Spin

Spin

In classical mechanics, a rigid object admits two kinds of angular momentum: orbital ($L = r \times p$), associated with the motion of the center of mass, and spin ($S = I\omega$) associated with motion about the center of mass.

Spin

Also in quantum mechanics, in addition to orbital angular momentum, associated with the motion of the electron around the nucleus (for the H atom), the electron also carries another form of angular momentum.

But, it has nothing to do with motion in space :
elementary particles carry intrinsic angular momentum (S) in addition to their "extrinsic" angular momentum (L).

Spin operators

The algebraic theory of spin is is identical to the theory of orbital angular momentum, including commutator rules:

$$[S_x, S_y] = i\hbar S_z, \qquad [S_y, S_z] = i\hbar S_x, \qquad [S_z, S_x] = i\hbar S_y$$

We can represent the magnitude squared of the spin angular momentum vector by the operator:

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

It is not surprising that:

$$[S^2, S_x] = [S^2, S_y] = [S^2, S_z] = 0$$

Spin operators

By analogy with Orbital a.m., spin operators satisfy:

$$S_z |s,m\rangle = \hbar m |s\ m\rangle$$

$$S^2 |s,m\rangle = \hbar^2 s(s+1) |s,m\rangle$$

Spin operators

But here the eigenvectors are not spherical harmonics, and there is no a priori reason to exclude the half-integer values of s and m:

$$s = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots ; \qquad m = -s, -s+1, \ldots, s-1, s$$

Every elementary particle has a specific and immutable value of s, which we call the spin of that particular species: pi mesons have spin 0; electrons have spin 1/2; photons have spin 1; deltas have spin 3/2; gravitons have spin 2; and so on...

Spin 1/2

By far the most important case is $s = 1/2$, for this is the spin of the particles that make up ordinary matter such as protons, neutrons, and electrons.
There are just two eigenstates:

Spin up $(\uparrow)$ $\quad$ $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$

Spin down $(\downarrow)$ $\quad$ $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$

A general state of a particle with spin $\frac{1}{2}$, called spinor is:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\,\chi_+ + b\,\chi_-$$

with $\underbrace{\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{spin up}}$ $\quad$ $\underbrace{\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{spin down}}$

Spin 1/2

We can derive the Spin operator by investigating their effects on the eigen-spinor:

since $$S^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle$$

$$\Rightarrow S^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \quad \text{and} \quad S^2\chi_- = \frac{3}{4}\hbar^2\chi_-$$

Let's write $S^2 = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$

then from $S^2\chi_+$

$$\begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c \\ e \end{pmatrix} = \begin{pmatrix} 3/4\,\hbar^2 \\ 0 \end{pmatrix} \Rightarrow c = \frac{3}{4}\hbar^2$$

$$e = 0$$

Spin 1/2

We can derive the Spin operator by investigating their effects on the eigen-spinor:

and from $S^2 \chi_- = \frac{3}{4}\hbar^2 \chi_-$

$$\begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 3/4\,\hbar^2 \end{pmatrix} \Rightarrow \begin{array}{l} d = 0 \\ f = \frac{3}{4}\hbar^2 \end{array}$$

Thus: $$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Spin 1/2

Similarly for Sz:

$$S_z \chi_+ = \frac{\hbar}{2} \chi_+ \qquad S_z \chi_- = -\frac{\hbar}{2} \chi_-$$

One can derive:

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And exploiting $S_x \pm i S_y = \hbar \sqrt{s(s+1) - m(m \pm 1)} \; |s\,(m \pm 1)\rangle$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Spin 1/2

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Because they all have $\hbar/2$ factor, it's convenient to write:

$S = \frac{1}{2}\hbar\sigma$ and

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are the famous Pauli spin matrices

Spin 1/2

Thus, we have the eigenspinor of Sz:

$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eigenvalue $\frac{1}{2}\hbar$

$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalue $-\frac{1}{2}\hbar$

Because of:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = c_1 \chi_+ + c_2 \chi_- \Rightarrow c_1 = a \quad c_2 = b$$

$$P_1 = |a|^2 \quad P_2 = |b|^2$$

If you measure Sz on a particle in the general state X you could get $+\hbar/2$, with probability $|a|^2$, or $-\hbar/2$, with probability $|b|^2$.

Spin 1/2

One can show that the eigenvalue of Sx are still ½ℏ and −½ℏ with eigenvectors:

$$\chi_+^x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \text{ for } \frac{1}{2}\hbar \qquad \chi_-^x = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \text{ for } -\frac{1}{2}\hbar$$

Thus

$$\chi = \begin{pmatrix} \frac{c_1+c_2}{\sqrt{2}} \\ \frac{c_1-c_2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{cases} c_1+c_2 = a\sqrt{2} \Rightarrow a\sqrt{2} = c_1 + c_1 - b\sqrt{2} \\ c_1 - c_2 = b\sqrt{2} \end{cases} \Downarrow \; c_1 = \frac{a+b}{\sqrt{2}}$$

and we have $\frac{1}{2}|a+b|^2$ probability to get $+\frac{1}{2}\hbar$ and $\frac{1}{2}|a-b|^2$ for $-\frac{1}{2}\hbar$

Spin 1/2

One can show that the eigenvalue of S_y are still $\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ with eigenvectors:

$$\chi_+^y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \quad \text{for } \frac{1}{2}\hbar \qquad \chi_-^y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \quad \text{for } -\frac{1}{2}\hbar$$

Thus ...

and we have ... probability to get $+\frac{1}{2}\hbar$ and ... for $-\frac{1}{2}\hbar$

Spin magnetic moment

According to classical physics, a small current loop possesses a magnetic moment of magnitude:

$$|\vec{\mu}| = \underbrace{i}_{\text{current}}\ \underbrace{S}_{\text{Surface area}}$$

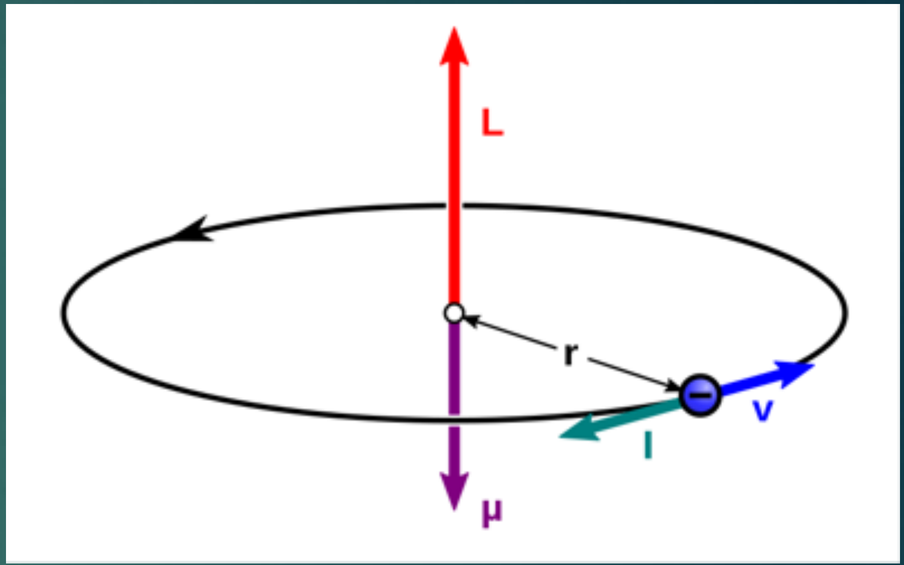
In a magnetic field B there, the magnetic dipole moment will experience a torque $\mu \times B$, which tends to line up the dipole to the field
The energy associated to this torque is:

$$\underbrace{E}_{\text{Energy}} = -\vec{\mu}\cdot\vec{B}$$

Spin magnetic moment

According to classical physics, a small current loop possesses a magnetic moment of magnitude:

$$|\vec{\mu}| = \underset{\text{current}}{i}\, \underset{\text{Surface area}}{S}$$



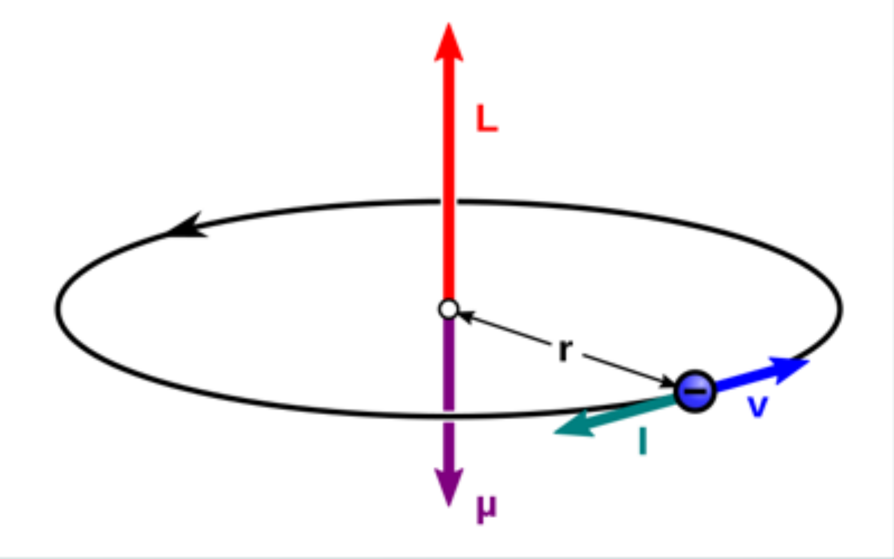
In a semi-classic picture the electron orbiting in the atom can be considered as a current loop:

$$T = \frac{\text{Circumf.}}{\text{velocity}} = \frac{2\pi r}{v} \qquad i = \frac{-e}{t} = \frac{-ev}{2\pi r}$$

$$\text{but } |\vec{\mu}| = iS = -\frac{ev}{2\pi r}\,\pi r^2 = -\frac{evr}{2}$$

Spin magnetic moment

In a semi-classic picture the electron orbiting in the atom can be considered as a current loop:



$$|\vec{\mu}| = \underbrace{i}_{\text{current}}\,\underbrace{S}_{\text{Surface area}}$$

$$|\vec{\mu}| = iS = -\frac{ev}{2\pi r}\pi r^2 = -\frac{evr}{2}$$

but

$$L = I\omega = m r^2 \frac{v}{r} = m r v$$

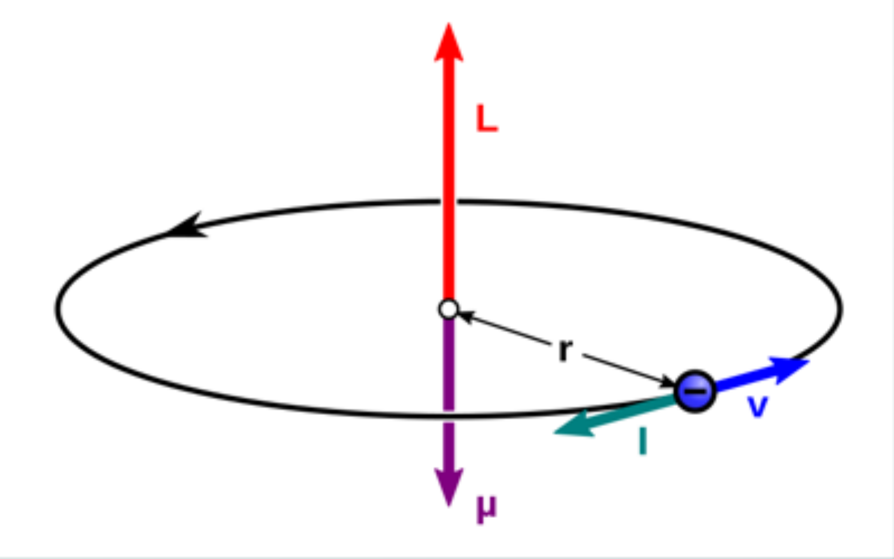
Thus

$$|\vec{\mu}_L^*| = -\frac{eL}{2m_e} \quad \text{and} \quad \vec{\mu}_L = -\frac{e\vec{L}}{2m_e}$$

