Formulario per l'esame del corso di Elementi di Fisica dei Plasmi*

December 20, 2022

1 Introduzione alla fisica dei plasmi

- Equazione di Saha-Boltzmann: $\frac{n_i}{n} = \left(k \frac{k_B T}{p} \frac{(2\pi m_e k_B T)^{3/2}}{h^3} \exp\left(-I/k_B T\right) \right)^{1/2}$
- Distribuzione di Maxwell-Boltzmann: $f(\mathbf{v}) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T} \right)$.
- Lunghezza di Debye (elettronica): $\frac{1}{\lambda_D^2} = \frac{n_e e^2}{\epsilon_0 k_B T_e}$
- Frequenza di plasma (elettronica): $\omega_P^2 = \frac{n_e e^2}{m \epsilon_0}$

2 Descrizione a singola particella: moto delle cariche in un plasma

- Raggio di Larmor: $r_L = \frac{mv_\perp}{qB}$
- Frequenza di Larmor: $\omega_L = \frac{qB}{m}$
- Velocità di deriva: $\mathbf{v_d} = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}$
- Velocità di polarizzazione: $\mathbf{v_P} = -\frac{m}{qB^2} \frac{d\mathbf{v_D}}{dt} \times \mathbf{B}$
- Momento magnetico: $\mu = \frac{mv_\perp^2}{2B}$
- Forza dovuta al gradiente del campo magnetico: $\mathbf{F_{\nabla B}} = -\mu \nabla B$
- Forza dovuta alla curvatura delle linee di campo magnetico: $\mathbf{F_c} = -mv_{||}^2 \left(\hat{\mathbf{b}} \cdot \nabla \right) \hat{\mathbf{b}}$
- Momento canonico: $p_j = \frac{\partial L}{\partial \dot{q}_j}$ $j = 1 \dots N$
- Equazioni di Lagrange: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$ $j = 1 \dots N$
- Integrale di azione: $I_j = \oint p_j dq_j$

*Per il significato dei simboli, si rimanda alle lezioni.

- Lagrangiana della particella carica in un campo elettromagnetico: $L = \frac{1}{2}mv^2 + q\mathbf{A}(\mathbf{x},t) \cdot \mathbf{v} - q\phi(\mathbf{x},t)$, con $\mathbf{B}(\mathbf{x},t) = \nabla \times \mathbf{A}(\mathbf{x},t)$ e $\mathbf{E}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t)$
- Angolo critico per il confinamento in uno specchio magnetico: $\sin^2\theta_C = \frac{B_{min}}{B_{max}}$

3 Il plasma come un fluido carico

- Equazione di continuità per la specie α: $\nabla \cdot (n_\alpha(\mathbf{x},t)\mathbf{u}_\alpha(\mathbf{x},t)) + \frac{\partial n_\alpha(\mathbf{x},t)}{\partial t} = 0$
- Equazione di conservazione del momento lineare per la specie α, senza collisioni: $mn_\alpha(\mathbf{x},t)\frac{d\mathbf{u}_\alpha(\mathbf{x},t)}{dt} = n_\alpha(\mathbf{x},t)q_\alpha(\mathbf{E} + \mathbf{u}_\alpha(\mathbf{x},t) \times \mathbf{B}) - \nabla p(\mathbf{x},t)$
- Equazione di stato per la specie α: $\frac{\nabla p_\alpha(\mathbf{x},t)}{p_\alpha(\mathbf{x},t)} = \gamma\frac{\nabla n_\alpha(\mathbf{x},t)}{n_\alpha(\mathbf{x},t)}$
- Equazioni di Maxwell, date $\rho(\mathbf{x},t) = \sum_\alpha q_\alpha n_\alpha(\mathbf{x},t)$ e $\mathbf{j}(\mathbf{x},t) = \sum_\alpha q_\alpha n_\alpha(\mathbf{x},t)\mathbf{u}_\alpha(\mathbf{x},t)$:

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{x},t) &= \frac{\rho(\mathbf{x},t)}{\varepsilon_0} \\
\nabla \times \mathbf{E}(\mathbf{x},t) &= -\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} \\
\nabla \cdot \mathbf{B}(\mathbf{x},t) &= 0 \\
\nabla \times \mathbf{B}(\mathbf{x},t) &= \mu_0\mathbf{J}(\mathbf{x},t) + \varepsilon_0\mu_0\frac{\partial \mathbf{E}(\mathbf{x},t)}{\partial t}
\end{aligned}$$

