Chapter 2
Ordered Sets and Complete Lattices
A Primer for Computer Science

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Abstract. These notes deal with an interconnecting web of mathematical techniques all of which deserve a place in the armoury of the well-educated computer scientist. The objective is to present the ideas as a self-contained body of material, worthy of study in its own right, and at the same time to assist the learning of algebraic and coalgebraic methods, by giving prior familiarization with some of the mathematical background that arises there. Examples drawn from computer science are only hinted at: the presentation seeks to complement and not to pre-empt other contributions to these ACMMPC Proceedings.

1 Introduction

Order enters into computer science in a variety of ways and at a variety of levels. At the most lowly level it provides terminology and notation in contexts where comparisons arise, of such things as

- size (of numbers),
- amount of information (after a number of steps of a computation, for example),
- degree of defined-ness (of partial maps).

Many areas of computer science use as models structures built on top of ordered sets. In some cases, only token familiarity with order-theoretic ideas is needed to study these, as is the case with CSP, for example. At the other extreme, domain theory uses highly sophisticated ordered structures as semantic domains (see for example Abramsky & Jung [2]). Indeed, the development of the theory of CPOs since the 1970s has led to new insights into the theory of ordered sets; see Gierz *et al.* [9] and, for a more recent perspective, Abramsky & Jung [2].

At an intermediate level come three notions exploited in computer science and closely linked to order-theoretic ideas: Galois connections, binary relations, and fixed points. A recurring theme here is the theory of **complete lattices**, and this account focusses on the way in which complete lattices can be described and their properties explored. Specifically we investigate the concepts in the diagram in Figure 1 and the arrows that link them. (An arrow pointing from A to B indicates that every object of type A gives rise, in a natural way, to an object of type B.

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We are not claiming that this is done in such a way that all triangles in the diagram commute.) Some of the links are principally of mathematical interest, but discussion of them has been included to complete the overall picture. For a summary of the results, see 7.10.

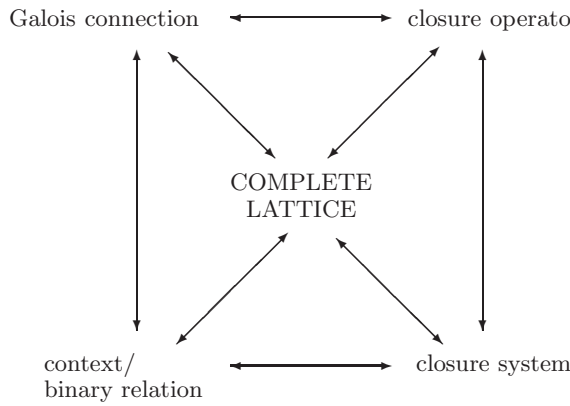


Fig. 1. A web of concepts

There is another, quite different, way in which order assists computer science. Category theory has established itself as a fundamental tool, and underpins much of the ACMMPC Workshop material. Ordered structures and the maps between them provide a wealth of examples of categories and functors. Equally importantly perhaps, every poset gives rise to a category in a natural way. Such categories are highly special (every set of arrows has at most one element) but very simple. As we hint in Section 9, elementary order-theoretic notions provide instances of more abstract categorical notions. For example, product, supremum, and infimum are instances of product, colimit, and colimit. Further, Galois connections between posets are instances of adjunctions between categories. Understanding categorical constructs in the special case of posets-as-categories can be helpful in cementing the general ideas.

As a final general comment on order in computer science we remark on the role of sets and powersets. Sets are a familiar concept and have accordingly been widely used, in such areas as the calculus of relations, for example. A powerset is ordered by its inclusion relation. As an ordered structure it possesses extremely nice properties, with infinitary disjunction and conjunction (union and intersection) available and interacting in a optimally well-behaved way. Powersets are too nice! Programs built on pure set models cannot capture all the behaviours that one might wish. Ordered set models are richer.

Probably some, but not all, of the ideas presented here will be familiar already to most readers. However, as befits concepts which have incarnations in a variety of disciplines, the concepts don different clothes in different settings.