Spin magnetic moment

In a semi-classic picture the electron orbiting in the atom can be considered as a current loop:

$$\vec{\mu}_L = \frac{-eL}{2m_e} \xrightarrow[\text{physics}]{\text{quantum}} \hat{\mu}_L = \frac{-e\hat{L}}{2m_e}$$



If a magnetic field B is present, I can calculate the energy linked to this magnetic moment:

$$E = -\vec{\mu}\cdot\vec{B}$$

Energy

$$\Rightarrow \quad \hat{H} = \frac{e}{2m_e}\hat{L}_z B \longrightarrow \text{we will see that} = \frac{e}{2m_e}\hbar m$$

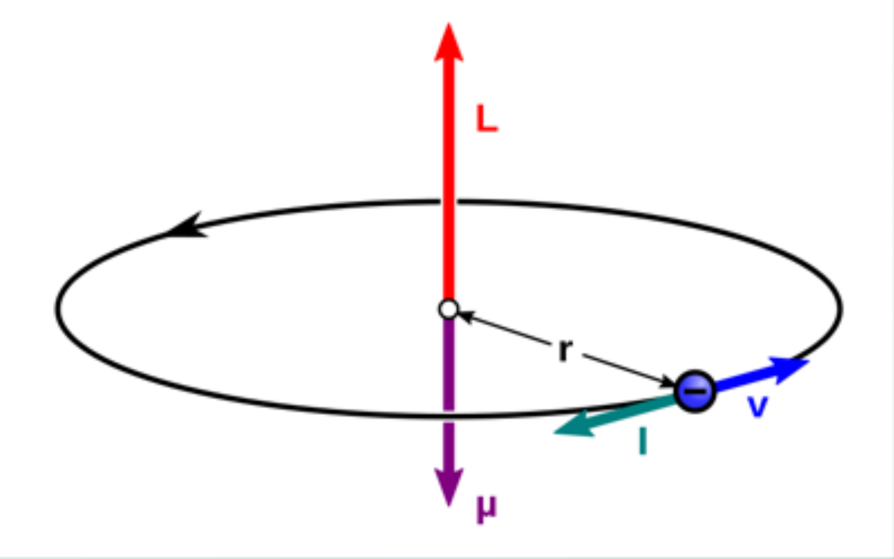
L reduces to Lz because B oriented along z

"split" for $m \neq 0$

Spin magnetic moment

In a semi-classic picture the electron orbiting in the atom can be considered as a current loop:

$$\vec{\mu}_L = \frac{-e\,L}{2\,m_e} \xrightarrow[\text{physics}]{\text{quantum}} \hat{\mu}_L = \frac{-e\,\hat{L}}{2\,m_e}$$



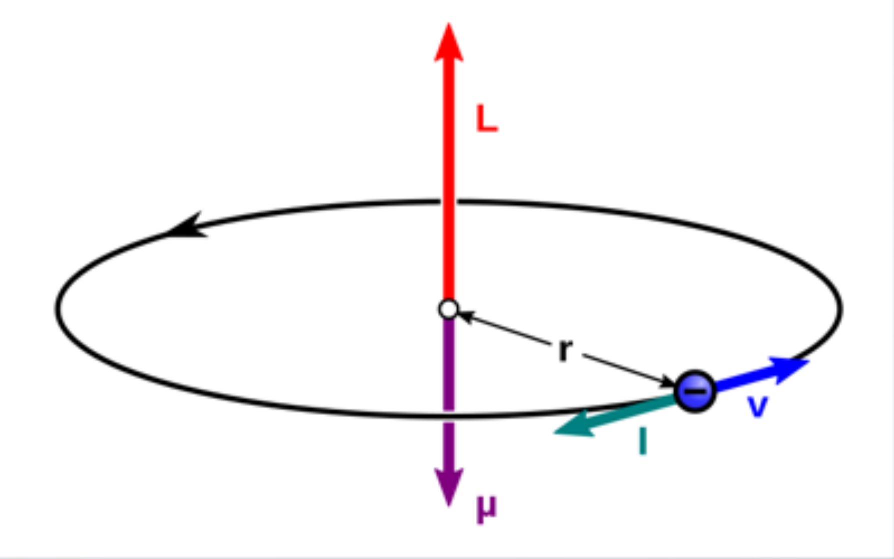
The analogy between spin and orbital angular momentum suggests that there may be a similar relationship between magnetic moment and spin angular momentum. We can write:

$$\boxed{\mu_S = -\frac{g\,e\,\vec{S}}{2\,m_e}}$$

Spin magnetic moment

Spin magnetic moment

$$\mu_s = -\frac{g_s e \vec{S}}{2 m_e}$$



where g is called the g-factor. Classically, we would expect $g=1$ but for reasons that could be explained by relativistic theory, here

$g_s=2.0023192$

We can also write:

$$\mu_s = \gamma_s S \quad \text{with } |\gamma_s| = \frac{|-e|}{2 m_e} g_s \quad (\sim 2 \text{ times } \gamma \text{ in classical systems})$$

gyromagnetic ratio

Spin magnetic moment

$$\mu_s = \gamma S$$

Also in this case we can find the energy associated to the dipole in presence of the magnetic field B:

$$\hat{H} = \frac{g_s e}{2 m_e} \vec{S}_z \cdot \vec{B} = -\gamma \vec{S}_z \cdot \vec{B}$$

Spin magnetic moment

Also in this case we can find the energy associated to the dipole in presence of the magnetic field B:

$$\hat{H} = \frac{g_s e}{2 m_e} \vec{S}_z \cdot \vec{B} = -\gamma \vec{S}_z \cdot \vec{B}$$

Imagine a particle of spin 1/2 at rest in a uniform magnetic field, which points in the z-direction:

$$B = B_0 \hat{z} \quad \Rightarrow \quad H = -\gamma B_0 S_z =$$

$$= -\frac{\gamma B_0 \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin magnetic moment

$$H = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenstates are the same of S_z, and the eigenvalues will have just an extra multiplicative term γB_0

$$\chi_+ \xrightarrow{\text{energy}} -\frac{\gamma B_0 \hbar}{2}$$

$$\chi_- \xrightarrow{\text{energy}} +\frac{\gamma B_0 \hbar}{2}$$

Spin magnetic moment

The general solutions must satisfy the t.d.S.E.:

$$i\hbar \frac{d\chi}{dt} = H\chi$$

And can be written in term of linear combination of stationary states:

$$\chi(t) = a\,\chi_+ e^{-iE_+t/\hbar} + b\,\chi_- e^{-iE_-t/\hbar}$$

Which can be written in the Pauli's notation as:

$$\chi(t) = \begin{pmatrix} a\, e^{i\gamma B_0 t/2} \\ b\, e^{-i\gamma B_0 t/2} \end{pmatrix}$$

Spin magnetic moment

$$\chi(t) = \begin{pmatrix} a\, e^{i\gamma B_0 t/2} \\ b\, e^{-i\gamma B_0 t/2} \end{pmatrix} \quad \text{for } t=0 \quad \chi(t) = \begin{pmatrix} a \\ b \end{pmatrix}$$

Thus $|a|^2 + |b|^2 = 1$ for the normalization condition of χ

so I could write

$$a = \cos(\alpha/2)$$

$$b = \sin(\alpha/2)$$

$$\Rightarrow \chi(t) = \begin{pmatrix} \cos(\alpha/2)\, e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)\, e^{-i\gamma B_0 t/2} \end{pmatrix}$$

Spin magnetic moment

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)\, e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)\, e^{-i\gamma B_0 t/2} \end{pmatrix} \qquad \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

Let's calculate the expectation value of Spin

$$\langle S_x \rangle = \chi(t)^* S_x \chi(t) = \left(c_+^*, c_-^*\right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} =$$

$$= \frac{\hbar}{2} \left(c_+^*, c_-^*\right) \begin{pmatrix} c_- \\ c_+ \end{pmatrix} = \frac{\hbar}{2} \left(c_+^* c_- + c_-^* c_+\right)$$

Spin magnetic moment

Let's calculate the expectation value of Spin

$$\langle S_x \rangle = \frac{\hbar}{2}\left(c_+^* c_- + c_-^* c_+\right) = \frac{\hbar}{2}\left(\cos(\alpha/2)\, e^{-i\gamma B_0 t/2} \sin(\alpha/2)\, e^{-i\gamma B_0 t/2} + \right.$$

$$\left. + \sin(\alpha/2)\, e^{i\gamma B_0 t/2} \cos(\alpha/2)\, e^{i\gamma B_0 t/2}\right)$$

$$\Rightarrow \langle S_x \rangle = \frac{\hbar}{2} \underbrace{\sin(\alpha/2)\cos(\alpha/2)}_{\frac{1}{2}\sin(\alpha)} \underbrace{\left(e^{-i\gamma B_0 t} + e^{i\gamma B_0 t}\right)}_{\left[\cos(\gamma B_0 t) - i\sin(\gamma B_0 t) + \cos(\gamma B_0 t) + i\sin(\gamma B_0 t)\right]}$$

$$= \frac{\hbar}{2} \quad \frac{1}{2}\sin(\alpha) \quad \left[\cos(\gamma B_0 t) - i\sin(\gamma B_0 t) + \cos(\gamma B_0 t) + i\sin(\gamma B_0 t)\right]$$