- Velocità di deriva diamagnetica: $\mathbf{u} = -\frac{\nabla p \times \mathbf{B}}{qnB^2}$
- Equazione di continuità in MHD: $\frac{d\rho}{dt} + \rho\nabla \cdot \mathbf{u} = 0$
- Equazione di conservazione del momento in MHD: $\rho\frac{d\mathbf{u}}{dt} = \mathbf{j} \times \mathbf{B} - \nabla p$
- Legge di Ohm per la MHD (resistiva): $\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta\mathbf{j}$
- Equazione di stato in MHD: $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$
- Equazioni di Maxwell per la MHD:

$$\begin{aligned}
Z_i n_i &= n_e \\
\nabla \times \mathbf{E}(\mathbf{x},t) &= -\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} \\
\nabla \cdot \mathbf{B}(\mathbf{x},t) &= 0 \\
\nabla \times \mathbf{B}(\mathbf{x},t) &= \mu_0\mathbf{J}(\mathbf{x},t)
\end{aligned}$$

4 Equilibri in MHD

- Equazione dell'equilibrio in MHD: $\nabla p = \mathbf{j} \times \mathbf{B}$
  oppure: $\nabla_\perp \left(p + \frac{B^2}{2\mu_0} \right) - \hat{\mathbf{k}} \frac{B^2}{\mu_0} = 0$ con $\hat{\mathbf{k}} = \left(\hat{\mathbf{b}} \cdot \nabla \right) \hat{\mathbf{b}}$
- Parametro β: $\beta = \frac{2\mu_0 p}{B^2}$
- Legge di gelo: $\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS} = 0$

5 Elementi di onde nei plasmi

- Serie di Fourier mediante esponenziali complessi: $f(x) = \sum_{m=-\infty}^{+\infty} c_m e^{i\frac{2\pi m x}{a}}$ con $c_m = \frac{1}{a} \int_{-a/2}^{a/2} dx f(x) e^{-i\frac{2\pi m x}{a}}$
- Trasformata di Fourier: $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$
- Anti-trasformata di Fourier: $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(k) e^{ikx} dk$
- Equazione delle onde elettromagnetiche nel vuoto: $\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} = 0$ (analogo per il vettore $\mathbf{B}$)
- Soluzione generale dell'equazione delle onde mediante somma di onde monocromatiche: $u(x,t) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} dk A(k) e^{ikx - \omega(k)t} + c.c. \right)$ con $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \left(u_0(x) + \frac{i}{\omega(k)} v_0(x) \right) e^{-ikx}$
- Velocità di fase $v_f = \frac{\omega(k)}{k}$ e di gruppo $v_g = \frac{d\omega(k)}{dk}|_{k_0}$
- Principio di indeterminazione: $\sigma_x \cdot \sigma_k \geq \frac{1}{2}$
- Relazione di dispersione per un plasma omogeneo privo di campo magnetico: $1 - \sum_s \frac{\omega_{P_s}^2}{\omega^2 - \gamma k^2 \frac{T_{0_s}}{m_s}} = 0$
- Relazione di dispersione delle onde iono-acustiche: $\omega^2 = k^2 c_s^2 + \frac{3}{2} k^2 v_{th_i}^2$ con $c_s^2 = \frac{T_e}{m_i}$
- Relazione di dispersione delle onde di Bohm-Gross: $\omega^2 = \omega_{P_e}^2 + \frac{3}{2} k^2 v_{th_e}^2$
- Relazione di dispersione delle onde elettromagnetiche: $\omega^2 = k^2 c^2 + \omega_{P_e}^2$
- Velocità di Alfven: $v_A^2 = \frac{B^2}{\mu_0 \rho}$; velocità del suono: $c_S^2 = \frac{\gamma p_0}{\rho_0}$.
- Relazione di dipersione delle onde di Alfven torsionali: $\omega = k_{||} v_A$.
- Relazione di dispersione delle onde di Alfven compressionali: $\omega = k v_A$.
- Relazione di dipersione delle onde sonore: $\omega = k c_S$.

A Identità e operatori vettoriali

Le formule nel seguito sono tratte da A. S Richardson, "2019 NRL PLASMA FORMULARY"

VECTOR IDENTITIES[4]

Notation: f, g, are scalars; **A**, **B**, etc., are vectors; ***T*** is a tensor; ***I*** is the unit dyad.