These notes are written by a mathematician, and the style reflects a mathematician's approach. We have however followed, though not slavishly, the calculational proof style favoured by functional programmers. This formalism contrasts with and is complemented by the pictorial dimension to order theory which gives the latter much of its appeal.

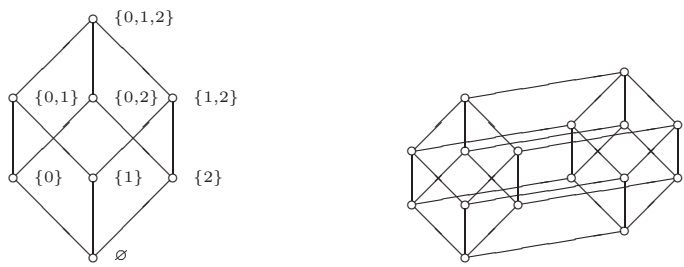
No prior knowledge of lattices or ordering will be presupposed, but basic facts concerning sets, maps, and relations are assumed. Many subsections contain 'Mini-exercises'. These serve both to record elementary results needed later on and as an invitation to the reader to reinforce understanding of the immediately preceding material; most of the verifications are of the 'follow-your-nose' variety. More substantial exercises are interspersed through the text. A background reference is the text *Introduction to Lattices and Order*; chapter numbers given are those in the second (2002) edition. This will henceforth be referred to simply as ILO2. Chapters 1–4 and 7–10 contain the material of primary relevance to this survey.

2 From Binary Relations to Diagrams

2.1 A Fundamental Example: Powersets

Very many of the structures we consider are families of subsets of some given set X, that is, they are members of the **powerset** of X. This powerset carries a natural ordering, namely set inclusion, $\subseteq$. We denote the set of all subsets of X by $\wp(X)$, and always regard this as equipped with the inclusion order. Alternatively (though this may seem perverse at this stage), we might order the subsets of X by reverse inclusion, $\supseteq$. When we wish to use reverse inclusion we shall write $\wp(X)^{\partial}$. (See 3.4 for a more general occurrence of the same idea.)

A recurrent theme hereafter will be the way in which ordered sets can be depicted diagrammatically. Ahead of considering this in a formal way, we give some illustrative examples. Let us first consider $X = \{0, 1, 2\}$. We can give a representation of $\wp(X)$, as shown in Figure 2(a). In (b) we show an unlabelled diagram for $\wp(\{0, 1, 2, 3\})$.



(a) (b)

Fig. 2. Some powersets, pictorially

Mini-exercise

(i) Draw a labelled diagram of $\wp(\{0,1,2\})^\partial$.
(ii) Label the diagram in Figure 2(b), and indicate a connected sequence of upward line segments from $\{3\}$ to $\{1,3,4\}$.

Any collection of subsets of a set X—not necessarily the full powerset—is also ordered by inclusion. For an example, see the diagram in Figure 8(b).

Mini-exercise

(i) Draw a diagram for the family $\{\{3\},\{1,3\},\{1,3,4\}\}$ in $\wp(\{1,2,3,4\})$.
(ii) Draw a similar picture for the following family of sets in $\wp(\{A,B,C,D,E\})$:

$$\varnothing, \{E\},\ \{A,E\},\ \{D,E\},\ \{C,D,E\},\ \{A,D,E\},$$
$$\{A,B,E\},\ \{A,C,D,E\},\ \{B,C,D,E\},\ \{A,B,C,D,E\}.$$

2.2 Input-Output Relations Pictorially

Consider a simple input-output relation with the inputs labelled by a,b,c,d,e and the outputs by A,B,C,D,E. The relation is indicated in the table in Figure 3. We may think of this as modelling a program which when started from a given input state p can terminate only in a output state Q for which $\times$ appears in position (p,Q) in the table. Note that the input state labelled e has no associated output states, and may be thought of as a way of capturing the possibility of non-termination.