Spin magnetic moment

$$\langle S_x \rangle = \frac{\hbar}{2} \underbrace{\sin(\alpha/2)\cos(\alpha/2)}_{\frac{1}{2}\sin(\alpha)} \underbrace{\left(e^{-i\gamma B_0 t} + e^{i\gamma B_0 t}\right)}_{\left[\cos(\gamma B_0 t) - \cancel{i\sin(\gamma B_0 t)} + \cos(\gamma B_0 t) + \cancel{i\sin(\gamma B_0 t)}\right]}$$

$$= \frac{\hbar}{2}$$

$$= \frac{\hbar}{2} \sin(\alpha)\cos(\gamma B_0 t)$$

Similarly:

$$\langle S_y \rangle = -\frac{\hbar}{2} \sin(\alpha)\sin(\gamma B_0 t)$$

$$\langle S_z \rangle = \frac{\hbar}{2} \cos(\alpha)$$

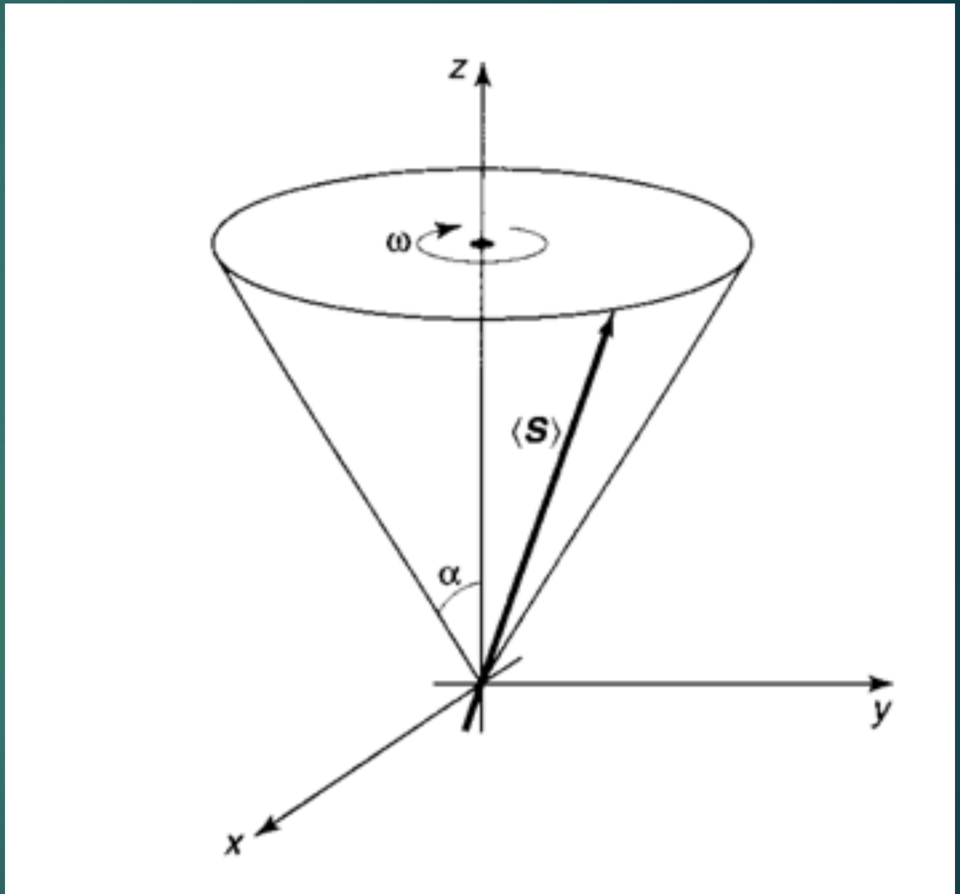
note that $\langle S_z \rangle$ is time indipendent

Spin precession

$$\langle S_x \rangle = \frac{\hbar}{2} \sin(\alpha) \cos(\gamma B_0 t)$$

$$\langle S_y \rangle = -\frac{\hbar}{2} \sin(\alpha) \sin(\gamma B_0 t)$$

$$\langle S_z \rangle = \frac{\hbar}{2} \cos(\alpha)$$



The expectation value of the spin angular momentum vector subtends a constant angle α with the z-axis, and precesses about this axis at the frequency

$$\omega = \gamma B_0 \simeq \frac{e B_0}{m_e}$$

Larmor Frequency

Spin precession

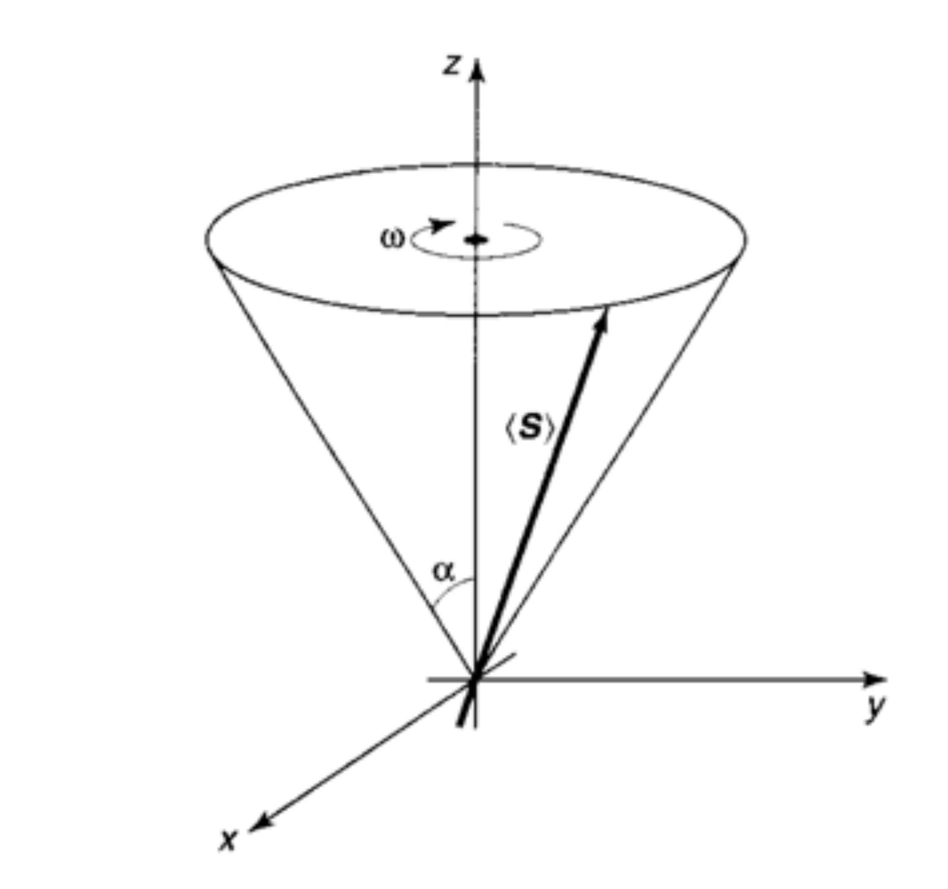
$$\langle S_x \rangle = \frac{\hbar}{2} \sin(\alpha) \cos(\gamma B_0 t)$$

$$\langle S_y \rangle = -\frac{\hbar}{2} \sin(\alpha) \sin(\gamma B_0 t)$$

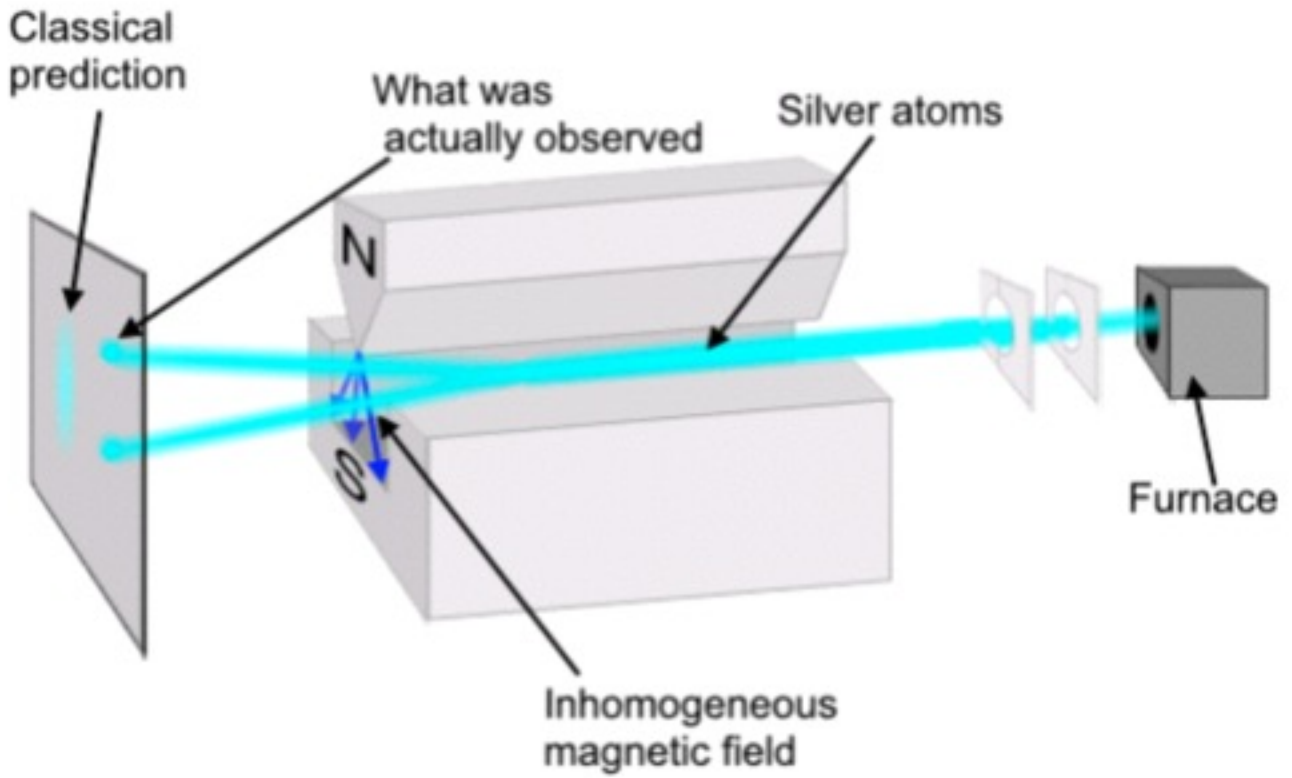
$$\langle S_z \rangle = \frac{\hbar}{2} \cos(\alpha)$$

$$\omega = \gamma B_0$$

This result is what one would expect classically, but with the expectation value of S instead of the classical momentum vector.