(1) $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A}$

$$= \mathbf{B}\times\mathbf{C}\cdot\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{C}\times\mathbf{A}\cdot\mathbf{B}$$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\times\mathbf{B})\times\mathbf{A} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

(3) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$

(4) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

(5) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D}$

(6) $\nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f$

(7) $\nabla\cdot(f\mathbf{A}) = f\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f$

(8) $\nabla\times(f\mathbf{A}) = f\nabla\times\mathbf{A} + \nabla f\times\mathbf{A}$

(9) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$

(10) $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(11) $\mathbf{A}\times(\nabla\times\mathbf{B}) = (\nabla\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(12) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(13) $\nabla^2 f = \nabla\cdot\nabla f$

(14) $\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$

(15) $\nabla\times\nabla f = 0$

(16) $\nabla\cdot\nabla\times\mathbf{A} = 0$

If $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are orthonormal unit vectors, a second-order tensor ***T*** can be written in the dyadic form

(17) $\boldsymbol{T} = \sum_{i,j} T_{ij}\mathbf{e}_i\mathbf{e}_j$

In cartesian coordinates the divergence of a tensor is a vector with components

(18) $(\nabla\cdot\boldsymbol{T})_i = \sum_j(\partial T_{ji}/\partial x_j)$

[This definition is required for consistency with Eq. (29)]. In general

(19) $\nabla\cdot(\mathbf{AB}) = (\nabla\cdot\mathbf{A})\mathbf{B} + (\mathbf{A}\cdot\nabla)\mathbf{B}$

(20) $\nabla \cdot (f\boldsymbol{T}) = \nabla f \cdot \boldsymbol{T} + f\nabla \cdot \boldsymbol{T}$

Let $\mathbf{r} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$ be the radius vector of magnitude r, from the origin to the point x, y, z. Then

(21) $\nabla \cdot \mathbf{r} = 3$

(22) $\nabla \times \mathbf{r} = 0$

(23) $\nabla r = \mathbf{r}/r$

(24) $\nabla(1/r) = -\mathbf{r}/r^3$

(25) $\nabla \cdot (\mathbf{r}/r^3) = 4\pi\delta(\mathbf{r})$

(26) $\nabla \mathbf{r} = \boldsymbol{I}$

If V is a volume enclosed by a surface S and $d\mathbf{S} = \mathbf{n}dS$, where $\mathbf{n}$ is the unit normal outward from V,

(27) $$\int_V dV \nabla f = \int_S d\mathbf{S} f$$

(28) $$\int_V dV \nabla \cdot \mathbf{A} = \int_S d\mathbf{S} \cdot \mathbf{A}$$

(29) $$\int_V dV \nabla \cdot \boldsymbol{T} = \int_S d\mathbf{S} \cdot \boldsymbol{T}$$

(30) $$\int_V dV \nabla \times \mathbf{A} = \int_S d\mathbf{S} \times A$$

(31) $$\int_V dV(f\nabla^2 g - g\nabla^2 f) = \int_S d\mathbf{S} \cdot (f\nabla g - g\nabla f)$$

(32) $$\int_V dV(\mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \nabla \times \mathbf{A}) = \int_S d\mathbf{S} \cdot (\mathbf{B} \times \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \times \mathbf{B})$$

If S is an open surface bounded by the contour C, of which the line element is $d\mathbf{l}$,

(33) $$\int_S d\mathbf{S} \times \nabla f = \oint_C d\mathbf{l} f$$

(34) $$\int_S d\mathbf{S} \cdot \nabla \times \mathbf{A} = \oint_C d\mathbf{l} \cdot \mathbf{A}$$

(35) $$\int_S (d\mathbf{S} \times \nabla) \times \mathbf{A} = \oint_C d\mathbf{l} \times \mathbf{A}$$

(36) $$\int_S d\mathbf{S} \cdot (\nabla f \times \nabla g) = \oint_C f\,dg = -\oint_C g\,df$$

DIFFERENTIAL OPERATORS IN CURVILINEAR COORDINATES[5]

Cylindrical Coordinates (r, θ, z)

Differential volume

$$d\tau = r\,dr\,d\theta\,dz$$

Line element

$$d\mathbf{l} = dr\,\mathbf{r} + r d\theta\,\boldsymbol{\theta} + dz\,\mathbf{z}$$