	A	B	C	D	E
a		$\times$	$\times$	$\times$	$\times$
b				$\times$	$\times$
c	$\times$	$\times$		$\times$	$\times$
d	$\times$	$\times$			$\times$
e					

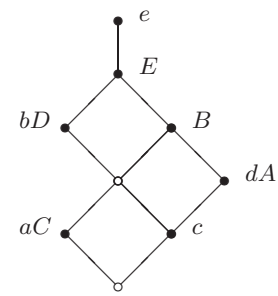


Fig. 3. An input-output relation and a diagram for it

To reach a prescribed output state we have the following sets of starting states:

$$A^\triangleleft = \{c,d\} \qquad C^\triangleleft = \{a\} \qquad E^\triangleleft = \{a,b,c,d\}$$
$$B^\triangleleft = \{a,c,d\} \qquad D^\triangleleft = \{a,b,c\}$$

We now take these five sets and all sets we can form from them by closing up under intersections. This adds to the original collection $\{c\}$, $\{b, c\}$, $\varnothing$, and the empty intersection (that is, the intersection of no sets); for example, $\{c\} = A^{\triangleleft} \cap D^{\triangleleft}$, indicating that c is the only input state from which it is possible to terminate in either A or D.

In a similar way we can write down the set of output states that can be reached from each input state:

$$a^{\triangleright} = \{B, C, D, E\} \qquad c^{\triangleright} = \{A, B, D, E\} \qquad e^{\triangleright} = \{E\}$$
$$b^{\triangleright} = \{D, E\} \qquad d^{\triangleright} = \{A, B, E\}$$

Conjuring a rabbit out of a hat we exhibit the picture in Figure 3. Leaving aside for now how we derived the labelling, we can see rather easily how this encodes the information in the input-output table. To see which output states are attainable from a given input state, say p, simply locate the output states Q which lie above p, in the sense that there is a connected sequence of upward line segments from p to Q. For $p = a$, for example, this gives C and D as the possibilities for Q. Similarly, to find the input states from which it is possible to terminate in a given output state Q, look downwards along connected line segments to find the input states below Q.

Mini-exercise Carry out the procedure above for each of the input and output states, and so reconstruct the original input-output table from Figure 3.

Now consider the central unshaded point in Figure 3. Looking upward along line segments for points labelled with output states we find B, D, E and looking downward for points labelled with input states we see a, c. Observe that B, D and E are precisely the output states attainable from both the input states a and c, while a and c are exactly the possible initial states if the program is to be guaranteed to terminate in one of B, C, or D.

We make no claim that this diagrammatic way of viewing an input-output relation has any merits from a computer scientist's standpoint. Indeed quite the reverse might be said, since it goes counter to the philosophy of operating according to fixed calculational rules. However we shall with profit return to this very simple example later, and indicate that analysis of more complicated examples can yield information which is not otherwise easy to obtain.

2.3 Exercise

Analyse the input-output relation with table shown in Table 1. See if you can work out how to draw a labelled diagram encoding the input-output table, and interpret the diagram in the same way as in the example in 2.2.

Table 1. Input-output relation for Exercise 2.3

	A	B	C	D	E
a			×	×	
b	×		×	×	
c	×				
d	×	×	×		
e	×	×	×	×	

2.4 Binary Relations and Their Polars

Let G and M be sets, and let $R \subseteq G \times M$ be a binary relation. Changing the perspective from that we adopted in 2.2, we think of G as a set of **objects**, M as a set of **attributes** and the relationship $(g, m) \in R$ (sometimes written alternatively as gRm) as asserting that 'object g has attribute m'. We shall refer to the triple (G, M, R) as a **context**. The choice of letters G and M comes from the German (*Gegenstände* and *Merkmale*—the theory of concept lattices having been principally developed by R. Wille and his group at TH Darmstadt. From its beginnings 20 years ago this theory has evolved into a commercially applicable tool for data analysis through the TOSCANA software. The concept lattice associated with a context reveals inherent hierarchical structure and thence natural groupings and dependencies among the objects and the attributes. Introductory accounts of the theory, with illustrations, appear in ILO2 Chapter 3, and in Ganter & Wille [8].