- The Stern-Gerlach experiment (1922):



Beam of Silver atoms:

$$Ag : \underbrace{[Kr]\, 4d^{10}}_{\text{All inner electrons are paired} \Downarrow \text{ their } \hat{S} \text{ and } \hat{L} \text{ cancel}} \; 5s^1$$

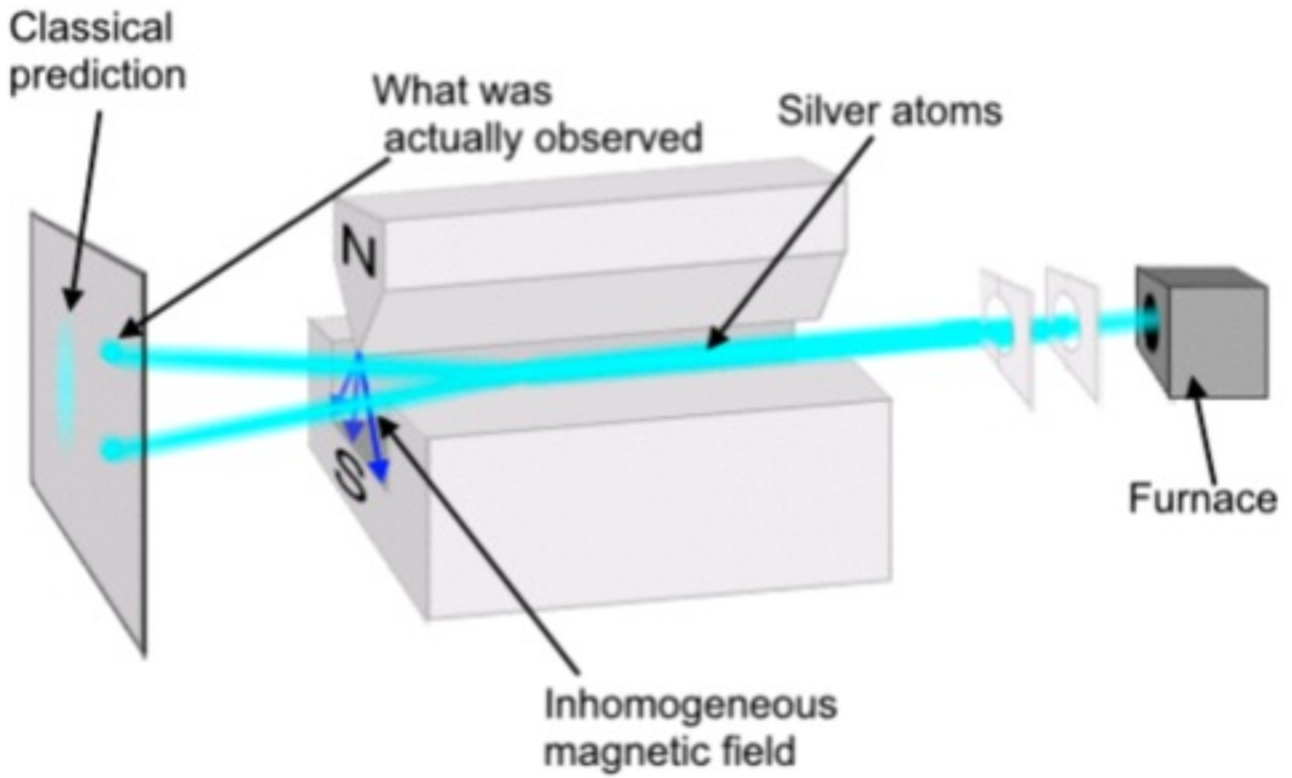
$5s^1$ ⇓ like H case in term of L and S

In homogeneous magnetic field in z-direction:

$$E = -\mu_z B \qquad -\frac{\delta E}{\delta z} = F_z = 0$$

- The Stern-Gerlach experiment (1922):

z



Beam of Silver atoms:

$Ag : \underbrace{[Kr]\, 4d^{10}}\; 5s^1$

All inner electrons are paired $\Downarrow$ their $\hat{S}$ and $\hat{L}$ cancel

$\Downarrow$ Like H case in term of L and S

In inhomogeneous magnetic field

$E = -\mu_z B \qquad -\dfrac{\delta E}{\delta z} = F_z \neq 0$

$\quad \| \; \mu_z B_I$

with $B = \underline{-B_I x \hat{x}} + (B_0 + B_I z)\hat{z}$

$\Downarrow$ we must have it to satisfy $\nabla B = 0$

The Stern Gerlach experiment

In inhomogeneous magnetic field

$$E = -\mu_z B \qquad -\frac{\delta E}{\delta z} = F_z \neq 0$$

$$\parallel \; \mu_z B_i$$

with $B = -B_i x \hat{x} + (B_0 + B_i z)\hat{z}$

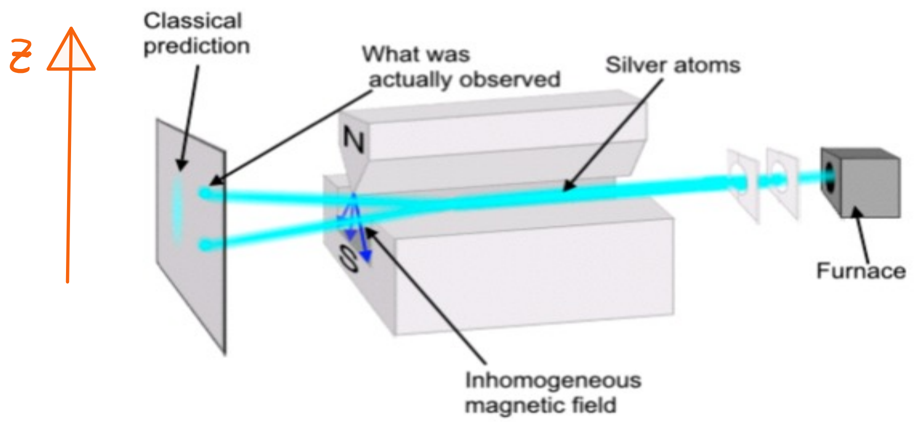
$$\mu_L = -\frac{e}{2m} L_z B$$

$$\Downarrow$$

$$F_z = -\frac{e}{2m} L_z B_i \longrightarrow$$

Classically, because L of atoms in the Furnace is oriented randomly, I would expect any value for L_z and thus F_z can have any value

- The Stern-Gerlach experiment (1922):



The Stern Gerlach experiment

In inhomogeneous magnetic field

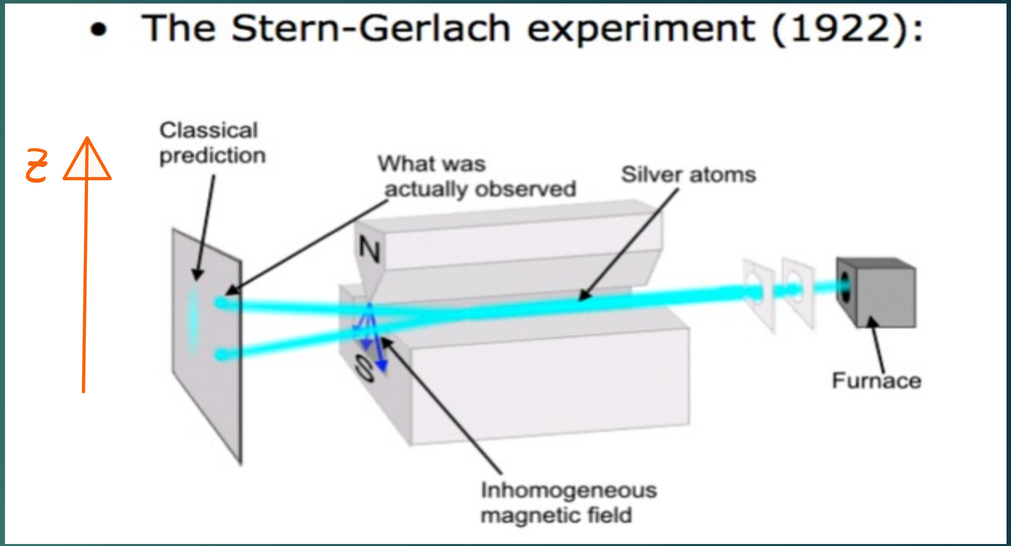
$$E = -\mu_z B \qquad -\frac{\delta E}{\delta z} = F_z \neq 0$$
$$F_z = \mu_z B_i$$

with $B = -B_i\, x\, \hat{x} + (B_0 + B_i\, z)\hat{z}$

$$\mu_L = -\frac{e}{2m_e} L_z$$
$$\Downarrow$$

$$F_z = -\frac{e}{2m_e} L_z B_i \longrightarrow$$ Even considering quantization of L_z,

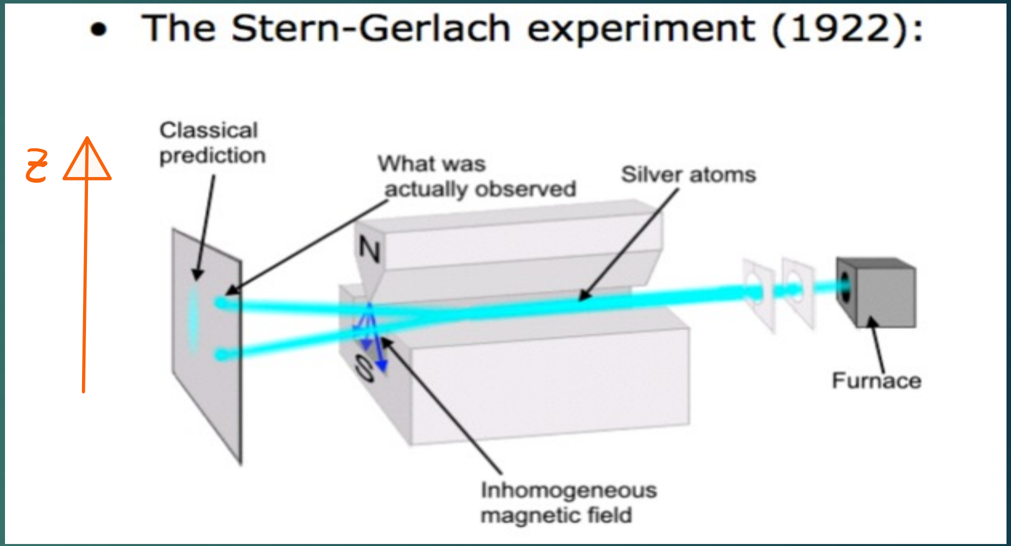
$$\hat{H} = \frac{e}{2m_e}\hat{L}_z (B_0 + B_i\, z) \Rightarrow E = \frac{e}{2m_e}\hbar m\,(B_0 + B_i\, z)$$
$$F_z = -\frac{e}{2m_e}\hbar m B_i$$



