Velocità di fase e di gruppo

$$v_f = \frac{\omega(k)}{k} \qquad u(x,t) = u_0 e^{i(kx - \overset{(k)}{\omega} t)} = u_0 e^{ik(x - \overset{\to v_f}{\frac{\omega(k)}{k}} t)} = f(x - vt)$$

In un mezzo non dispersivo: v_f = const $\forall k \in$ pacchetto. Il pacchetto viaggia con v_f.

" " " dispersivo: $v_f = v_f(k)$ In generale: deformazione del pacchetto

$[\, v_0(x) = 0 \,]$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \mathrm{Re}\left[\int_{-\infty}^{+\infty} dk\, A(k)\, e^{i(kx - \omega(k) t)}\right]$$

Mezzo non disp.

$$\frac{\omega}{k} = \underset{\llcorner \text{const}}{v_f} \Rightarrow \omega = v_f \cdot k$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \mathrm{Re}\left[\int_{-\infty}^{+\infty} dk\, A(k)\, e^{i(kx - \omega(k_0)t - v_g(k - k_0)t}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \mathrm{Re}\left[\left(\int_{-\infty}^{+\infty} dk\, A(k)\, e^{ik(x - v_g t)}\right) e^{i k_0 v_g t - \omega(k_0) t)}\right]$$

Mezzo disp. ω ha termini non

$\omega = k \cdot v_f(k)$ $\to v_f$ prop. a k

$$\omega(k) \simeq \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k_0} (k - k_0) + \dots$$

k_0: numero d'onda dominante $\Rightarrow A(k_0) >> A(k)$ $k \neq k_0$

Se $v_0(x) = 0$; $\quad A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, u_0(x) \, e^{-ikx}$

e $\quad u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \, A(k) e^{ikx}$

$$u(x,t) \simeq R_e \Big[\underbrace{u_0(x - v_g t)} \cdot e^{i k_0 v_g t - \omega(k_0) t} \Big] = \underbrace{u_0(x - v_g t)}_{\substack{\text{Profilo iniziale} \\ \text{dell'onda spostato} \\ \text{di } v_g t}} \; \underbrace{\cos\big[(k_0 v_g - \omega(k_0))t\big]}_{\substack{\text{modulazione} \\ \text{temporale} \\ \text{dell'ampiezza}}}$$

$v_g = \frac{d\omega}{dk}\Big|_{k=k_0}$

Se $v_g = \frac{\omega(k_0)}{k_0} = v_f(k_0)$

allora

$\Rightarrow \cos(0) = 1$

Se il mezzo è non dispersivo

$v_f = \frac{\omega}{k} \rightarrow \omega = v_f k$

$v_g = \frac{d\omega}{dk}\Big|_{k_0} = v_f$

Principio di indeterm. per onde

$$u(x)$$

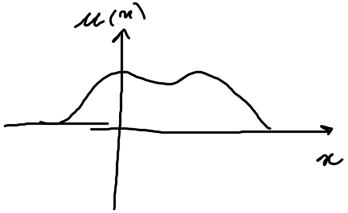
$$\tilde{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, u(x) \, e^{-ikx}$$

$$\sigma_x = \sqrt{\frac{\sum (x - \langle x \rangle)^2}{N}} = \sqrt{\langle x^2 \rangle - (\langle x \rangle)^2}$$

$$= \sqrt{\langle x^2 \rangle}$$

Scegliamo gli assi in modo che $\langle x \rangle = 0$

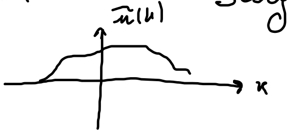
$$\langle x \rangle = \frac{\int_{-\infty}^{+\infty} x \, \overbrace{u^2 dx}^{\text{energia}} \cancel{dS}}{\int_{-\infty}^{+\infty} u^2 dx \, \cancel{dS}} = 0 \quad \text{impongo}$$



$$\begin{tikzcd} & E \\ u \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd}$$

$$\langle x^2 \rangle = \frac{\int_{-\infty}^{+\infty} x^2 u^2 dx}{\int_{-\infty}^{+\infty} u^2 dx}$$

