0}{2} + \sum_{m=1}^{+\infty} \underbrace{\frac{a_m - ib_m}{2}}_{c_m} z_m + \sum_{m=1}^{+\infty} \frac{\overbrace{a_m}^{a_{-m}} + \overbrace{i b_m}^{-ib_{-m}}}{2} \underbrace{z_m^*}_{z_{-m}}$$

$$c_m \overset{def}{=} \frac{a_m - ib_m}{2}$$

$$m' = -m$$

$$\sum_{m'=-\infty}^{-1} \frac{a_{m'} - i b_{m'}}{2} z_{m'}$$

$$= \sum_{m=-\infty}^{+\infty} c_m e^{i\frac{2\pi m x}{a}}$$

$$c_m = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx\, f(x) \left[\cos\left(\frac{2\pi m x}{a}\right) + - i \,\text{sen}(\ )\right]$$

$$= \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx\, f(x)\, e^{-\frac{i 2\pi m x}{a}}$$

Serie di Fourier : $a \longrightarrow +\infty$

$$\left(-\frac{a}{2}, \frac{a}{2}\right) \longrightarrow (-\infty, +\infty)$$

$$\underbrace{\Delta k}_{dk} = \frac{2\pi}{a} \xrightarrow{a \to +\infty} 0$$

$$C_m = \frac{1}{2\pi} \int_{-a/2}^{a/2} dx\, f(x)\, e^{-i \overbrace{\frac{2\pi m}{a}}^{k} x} \qquad \frac{2\pi}{a} \longrightarrow dk$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} dx \frac{f(x)}{\sqrt{2\pi}} e^{-ikx}}_{C(k)} dk$$

$$f(x) = \cancel{\sum_{m=-\infty}^{+\infty}} \frac{1}{\sqrt{2\pi}} dk\, C(k)\, e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, C(k)\, e^{ikx}$$

$$C(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, f(x)\, e^{-ikx}$$

Onde e.m. nel vuoto

$\rho = 0 \quad \underline{j} = \underline{0}$

$$\nabla \cdot \underline{E} = 0 \qquad \nabla \times \underline{E} = -\partial \underline{B}/\partial t$$
$$\nabla \cdot \underline{B} = 0 \qquad \nabla \times \underline{B} = \varepsilon_0 \mu_0 \partial \underline{E}/\partial t$$

$$\underbrace{\nabla \times (\nabla \times \underline{E})}_{\nabla(\cancel{\nabla \cdot \underline{E}} - \nabla^2 \underline{E}),\ \nabla \cdot \underline{E} = 0} = -\frac{\partial}{\partial t}(\nabla \times \underline{B}) = -\frac{\partial}{\partial t}\left(\varepsilon_0 \mu_0 \partial \underline{E}/\partial t\right);$$

$$\nabla^2 \underline{E} - \frac{\partial^2 \underline{E}}{\partial t^2} \cdot \frac{1}{c^2} = 0 \qquad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\underbrace{\nabla \times (\nabla \times \underline{B})}_{\nabla(\cancel{\nabla \cdot \underline{B}} - \nabla^2 \underline{B}),\ \nabla \cdot \underline{B} = 0} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \nabla \times \underline{E} = -\varepsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2} \Rightarrow \nabla^2 \underline{B} - \frac{1}{c^2} \partial^2 \underline{B}/\partial t^2 = 0$$

onda progressiva

$f(x \mp vt)$

onda regressiva

è soluzione dell'eq. delle onde

Oss

$$u(x,t) = u_0 e^{i(kx \mp \omega t)}$$

Verifica: 1D $\qquad \nabla^2 = \frac{d^2}{dx^2}$

$$\frac{d}{dx}\left(u_0 e^{i(kx \mp \omega t)}\right) = ik\, u_0 e^{i(kx \mp \omega t)}$$

$$\frac{d^2}{dx^2}\left(\quad\right) = (ik)^2 u_0 e^{i(kx \mp \omega t)} = -k^2 u_0 e^{i(kx \mp \omega t)}$$

$$\frac{\partial}{\partial t}\left(\quad\right) = \mp i\omega\, u_0 e^{i(kx \mp \omega t)}$$

$$\frac{\partial^2}{\partial t^2}\left(\quad\right) = (i\omega)^2 u_0 e^{i(kx \mp \omega t)} = -\omega^2 \left[u(x,t)\right]$$

$$-k^2 \cancel{u(x,t)} + \frac{\omega^2}{c^2} \cancel{u(x,t)} = 0$$

$$u(x,t)\left[\frac{\omega^2}{c^2} - k^2\right] = 0 \Rightarrow \omega/k = c \,; \; \omega = k \cdot c$$