The Stern Gerlach experiment

In inhomogeneous magnetic field

$$F_z = -\frac{e}{2m_e} L_z B_i$$



Even considering quantization of L_z,

$$\hat{H} = \frac{e}{2m_e}\hat{L}_z(B_0 + B_i z)$$

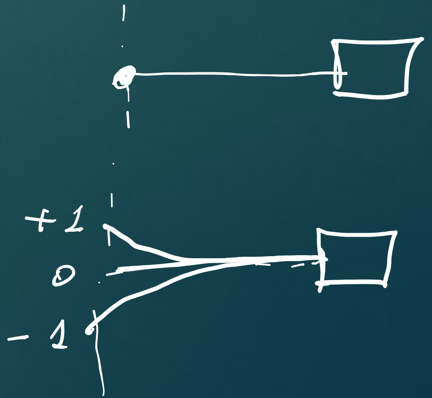
$$\Rightarrow \quad E = \frac{e}{2m_e}\hbar m (B_0 + B_i z)$$

$$F_z = -\frac{e}{2m_e}\hbar m B_i$$

for $m=0$ 1 spot

for $m=0,\pm 1$ 3 spot

$2n+1$ spot



UNIVERSITÀ DEGLI STUDI DI MILANO BICOCCA

The Stern Gerlach experiment

In inhomogeneous magnetic field

Considering the net spin of the atoms, that of the outer shell electron (5s¹)

$$S = \frac{1}{2} \qquad m_s = -\frac{1}{2}, +\frac{1}{2}$$

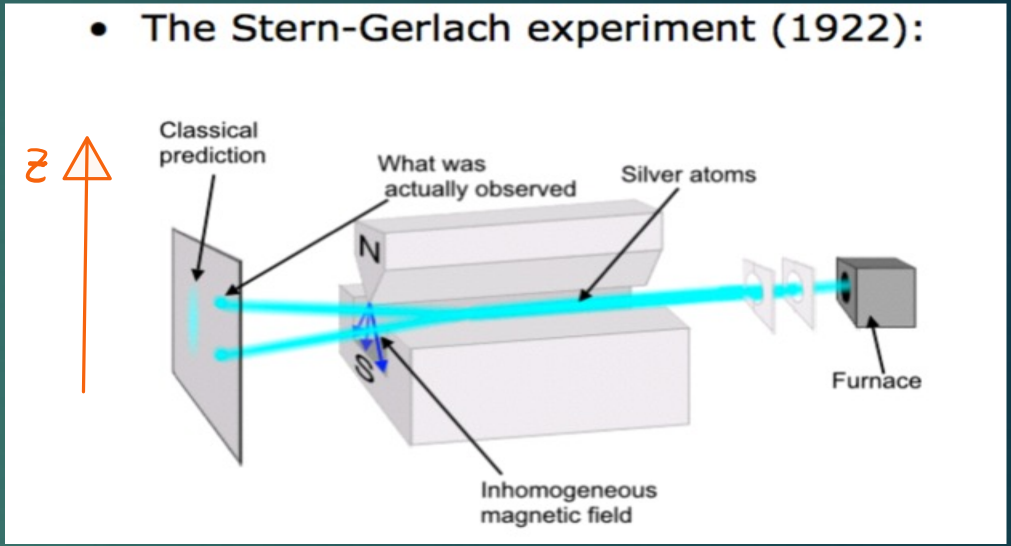
$$E_\pm = \mp \gamma \left(B_0 + B_i z\right) \frac{\hbar}{2}$$

$$\Downarrow$$

$$F_z = \gamma B_i S_z \quad \begin{cases} \gamma B_i \hbar/2 \\ -\gamma B_i \frac{\hbar}{2} \end{cases}$$

$$\Downarrow$$

$\boxed{\text{2 BEAMS expected}}$



Spin-Orbit

The three components of the orbital angular momentum operator, Lx, Ly, and Lz, obey the commutation relations that we have seen previously, which can be written in the convenient vector form:

$$L \times L = i\hbar L$$

e.g. $[L_x, L_y] = i\hbar L_z$

Similarly for spin angular momentum operators:

$$S \times S = i\hbar S$$

One can also see that:

$$[L_i, S_j] = 0 \qquad \text{with } i,j = 1,2,3 \longrightarrow x,y,z$$

N.B. the two types of "motion" represented by the angular momentum operators are unrelated, thus it is reasonable to suppose that the two sets of operators commute with one another...

Spin-Orbit

Let us now consider the electron's total angular momentum vector:

$$J = L + S$$

One can show that:

$$J \times J = i\hbar J$$

It is thus evident that the three components of the total angular momentum operator obey analogous commutation relations to the corresponding orbital and spin angular momentum operators...it is obvious that the total angular momentum has similar properties to the orbital and spin angular momenta. Thus, it is only possible to simultaneously measure the magnitude squared of the total angular momentum vector, and only one of its single components:

$$[J^2, J_z] = 0 \qquad J^2 = J_x^2 + J_y^2 + J_z^2$$

Spin-Orbit

Simultaneous eigenstates of J_z and J^2 satisfy:

$$\boxed{\begin{aligned} J_z \psi_{j,m_j} &= m_j \hbar \psi_{j,m_j} \\ J^2 \psi_{j,m_j} &= j(j+1)\hbar^2 \psi_{j,m_j} \end{aligned}}$$

$$m_j = 0, \pm 1, \pm 2, \ldots \pm j$$

identical to the spin operators

L	an S	$\rightarrow$	J
$\Downarrow$	$\Downarrow$		$\Downarrow$
l	s		j
m	m_s		m_j

Spin-Orbit

We can also write:

$$J^2 = [L+S]\cdot[L+S] = L^2 + S^2 + 2\,L\cdot S$$

And one can show that:

$$[J^2, S^2] = 0 \qquad [J^2, L^2] = 0 \qquad [J^2, L_z] \neq 0 \qquad [J^2, S_z] \neq 0$$

We can measure simultaneously

J^2, S^2, L^2 and J_z or L^2, S^2, L_z, S_z, J_z

Spin-Orbit

We can measure simultaneously:

L^2, S^2, L_z, S_z, J_z

Thus Jz has simultaneous eigenstate with L² S² Lz and Sz:

$$L^2 \psi_{n,l,m,m_s} = l(l+1)\hbar^2 \psi_{n,l,m,m_s}$$

$$S^2 \psi_{n,l,m,m_s} = s(s+1)\hbar \psi_{n,l,m,m_s}$$

$$L_z \psi_{n,l,m,m_s} = m \hbar \psi_{n,l,m,m_s}$$

$$S_z \psi_{n,l,m,m_s} = m_s \hbar \psi_{n,l,m,m_s}$$

degeneracy index

$$|n, l, s, m, m_s\rangle = |n, l, \tfrac{1}{2}, m, m_s\rangle = |n, l, m, m_s\rangle$$

$$\psi_{n,l,m,m_s}$$

Thus:

$$J_z \psi_{n,l,m,m_s} = (L_z + S_z)\psi_{n,l,m,m_s} = (m + m_s)\hbar \psi_{n,l,m,m_s} = m_j \hbar \psi_{n,l,m,m_s}$$

Spin-Orbit

Thus

$$m_j = m + m_s$$

But Jz has simultaneous eigenstate with L² S² and J²:

$$L^2 \psi_{n,l,j,m_j} = l(l+1)\hbar^2 \psi_{n,l,j,m_j}$$

$$S^2 \psi_{n,l,j,m_j} = s(s+1)\hbar \psi_{n,l,j,m_j}$$

$$J^2 \psi_{n,l,j,m_j} = j(j+1)\hbar \psi_{n,l,j,m_j}$$

$$J_z \psi_{n,l,j,m_j} = m_j \hbar \psi_{n,l,j,m_j}$$

Thus ψ_{n,l,j,m_j} are simult. eigenstate for L^2, S^2, J^2, J_z

Spin-Orbit Hydrogen atom

Simultaneous eigenstate for L^2 S^2 S_z and L_z in separable form:

$$\psi_{n,\ell,m,\frac{1}{2}} = R_{n,\ell}(r)\, Y_{\ell,m}(\theta,\varphi)\, \chi_+$$

$$\psi_{n,\ell,m,-\frac{1}{2}} = R_{n,\ell}(r)\, Y_{\ell,m}(\theta,\varphi)\, \chi_-$$

we can express the eigenstates $\psi_{n,l,1/2;j,mj}$ as linear combinations of them
(let me omit the Radial Part and index n, which are common to all these states)

$$\psi_{\ell,j,\underbrace{m\pm\frac{1}{2}}_{m_j}} = \alpha\, \psi_{\ell,m,\frac{1}{2}} + \beta\, \psi_{\ell,m,-\frac{1}{2}}$$

with $\alpha^2+\beta^2=1$

Spin-Orbit Hydrogen atom

Finally I obtain:

$$\psi_{l+1/2,m+1/2} = \left(\frac{l+m+1}{2\,l+1}\right)^{1/2} \psi_{m,1/2} + \left(\frac{l-m}{2\,l+1}\right)^{1/2} \psi_{m,-1/2}$$

$$\psi_{l-1/2,m+1/2} = \left(\frac{l-m}{2\,l+1}\right)^{1/2} \psi_{m,1/2} - \left(\frac{l+m+1}{2\,l+1}\right)^{1/2} \psi_{m,-1/2}$$

eigenstate J

eigenstate L, S

Clebsch-Gordon coefficients

We will see them better in a while

Spin-Orbit Hydrogen atom

But why I introduced J, total angular momentum operator?

...let's see the effect of a magnetic field on the H atom:
The effect on the Hamiltonian will be?

$$H_z = (eB_0/2m_e)\,(\hat{L}_z + g_s \hat{S}_z)$$

The new H will be

$H' = H + H_z$ but still $H'|n,\ell,m,m_s\rangle = E_n|n,\ell,m,m_s\rangle$

My old eigenfuctions are still good!

Spin-Orbit Hydrogen atom

But why I introduced J, total angular momentum operator?

...let's see the effect of a magnetic field on the H atom:
The effect on the Hamiltonian will be?

$$\underline{H'|n,l,m,m_s\rangle = E_n|n,l,m,m_s\rangle}$$

$$(H+H_z)|n,l,m,m_s\rangle = E_n|n,l,m,m_s\rangle + (eB_0/2m_e)(L_z+g_sS_z)|n,l,m,m_s\rangle$$

$$= E_n|n,l,m,m_s\rangle + (eB_0\hbar/2m_e)(m+g_sm_s)|n,l,m,m_s\rangle$$

$$= [E_n + \mu_B B_0(m+g_sm_s)]|n,l,m,m_s\rangle$$

Spin-Orbit Hydrogen atom

But why I introduced J, total angular momentum operator?