$$\sigma_k = \sqrt{\langle k^2 \rangle}$$



Scelgo S.R. nello spazio dei numeri d'onda in modo che $\langle k \rangle = 0$

$$\langle k^2 \rangle = \frac{\int_{-\infty}^{+\infty} k^2 \overbrace{\tilde{u}^2(k)}^{\text{energia associata a } k} dk}{\int_{-\infty}^{+\infty} \tilde{u}^2(k) \, dk}$$

$$\langle f|g\rangle \overset{def}{=} \int_{-\infty}^{+\infty} f(x)\, g^*(x)\, dx$$

Def
$$f(x) = u(x)\cdot x$$
$$\tilde{g}(k) = k\cdot \tilde{u}(k)$$

$$f(x) \longrightarrow \tilde{f}(k)$$

$$\sigma_x^2 = \frac{\langle f|f\rangle}{\langle u|u\rangle} \qquad \sigma_k^2 = \frac{\langle \tilde{g}|\tilde{g}\rangle}{\langle \tilde{u}|\tilde{u}\rangle}$$

$$\int |f|^2 dx = \int |\tilde{f}|^2 dk$$ Th Parseval

$$\sigma_x^2\,\sigma_k^2 = \frac{\langle f|f\rangle}{\langle u|u\rangle}\,\frac{\langle \tilde{g}|\tilde{g}\rangle}{\langle \tilde{u}|\tilde{u}\rangle} \overset{\uparrow}{=} \frac{\langle f|f\rangle\,\langle g|g\rangle}{\langle u|u\rangle\langle u|u\rangle} \geq \frac{|\langle f|g\rangle|^2}{\langle u|u\rangle\langle u|u\rangle}$$

Th Parseval

$$\|f\|^2\,\|g\|^2 \geq (\langle f|g\rangle)^2$$ Dis. Schwarz

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\; k\,\tilde{u}(k)\, e^{ikx}$$

$$\tilde{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\xi\; \underset{a(\xi)}{u(\xi)}\; \underset{b'(\xi) = e^{-ik\xi}}{e^{-ik\xi}}$$ Int. per parti

$$= \frac{1}{\sqrt{2\pi}}\left(\cancel{-\frac{1}{ik}\, u\, e^{-ik\xi}\Big|_{-\infty}^{+\infty}} + \frac{1}{ik}\int_{-\infty}^{+\infty} d\xi\; \frac{du}{d\xi}\, e^{-ik\xi} \right)$$

$$\tilde{u}(k) = \frac{1}{ik}\,\widetilde{\frac{d\,u(\xi)}{d\xi}}$$

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\,\cancel{k}\,\frac{1}{i\cancel{k}}\,\widetilde{\frac{du}{dx}}\,e^{ikx} = \frac{du}{dx}\cdot\frac{1}{i} = -i\,\frac{du}{dx}$$

$$\langle f|g\rangle = i\int_{-\infty}^{+\infty} dx\;x\,u\,\overbrace{\frac{du}{dx}}^{\frac{d}{dx}\frac{u^2}{2}} =$$

per parti

$$= \frac{i}{2}\int_{-\infty}^{+\infty} dx\;x\,\underbrace{\frac{d}{dx}u^2} = \frac{i}{2}\left[\cancel{x u^2\Big|_{-\infty}^{+\infty}} - \int dx\,u^2\right] = -\frac{i}{2}\|u\|^2$$

$$\sigma_x^2\sigma_k^2 \geq \frac{\|\langle f|g\rangle\|^2}{\|u\|^2\|u^2\|} = \frac{1}{4}\,\frac{\cancel{\|u\|^2}\cancel{\|u\|^2}}{\cancel{\|u\|^2}\cancel{\|u\|^2}} \Rightarrow \boxed{\sigma_x\sigma_k \geq \frac{1}{2}}$$

Ipotesi De Broglie

MQ $p = \hbar k$

$\sigma_p = \hbar\,\sigma_k$

$$\boxed{\sigma_x\sigma_p \geq \frac{\hbar}{2}}$$