Given the context (G, M, R) we define

$$A^{\triangleright} := \{\, m \in M \,:\, (\forall g \in A)\,(g, m) \in R \,\}, \quad \text{for } A \subseteq G,$$
$$B^{\triangleleft} := \{\, g \in G \,:\, (\forall m \in B)\,(g, m) \in R \,\} \quad \text{for } B \subseteq M,$$

called the **polars** of A, B, respectively. For singleton sets we drop the set brackets and write $g^{\triangleright}$ instead of $\{g\}^{\triangleright}$ and $m^{\triangleleft}$ instead of $\{m\}^{\triangleleft}$. The polar map $^{\triangleright}$ takes subsets of G to subsets of M—it takes a set A of objects to the set of attributes common to all the objects in A; likewise, $^{\triangleleft}$ maps subsets of M to subsets of G, taking a set of attributes B to the set of all objects which possess all of the attributes in B. Of course the bigger A is, the fewer attributes all its members will share and the bigger B is, the fewer objects will share all the attributes demanded by B. It is therefore natural to reverse the ordering on one side and to regard $^{\triangleright}$ as mapping from $\wp(G)$ to $\wp(M)^{\partial}$ and $^{\triangleleft}$ as mapping from $\wp(M)^{\partial}$ to $\wp(G)$. Flipping the order on $\wp(M)$ like this makes $^{\triangleright}$ and $^{\triangleleft}$ monotone (that is, order-*preserving*), rather than order-*reversing*). We also have

$$\begin{aligned} A \subseteq B^{\triangleleft} &\iff (\forall g \in A)(\forall m \in B)(g, m) \in R && \text{(definition of } ^{\triangleleft}) \\ &\iff (\forall m \in B)(\forall g \in A)(g, m) \in R && \text{(predicate calculus)} \\ &\iff A^{\triangleright} \supseteq B && \text{(definition of } ^{\triangleright}). \end{aligned}$$

Note the reversal, as expected, of the order. (Those already in the know will recognize that we have established that $({}^{\triangleright}, {}^{\triangleleft})$ sets up a Galois connection between $\wp(G)$ and $\wp(M)^{\partial}$.)

The polars give rise to related notions. Write

$$g \dashv A \iff g \in A^{\triangleright\triangleleft} \quad (g \in G,\ A \subseteq G),$$
$$B \vdash m \iff m \in B^{\triangleleft\triangleright} \quad (m \in M,\ B \subseteq M);$$

$\dashv$ and $\vdash$ may be referred to as the **emulation** operator and the **semantic consequence** operator, respectively. For clarity we worked in 2.2 with the set of input states and the set of output states distinguished. Taking $G = M = S$ and $R \subseteq S \times S$ we may regard (S, S, R) as defining a (**non-deterministic**) **transition system** on S. Such systems provide a point of entry into the theory of coalgebras; see Rutten [16], and other papers by the same author.

Given $A \subseteq G$ and $B \subseteq M$ we call (A, B) a **concept** if

$$A = B^{\triangleleft} \quad \text{and} \quad A^{\triangleright} = B.$$

Concepts are ordered by inclusion on the first co-ordinate, reverse inclusion on the second (this is just the order inherited from the co-ordinatewise ordered product $\wp(G) \times \wp(M)^{\partial}$; see 3.7). The set of all concepts ordered in this way is denoted $\mathfrak{B}(G, M, R)$. What we have drawn in Figure 3 is a pictorial representation of $\mathfrak{B}(G, M, R)$ for the context given in 2.2.

2.5 Exercise

Let X be a set and let $R_{=}$ and $R_{\neq}$ denote $=$ and $\neq$ regarded as binary relations, that is, as subsets of $X \times X$.

(i) Consider the context $(X, X, R_{\neq})$. Identify the polars $A^{\triangleright}$ and $B^{\triangleleft}$ for $A, B \subseteq X$, and show that $\mathfrak{B}(X, X, R_{\neq}) = \{\, (A, X \backslash A) \,:\, A \subseteq X \,\}$.

(ii) Consider the context $(X, \wp(X), \in)$. Show that

$$\mathfrak{B}(X, \wp(X), \in) = \{\, (A, \{\, B \in \wp(X) \,:\, A \subseteq B \,\}) \,:\, A \subseteq X \,\}.$$

2.6 Summing Up So Far

Binary relations, and their associated contexts, are versatile creatures, with a wide spectrum of semantic interpretations. Any context has associated with it a pair of polar maps, forming what is known as a Galois connection. In a way we have yet fully to explore, these maps in turn give rise to an ordered set of objects we call concepts, and this encodes the original binary relation. But we are rushing ahead. Before we can make all this precise we need to know quite a lot about ordered sets.