...let's see the effect of a magnetic field on the H atom:
The effect on the Hamiltonian will be?

$$H'|n,l,m,m_s\rangle = E_n|n,l,m,m_s\rangle$$

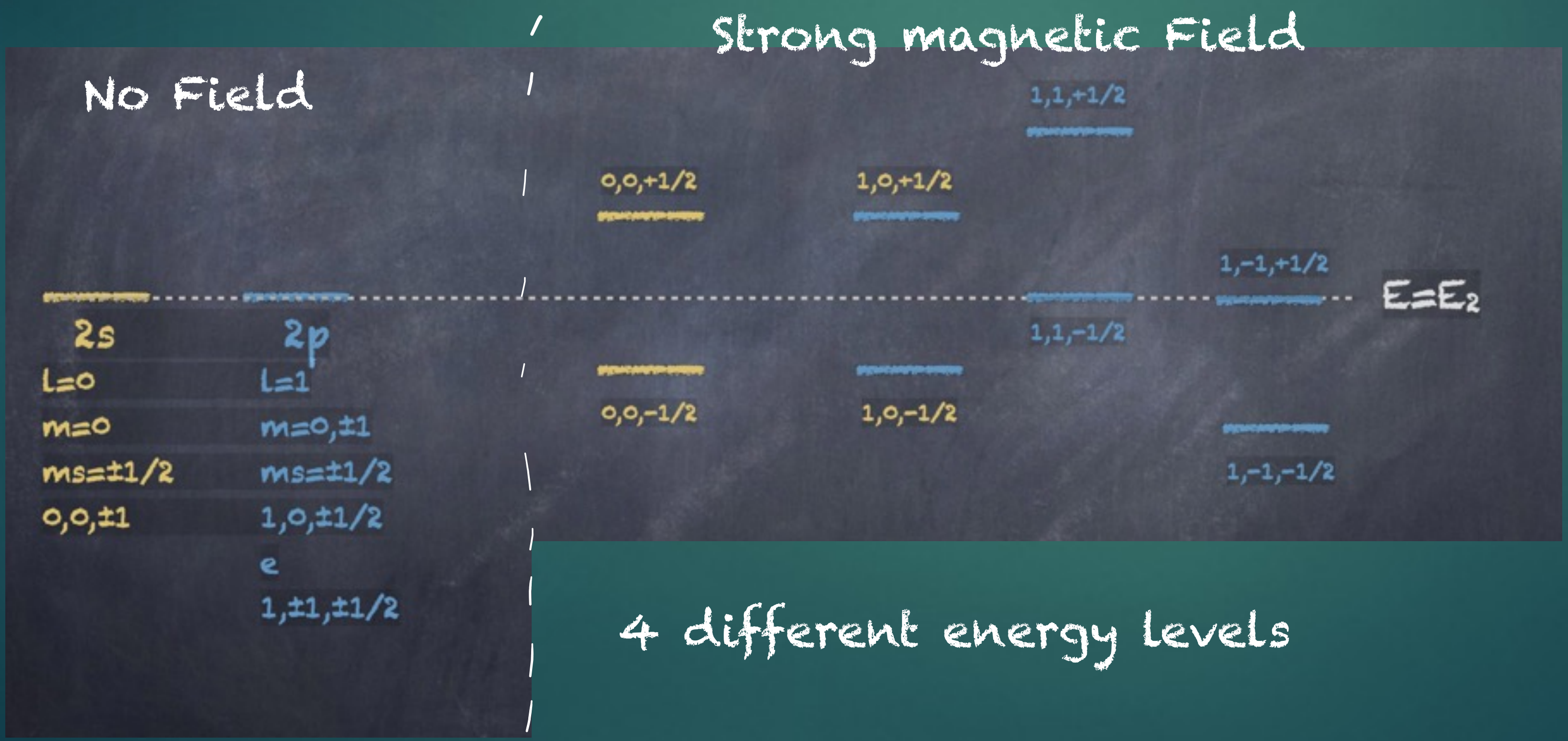
$$(H+H_z)|n,l,m,m_s\rangle = [E_n + \mu_B B_0 (m + g_s m_s)]|n,l,m,m_s\rangle$$

$$\mu_B = \frac{e\hbar}{2m_e} \longrightarrow \text{Bohr magneton}$$

Spin-Orbit Hydrogen atom

...let's see the effect of a magnetic field on the H atom:
The effect on the Energy will be?

$$E_{n,m,m_s} = E_n + \mu_B B_0 (m + g_s m_s)$$



...but what does it mean strong?

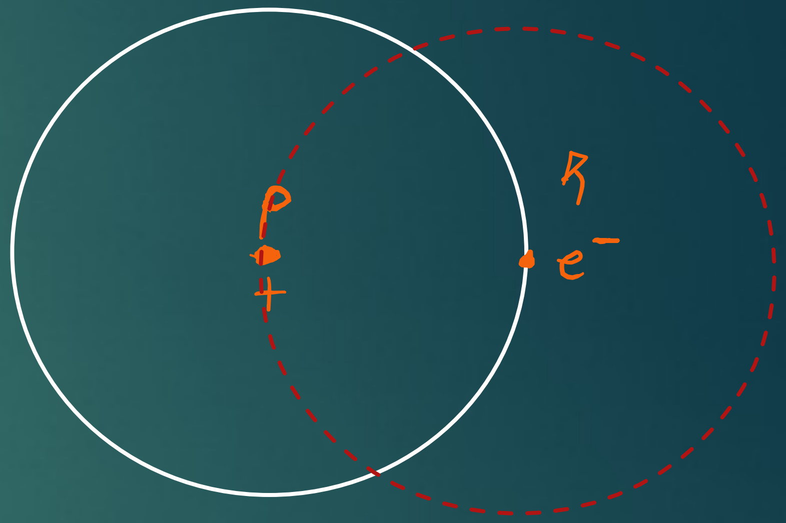
4 different energy levels

Spin-Orbit Hydrogen atom

...but what does it mean strong magnetic field?

From the "point of view" of the electron, it looks like the proton is orbiting around the electron.

According to the Biot-Savart law, the magnetic field Due to the current (charge motion) is:



$$B = \frac{\mu_0 i}{2r} = \frac{\mu_0 e}{2r T^{*}} = \frac{\mu_0 e L}{4\pi r^3 m}$$

but $c^2 = \frac{1}{\mu_0 \varepsilon_0}$ $\Rightarrow$ $\vec{B} = \frac{e \vec{L}}{4\pi \varepsilon_0 r^3 m_e c^2}$

* T is identical to the period of electron revolution

$L = m r v$

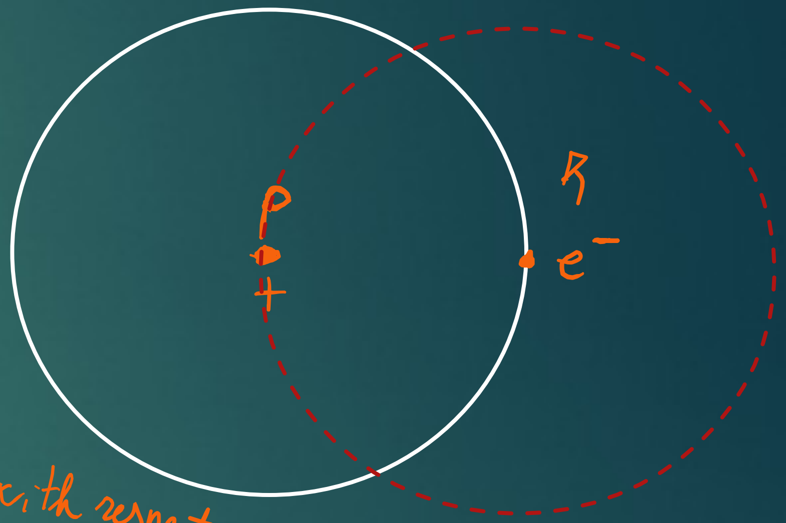
$T = \frac{2\pi r}{v}$

Spin-Orbit Hydrogen atom

…but what does it mean strong magnetic field?

From the "point of view" of the electron, it looks like the proton is orbiting around the electron.

According to the Biot-Savart law, the magnetic field Due to the current (charge motion) is:



$$B = \frac{\mu_0 i}{2r} = \frac{\mu_0 e}{2r T^{*}} = \frac{\mu_0 e L}{4\pi r^3 m}$$

$$\text{but } c^2 = \frac{1}{\mu_0 \varepsilon_0} \Rightarrow \vec{B} = \frac{e\vec{L}}{4\pi \varepsilon_0 r^3 m_e c^2}$$

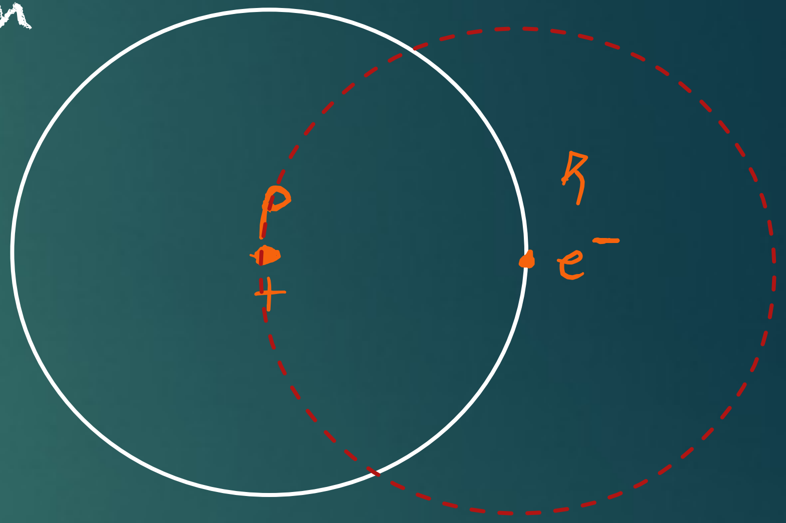
Strong with respect to this intrinsic B

*T is identical to the period of electron revolution

Spin-Orbit Hydrogen atom

…Thus

Even without any external magnetic field, the electron of the H atom will experiences the "internal" magnetic field due to its motion around the positive charge of nucleus. This internal field will be coupled to the spin angular momentum:



$$\vec{B} = \frac{e\vec{L}}{4\pi\varepsilon_0 r^3 m_e c^2}$$

$$H_{so} = \frac{e}{m_e}\vec{S}\cdot\vec{B} = \frac{e^2}{4\pi\varepsilon_0}\,\frac{1}{m_e^2 c^2 r^3}\,\vec{S}\cdot\vec{L} = f(r)\,\vec{S}\cdot\vec{L}$$

spin-orbit interaction

Spin-Orbit Hydrogen atom

The total Hamiltonian will be:

$$H = H_0 + H_{SO}$$

H_0: term we have seen without spin consideration

And what about the new energy values?

$$H_{SO} = f(r)\,\vec{S}\cdot\vec{L}\,, \quad \vec{S}\cdot\vec{L} = S_x L_x + S_y L_y + S_z L_z$$

$\Rightarrow$ while $[H_0, L_z] = 0$ $[H, L_z] \neq 0$

$[H_0, S_z] = 0$ $[H, S_z] \neq 0$

with s.o. interaction

$\Rightarrow$ m_l, m_s are not "good" quantum numbers!