Usiamo pr. sovrapposizione

$$u(x,t) = \int_{-\infty}^{+\infty} dk \, \tilde{u}(k) \, e^{i(kx-\omega t)}$$

è soluzione

include sia onde progressive

sia onde regressive

$\tilde{u}(k)$?

$$\underline{E}(x,t) = \underline{\varepsilon}_1 E_0 \, e^{i(kx-\omega t)}$$

$\underline{\varepsilon}_1 \; ? \; \underline{\varepsilon}_2$

$$\underline{B}(x,t) = \underline{\varepsilon}_2 B_0 \, e^{i(kx-\omega t)}$$

se $\rho \neq 0 \rightarrow \underline{k} \cdot \underline{\varepsilon}_1 \neq 0$

$$\underline{\nabla} \cdot \underline{E} = 0$$

$$k \cdot \underline{\varepsilon}_1 = 0 \Rightarrow \underline{\varepsilon}_1 \perp \underline{k}$$

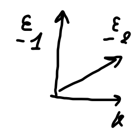
$$\underline{\nabla} f^{ikx} = i \underline{k} \cdot (f)$$

$$\underline{\nabla} \cdot \underline{B} = 0,$$

$$\underline{k} \cdot \underline{\varepsilon}_2 = 0 \Rightarrow \underline{\varepsilon}_2 \perp \underline{k}$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$i\underline{k}$



$$i \underline{k} \times \underline{E} = + i \omega \underline{B} \, ;$$

$$\underline{k} \times \underline{\varepsilon}_1 E_0 = \omega \underline{\varepsilon}_2 B_0 \, ;$$

$$\underline{\varepsilon}_2 = \frac{\underline{k} \times \underline{\varepsilon}_1}{\omega} \, E_0 / B_0$$

$\underline{\varepsilon}_2 \perp \underline{k}, \underline{\varepsilon}_1$

Considero il modello:

$$1 = \frac{k}{\omega}\frac{E_0}{B_0} \Rightarrow c = \frac{\omega}{k} = \frac{E_0}{B_0}$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Def Mezzo <u>non</u> dispersivo: $\frac{\omega}{k} = \text{cost}$ $\qquad \frac{\omega}{k} = \text{vel. fase} = v_f$

" dispersivo: $\omega/k = \text{funzione di } k = v_f(k)$

$\omega = \underline{\omega(k)}$

$e^{i(kx - \omega(k)t)}$ rimane soluzione

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\, A(k)\, e^{i(kx - \omega(k)t)}$$ rimane soluzione

Supponiamo che $u(x, t=0) = u_0(x)$ assegnato

$$u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, A(k)\, e^{ikx}$$

$$\Rightarrow \quad A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, u_0(x)\, e^{-ikx}$$ T.F. di $u_0(x)$

$u_0(x)$ è reale $\Rightarrow$ $A^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, u_0(x)\, e^{ikx} = A(-k)$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} dk\, A(k)\, e^{i(kx-\omega t)} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} dk\, A(k)\, e^{i(kx-\omega t)}$$

$\underbrace{\omega(k) = \omega(-k)}_{\text{onde progressive e regressive propagano nello stesso modo}}$

$k' = -k$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} dk \left(\overbrace{A(k)\, e^{ikx} + A^*(k)\, e^{-ikx}}^{\text{Reale}} \right) \overbrace{e^{-i\omega t}}^{\text{Complesso}} \qquad \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} dk'\, \underbrace{A(-k')}_{A^*(k')}\, e^{-ik'x}\, e^{-i\overbrace{\omega(-k')}^{\omega(k')}t}$$

Soluzione fisica

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} dk\, \underline{A(k)}\, e^{i(kx-\omega(k)t)} + c.c.\right]$$

Vera se $\frac{\partial}{\partial t} u(x,t=0) = 0$

E se $\begin{cases} \frac{\partial}{\partial t} u(x,t=0) = v_0(x) \\ u(x,t=0) = u_0(x) \end{cases}$

Comunque

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} dk\, A(k)\, e^{i(kx-\omega(k)t)} + c.c.\right]$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} dk \left[-i\omega A(k)e^{ikx} + i\omega A^*(k)e^{-ikx}\right]\right]$$

$$u_0(x) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \left(A(k) + A^*(-k) \right) e^{ikx}$$

$$v_0(x) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \quad -i\omega(k) \left(A(k) - A^*(-k) \right) e^{ikx}$$

$$\frac{A(k) + A^*(-k)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, u_0(x) e^{-ikx}$$

$$-\frac{i\omega}{2} \left(A(k) - A^*(-k) \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, v_0(x) e^{-ikx}$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \left(u_0(x) + \frac{i}{\omega(k)} v_0(x) \right) e^{-ikx}$$