3 Order, Order, Order, ...

We have mentioned order in an informal way in the preceding section, and have made use of the inclusion and reverse inclusion orderings on a powerset. Now we need to be more formal. What exactly do we mean by order? an order relation? an ordered set?

3.1 Partial Order

Let us consider ordering in the context of some familiar datatypes:

(a) $<$ on the natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$, with $1 < 2 < 3 < \dots$;
(b) $\subseteq$ ('is a subset of') on the powerset $\wp(X)$ of all subsets of a set X;
(c) the relation $\mathbf{F} < \mathbf{T}$ on the set of booleans $\{\mathbf{F}, \mathbf{T}\}$;
(d) the prefix order on binary strings—here $0110 < 011001100000$, for example;
(e) the relation 'is more defined than' on partial maps π from $\mathbb{N}$ to $\mathbb{N}$ (so the domain and range of π are subsets of $\mathbb{N}$); for example, for $\pi_k \colon \{1, \dots, k\} \to \mathbb{N}$ given by $\pi_k(n) = n + 1$ for $n = 1, \dots, k$, we have π_k less defined than π_{k+1} for each k.

These examples confirm that order concerns comparison between pairs of objects: 1 is smaller than 2 in $\mathbb{N}$, etc. In mathematical terms, an ordering is a **binary relation** on a set of objects.

Order relations are of two types: strict and non-strict. Outside mathematics, the strict notion is more common. When comparing people's heights, the relation 'is taller than' is generally taken to mean 'is strictly taller than'. Mathematicians usually allow equality and write, for instance, $3 \leqslant 3$ and $3 \leqslant 22/7$. We shall deal mainly with non-strict order relations.

What distinguishes a strict order relation amongst binary relations? Firstly, it is transitive. From the facts that $0 < 1$ and $1 < 10^{23}$ we can deduce that $0 < 10^{23}$. Secondly, a strict order is antisymmetric: 5 is bigger than 3 but 3 is not bigger than 5. Formally: a binary relation $<$ on a set P is a **strict partial order** if it satisfies

(spo1) **antisymmetry**: for $x, y \in P$, if $x < y$ holds, then $y < x$ does not hold;
(spo2) **transitivity**: for $x, y, z \in P$, $x < y$ and $y < z$ implies $x < z$.

A (**non-strict**) **partial order**, $\leqslant$, on P is a binary relation satisfying

(po1) **reflexivity**: for $x \in P$, $x \leqslant x$;
(po2) **antisymmetry**: for $x, y \in P$, $x \leqslant y$ and $y \leqslant x$ imply $x = y$;
(po3) **transitivity**: for $x, y, z \in P$, $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$.

Relaxing the conditions by dropping antisymmetry leads to what is known as a **pre-order** or **quasi-order**.

Formally, a binary relation R on the set P is a subset of $P \times P$. With this interpretation the equality relation, $=$, is $R_= := \{(x, x) : x \in P\}$. Given a partial order $\leqslant$ on P, the associated subset of $P \times P$ is $R_\leqslant := \{(x, y) : x, y \in P, x \leqslant y\}$,

and likewise for $<$. Then $R_{\leqslant} = R_{<} \cup R_{=}$, the union being disjoint. This is a fancy way of expressing the way in which $<$ and $\leqslant$ are related.

A set P equipped with a partial order, strict or non-strict, is called in ILO2 an **ordered set**. Here we shall use the snappier term **poset**. Usually we shall be a little slovenly and say simply 'P is a poset'. When we wish to make the order explicit we write $\langle P; \leqslant \rangle$ and when working with more than one poset we shall sometimes write $\leqslant_P$ for the order on poset P.