Spin-Orbit Hydrogen atom

The total Hamiltonian will be:

$$H = H_0 + H_{SO}$$

term we have seen without spin consideration

$$H_{SO} = f(r)\,\vec{L}\cdot\vec{S}$$

$$\vec{S}\cdot\vec{L} = S_x L_x + S_y L_y + S_z L_z$$

$$[H_0, L_z] = 0 \qquad [H, L_z] \neq 0$$

$$[H_0, S_z] = 0 \qquad [H, S_z] \neq 0$$

with s.o. interaction

$\Rightarrow$ m_l, m_s are not "good" quantum numbers!

And what about the new energy values?
I would need for good quantum states of H...
One can easily check that

$$[J_z, H_{SO}] = 0 \qquad [J^2, H_{SO}] = 0$$

j, m_j are "good" quantum numbers!

Spin-Orbit

More in general we can write a new basis set for J, as:

$$|l,\tfrac{1}{2},j,m_j\rangle = \sum_{\substack{m,m_s \\ m_j=m+m_s}} C^{l,j}_{m,m_s,m_j}\,|l,m,\tfrac{1}{2},m_s\rangle$$

eigenstate J

eigenstate L, S

$\boxed{\text{note that}}$ $m_j = m + m_s$ and $-j \le m_j \le j$

but $m \in [-l,+l] \Rightarrow \min[m_j] = -j = -l + \frac{1}{2}$ ($m_s = \frac{1}{2}$) or $-l - \frac{1}{2}$ ($m_s = -\frac{1}{2}$)

$\max[m_j] = j = l + \frac{1}{2}$ $l - \frac{1}{2}$

Spin-Orbit

More in general:

$$|l,\tfrac{1}{2},j,m_j\rangle = \sum_{\substack{m,m_s \\ m_j=m+m_s}} C^{l,j}_{m,m_s,m_j}\,|l,m,\tfrac{1}{2},m_s\rangle$$

Thus

$$j \in \left[-l-\tfrac{1}{2},\ l+\tfrac{1}{2}\right]$$

$$m_j \in [-j,\ j]$$

Sum of angular momenta

Let's generalize even more, and suppose that we have two angular momenta operators:

$$\hat{J}_1 \text{ and } \hat{J}_2$$

And let's build a new operator, being the total angular momentum operator:

$$\hat{J} = \hat{J}_1 + \hat{J}_2$$

$$\begin{array}{llllll}
j & j_1 & j_2 & \rightarrow & \text{eigenvalues} & J_1^2, J_2^2 \\
m_j & m_1 & m_2 & \rightarrow & \text{"} & J_{z1}, J_{z2}
\end{array}$$

Sum of angular momenta

Let's generalize even more, and suppose that we have two angular momenta operators:

$$\hat{J} = \hat{J}_1 + \hat{J}_2$$

Clebsch-Gordon coefficients

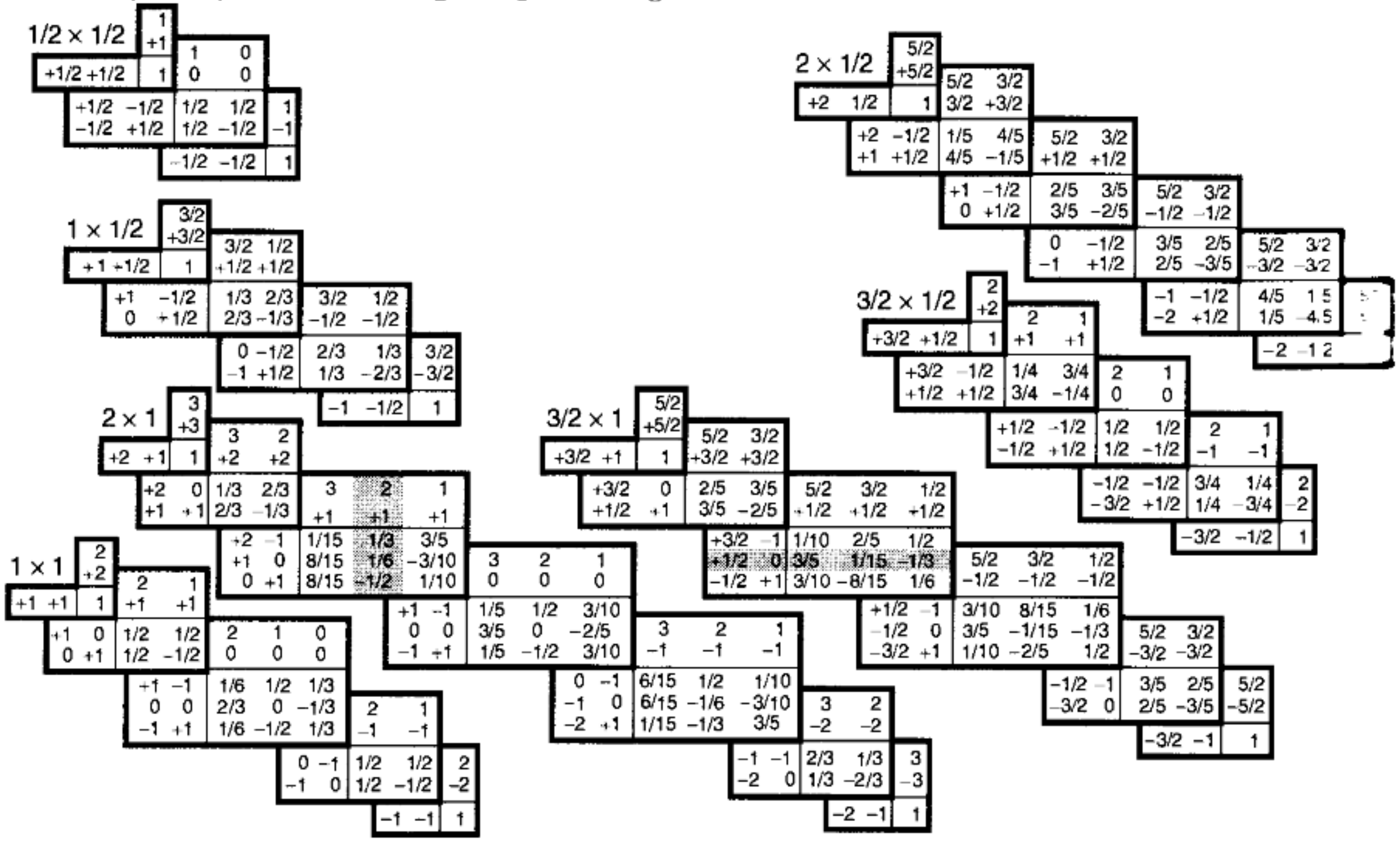
$$|j_1, j_2, j, m_j\rangle = \sum_{\substack{m_1, m_2 \\ m_j = m_1 + m_2}} C^{j_1, j_2, j}_{m_1, m_2, m_j} |j_1, m_1, j_2, m_2\rangle$$

with $\left.\begin{matrix} J_1 = L \\ J_2 = S \end{matrix}\right\}$ we have same as before

Sum of angular momenta

Clebsch-Gordon coefficients, how to read the table:

Table 4.7: Clebsch-Gordan coefficients. (A square root sign is understood for every entry; the minus sign, if present, goes *outside* the radical.)



Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

$$\left|\tfrac{1}{2},\tfrac{1}{2},j,m_j\right\rangle = \sum_{\substack{m_{s1},m_{s2}\\ m_j=m_{s1}+m_{s2}}} C^{j}_{m_{s1},m_{s2},m_j} \left|\tfrac{1}{2},m_{s1},\tfrac{1}{2},m_{s2}\right\rangle$$

$j_1=s_1$ $\quad j_2=s_2$ $\quad s_1$ $\quad s_1$

$$m_j = m_{s1}+m_{s2} \Rightarrow \quad \max[m_j] = \tfrac{1}{2}+\tfrac{1}{2} = 1 \quad \min[m_j] = -1 \longrightarrow -j < m_j < j \Downarrow \max[j] = 1$$

$$j \in [0,1] \qquad m_j \in [j, j-1, \dots -j]$$

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

$$\left|\tfrac{1}{2},\tfrac{1}{2},j,m_j\right\rangle = \sum_{\substack{m_{s1},m_{s2} \\ m_j = m_{s1}+m_{s2}}} C^{j}_{m_{s1},m_{s2},m_j} \left|\tfrac{1}{2}, m_{s1}, \tfrac{1}{2}, m_{s2}\right\rangle$$

$j_1 = s_1$ $j_2 = s_2$ s_1 s_1

$j \in [0,1]$ $m_j \in [j, j-1, \dots, -j]$

Case 1: $J = 1$ $m_j = 1$

$$\left|\tfrac{1}{2},\tfrac{1}{2},1,1\right\rangle = \ ? \qquad \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle \equiv \uparrow\uparrow$$

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

Notation:		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
⋮	⋮			

1/2 × 1/2		1			
		+1	1	0	
+1/2	+1/2	1	0	0	
+1/2	−1/2		1/2	1/2	1
−1/2	+1/2		1/2	−1/2	−1
			−1/2	−1/2	1

means that I'm summing two angular momentum both with value $j_1 = j_2 = \frac{1}{2}$

$\left|\frac{1}{2}, \frac{1}{2}, 1, 1\right\rangle =$?