Relation to cartesian coordinates

$$x = r\cos\theta \qquad \mathbf{x} = \cos\theta\,\mathbf{r} - \sin\theta\,\boldsymbol{\theta}$$

$$y = r\sin\theta \qquad \mathbf{y} = \sin\theta\,\mathbf{r} + \cos\theta\,\boldsymbol{\theta}$$

$$z = z \qquad \mathbf{z} = \mathbf{z}$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\phi = \frac{1}{r}\frac{\partial f}{\partial \phi}; \quad (\nabla f)_z = \frac{\partial f}{\partial z}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}$$

$$(\nabla \times \mathbf{A})_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\nabla \times \mathbf{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}$$

Laplacian

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian of a vector

$$(\nabla^2 \mathbf{A})_r = \nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi} - \frac{A_r}{r^2}$$

$$(\nabla^2 \mathbf{A})_\phi = \nabla^2 A_\phi + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r^2}$$

$$(\nabla^2 \mathbf{A})_z = \nabla^2 A_z$$

Components of $(\mathbf{A}\cdot\nabla)\mathbf{B}$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_r}{\partial \phi} + A_z\frac{\partial B_r}{\partial z} - \frac{A_\phi B_\phi}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\phi = A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_\phi}{\partial \phi} + A_z\frac{\partial B_\phi}{\partial z} + \frac{A_\phi B_r}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_z = A_r\frac{\partial B_z}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_z}{\partial \phi} + A_z\frac{\partial B_z}{\partial z}$$

Divergence of a tensor

$$(\nabla\cdot\boldsymbol{T})_r = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) + \frac{1}{r}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{\partial T_{zr}}{\partial z} - \frac{T_{\phi\phi}}{r}$$

$$(\nabla\cdot\boldsymbol{T})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rT_{r\phi}) + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{z\phi}}{\partial z} + \frac{T_{\phi r}}{r}$$

$$(\nabla\cdot\boldsymbol{T})_z = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rz}) + \frac{1}{r}\frac{\partial T_{\phi z}}{\partial \phi} + \frac{\partial T_{zz}}{\partial z}$$

Spherical Coordinates (r, θ, ϕ)

Differential volume

$$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Line element

$$d\mathbf{l} = dr\,\mathbf{r} + r d\theta\,\boldsymbol{\theta} + r\sin\theta d\phi\,\boldsymbol{\phi}$$

Relation to cartesian coordinates

$$x = r\sin\theta\cos\phi \qquad \mathbf{x} = \sin\theta\cos\phi\,\mathbf{r} + \cos\theta\cos\phi\,\boldsymbol{\theta} - \sin\phi\,\boldsymbol{\phi}$$

$$y = r\sin\theta\sin\phi \qquad \mathbf{y} = \sin\theta\sin\phi\,\mathbf{r} + \cos\theta\sin\phi\,\boldsymbol{\theta} + \cos\phi\,\boldsymbol{\phi}$$

$$z = r\cos\theta \qquad \mathbf{z} = \cos\theta\,\mathbf{r} - \sin\theta\,\boldsymbol{\theta}$$

Divergence

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}; \quad (\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}$$

$$(\nabla \times \mathbf{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}$$

Laplacian

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Laplacian of a vector

$$(\nabla^2\mathbf{A})_r = \nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta A_\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$(\nabla^2\mathbf{A})_\theta = \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$(\nabla^2\mathbf{A})_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}$$

Components of $(\mathbf{A}\cdot\nabla)\mathbf{B}$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\theta = A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi} + \frac{A_\theta B_r}{r} - \frac{\cot\theta A_\phi B_\phi}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\phi = A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\phi}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi} + \frac{A_\phi B_r}{r} + \frac{\cot\theta A_\phi B_\theta}{r}$$

Divergence of a tensor

$$(\nabla\cdot\boldsymbol{T})_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta T_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial\phi} - \frac{T_{\theta\theta} + T_{\phi\phi}}{r}$$

$$(\nabla \cdot \boldsymbol{T})_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta T_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial\phi} + \frac{T_{\theta r}}{r} - \frac{\cot\theta T_{\phi\phi}}{r}$$

$$(\nabla \cdot \boldsymbol{T})_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta T_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{T_{\phi r}}{r} + \frac{\cot\theta T_{\phi\theta}}{r}$$