$\left|\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle$

$$|j_1, j_2, j, m_j\rangle = \sum_{\substack{m_1, m_2 \\ m_j = m_1 + m_2}} C^{j_1, j_2, j}_{m_1, m_2, m_j} |j_1, m_1, j_2, m_2\rangle$$

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
⋮	⋮			

$1/2 \times 1/2$

		1	1	0	1
		+1	0	0	−1
+1/2	+1/2	1			
+1/2	−1/2		1/2	1/2	
−1/2	+1/2		1/2	−1/2	
−1/2	−1/2				1

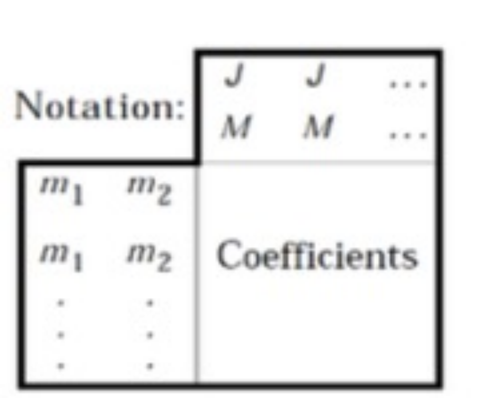
means that I'm summing two angular momentum both with value $j_1=j_2=\frac{1}{2}$

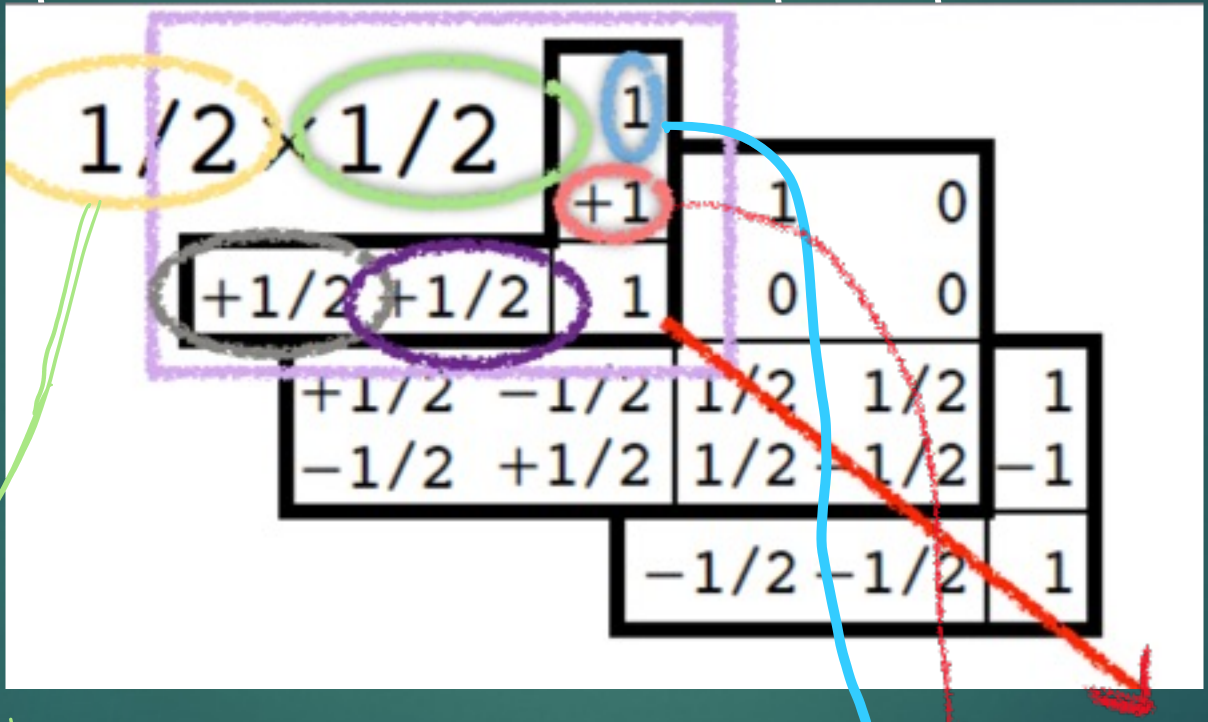
$$\left|\tfrac{1}{2},\tfrac{1}{2},1,1\right\rangle = \ ? \qquad \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$|j_1,j_2,j,m_j\rangle = \sum_{\substack{m_1,m_2\\ m_j=m_1+m_2}} C^{j_1,j_2,j}_{m_1,m_2,m_j}\,|j_1,m_1,j_2,m_2\rangle$$

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :





means that I'm summing two angular momentum both with value $j_1 = j_2 = \frac{1}{2}$

$$\left|\tfrac{1}{2}, \tfrac{1}{2}, 1, 1\right\rangle = \ ? \qquad \left|\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

$$|j_1, j_2, j, m_j\rangle = \sum_{\substack{m_1, m_2 \\ m_j = m_1 + m_2}} C^{j_1, j_2, j}_{m_1, m_2, m_j} \, |j_1, m_1, j_2, m_2\rangle$$

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

Notation:	J	J	...
	M	M	...
m_1 m_2	Coefficients		
m_1 m_2			
⋮ ⋮			

1/2 × 1/2	1			
	+1	1	0	
+1/2 +1/2	1	0	0	
+1/2 −1/2		1/2	1/2	1
−1/2 +1/2		1/2	−1/2	−1
−1/2 −1/2				1

means
that I'm summing
two angular momentum
both with value $j_1=j_2=\frac{1}{2}$

$$\left|\tfrac{1}{2},\tfrac{1}{2},1,1\right\rangle = \ ? \qquad \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$|j_1,j_2,j,m_j\rangle = \sum_{\substack{m_1,m_2 \\ m_j=m_1+m_2}} C^{j_1,j_2,j}_{m_1,m_2,m_j}\,|j_1,m_1,j_2,m_2\rangle$$

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

Notation:		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
⋮	⋮			

$1/2 \times 1/2$		1			
		+1	1	0	
+1/2	+1/2	1	0	0	
+1/2	−1/2		1/2	1/2	1
−1/2	+1/2		1/2	−1/2	−1
			−1/2	−1/2	1

means that I'm summing two angular momentum both with value $j_1 = j_2 = \frac{1}{2}$

$$\overset{j\ \ m_j}{|1,1\rangle} = \uparrow\uparrow$$

$$\left|\tfrac{1}{2},\tfrac{1}{2},1,1\right\rangle = 1 \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

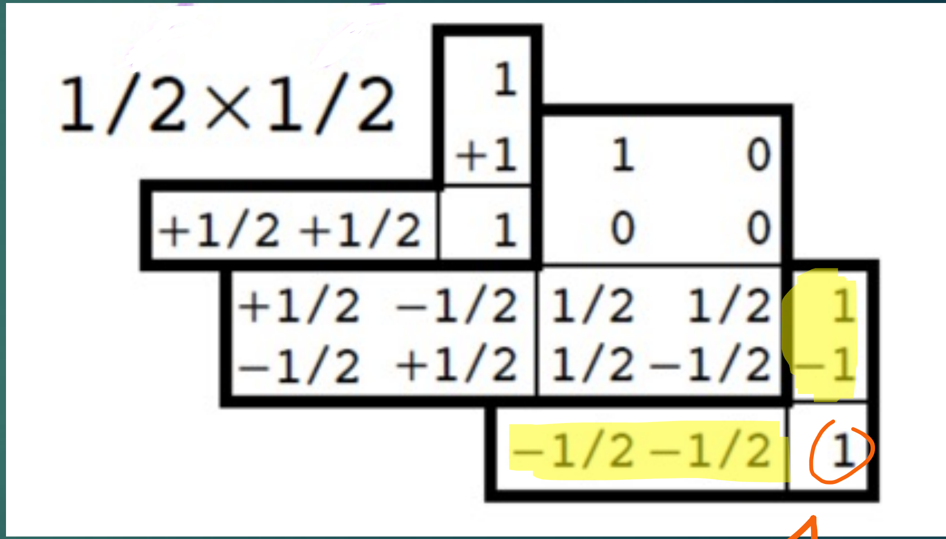
$$|j_1, j_2, j, m_j\rangle = \sum_{\substack{m_1, m_2 \\ m_j = m_1 + m_2}} C^{j_1, j_2, j}_{m_1, m_2, m_j} |j_1, m_1, j_2, m_2\rangle$$

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

$$\left|\tfrac{1}{2},\tfrac{1}{2},j,m_j\right\rangle = \sum_{\substack{m_{s1},m_{s2}\\ m_j=m_{s1}+m_{s2}}} C^{j}_{m_{s1},m_{s2},m_j} \left|\tfrac{1}{2},m_{s1},\tfrac{1}{2},m_{s2}\right\rangle$$

$j_1=s_1$ $j_2=s_2$ s_1 s_1



$j \in [0,1]$ $m_j \in [j, j-1, \ldots, j]$

Case 2 : $J=1$ $m_j=-1$

$\left|\tfrac{1}{2},\tfrac{1}{2},1,-1\right\rangle = \;?\; \left|\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$ $\Rightarrow |1\,-1\rangle = \downarrow\downarrow$

$= \downarrow\downarrow$

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

$$\left|\tfrac{1}{2},\tfrac{1}{2},j,m_j\right\rangle = \sum_{\substack{m_{s1},m_{s2}\\ m_j=m_{s1}+m_{s2}}} C^{\,j}_{m_{s1},m_{s2},m_j}\left|\tfrac{1}{2},m_{s1},\tfrac{1}{2},m_{s2}\right\rangle$$

$j_1=s_1$ $\quad j_2=s_2$ $\quad s_1$ $\quad s_1$

$j\in[0,1]$ $\qquad m_j\in[j,j-1,\dots,-j]$

1/2×1/2	1			
	+1	1	0	
+1/2 +1/2	1	0	0	
+1/2 −1/2		1/2	1/2	1
−1/2 +1/2		1/2	−1/2	−1
−1/2 −1/2				1

Case 3: $J=1$ $\quad m_j=0$

$\left|\tfrac{1}{2},\tfrac{1}{2},1,0\right\rangle = \;?$ $\qquad \left|\tfrac{1}{2},\pm\tfrac{1}{2},\tfrac{1}{2},\mp\tfrac{1}{2}\right\rangle$

$\Rightarrow \left|\tfrac{1}{2},\tfrac{1}{2},1,0\right\rangle = \frac{1}{\sqrt{2}}\left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle$

$\equiv \downarrow\uparrow$ or $\uparrow\downarrow$

$\Rightarrow |1,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

$$\left|\tfrac{1}{2},\tfrac{1}{2},j,m_j\right\rangle = \sum_{\substack{m_{s_1},m_{s_2}\\ m_j=m_{s_1}+m_{s_2}}} C^{j}_{m_{s_1},m_{s_2},m_j}\left|\tfrac{1}{2},m_{s_1},\tfrac{1}{2},m_{s_2}\right\rangle$$

$j_1=s_1$ $\quad j_2=s_2$ $\quad s_1$ $\quad s_1$

$j\in[0,1]$ $\qquad m_j\in[j,j-1,\dots,-j]$

Case 4: $J=0$ $\quad m_j=0$

$\left|\tfrac{1}{2},\tfrac{1}{2},0,0\right\rangle=$? $\qquad \left|\tfrac{1}{2},\mp\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle$

$= \downarrow\uparrow$ or $\uparrow\downarrow$

$\Rightarrow \left|\tfrac{1}{2},\tfrac{1}{2},0,0\right\rangle = \frac{1}{\sqrt{2}}\left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left|\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle$

$\Rightarrow |0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

1/2×1/2

	1			
	+1	1	0	
+1/2 +1/2	1	0	0	
+1/2 −1/2		1/2	1/2	1
−1/2 +1/2		1/2	−1/2	−1
−1/2 −1/2				1

Sum of angular momenta

Suppose that the two operators J1 and J2 are spin operators with s=½ :

Thus, if $J = S = 1$

$$\left.\begin{array}{ll} m_j = 1 & |1\ 1\rangle = \uparrow\uparrow \\ m_j = 0 & |1\ 0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ m_j = -1 & |1\ {-1}\rangle = \downarrow\downarrow \end{array}\right\} \text{triplet state}$$

if $J = S = 0$

$$m_j = 0 \quad |0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \Big\} \text{ Singlet state}$$