Associated notation is predictable: $x \leqslant y$ and $y \geqslant x$ are used interchangeably, and $x \not\leqslant y$ means '$x \leqslant y$ is false', and so on. Of course,

$$R_{\geqslant} = \{\, (x, y) \in P \times P \,:\, (y, x) \in R_{\leqslant} \,\}$$

—the **converse** of $R_{\leqslant}$. Similarly, $R_{\not\leqslant} = (P \times P) \setminus R_{\leqslant}$. Any subset Q of a poset P inherits P's ordering: $x \leqslant_Q y$ if and only if $x, y \in Q$ and $x \leqslant_p y$. Note that $R_{\leqslant_Q}$ is just $R_{\leqslant_P} \cap (Q \times Q)$. The usual orderings of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ of natural numbers, integers, and rationals are obtained in this way from the ordering of the real numbers $\mathbb{R}$.

In the ordering $<$ on $\mathbb{R}$, any two distinct real numbers can be compared. This comparability property is possessed by many familiar orderings, but it is *not* universal. It is important to realize that under a partial order (strict or non-strict) we may have mutually incomparable elements. As examples, note that the sets $\{2\}$ and $\{1, 3\}$ in $\wp(\{0, 1, 2, 3\})$ are not comparable and the strings 101 and 010 are incomparable in the prefix order. A poset in which any two elements are comparable is called a **chain**, and the associated order relation a **linear** or **total** order. Of particular importance is the 2-element chain $\mathbf{2} = \{0, 1\}$ in which $0 < 1$: writing **F** (false) for 0 and **T** (true) for 1, we have the booleans, ordered by putting $\mathbf{F} < \mathbf{T}$. At the opposite extreme from a chain we have an **antichain**, in which $\leqslant$ coincides with $=$.

3.2 Information Orderings

We have already referred to binary strings and their prefix order. Strings may be thought of as information encoded in binary form: the longer the string the greater the information content. Let Σ^* be the set of all finite binary strings, that is, all finite sequences of 0s and 1s; the empty string is included. Adding the infinite sequences, we get the set of all finite or infinite sequences, which we denote by Σ^{**}. We order Σ^{**} by putting $u \leqslant v$ if and only if u is a prefix (that is, finite initial substring) of v. Given any string v, we may think of elements u with $u < v$ as providing approximations to v. In particular, any infinite string is, in a sense we later make more precise, the limit of its finite initial substrings.

The statement that some computed quantity r equals 1.35 correct to 2 decimal places may be re-expressed as the assertion that r lies in a particular closed interval in $\mathbb{R}$. We may accordingly treat the collection of all intervals $[\underline{x}, \overline{x}]$ (where $-\infty \leqslant \underline{x} \leqslant \overline{x} \leqslant \infty$) as defining a set P of approximations to the real numbers, with the intervals for which $\underline{x} = \overline{x}$ corresponding to exact values. The set P carries a very natural order: for $x = [\underline{x}, \overline{x}]$ and $y = [\underline{y}, \overline{y}]$ define $x \leqslant y$ if and only

if $\underline{x} \leqslant \underline{y}$ and $\overline{y} \leqslant \overline{x}$. Then $x \leqslant y$ means that y represents (or contains) at least as much information as x. These very simple ideas underlie the recent development of a method for doing exact computations with real numbers (see A. Edelat, [7] for an introductory survey).

Now consider partial maps. Let A, B be (non-empty) sets and denote by $A \multimap B$ the partial maps from A to B. Thus each element of $A \multimap B$ is a map π with domain $\operatorname{dom}\pi \subseteq A$ and range $\operatorname{ran}\pi \subseteq B$: π may be regarded as a recipe which assigns an output $\pi(x)$ in $\operatorname{ran}\pi$ to each input x in $\operatorname{dom}\pi$. Alternatively, and equivalently, π is determined by its graph,

$$\operatorname{graph}\pi := \{\, (x, \pi(x)) \,:\, x \in \operatorname{dom}\pi \,\},$$

a subset of $A \times B$. If $\pi \in A \multimap B$ is such that $\operatorname{dom}\pi = A$, then π is a map on X (or, for emphasis, a **total map**). Therefore $A \multimap B$ consists of all total maps from A to B and all partial determinations of them. This set is ordered in the following way: given partial maps π, σ, define $\pi \leqslant \sigma$ if and only if $\operatorname{dom}\pi \subseteq \operatorname{dom}\sigma$ and $\pi(x) = \sigma(x)$ for all $x \in \operatorname{dom}\pi$. Equivalently, $\pi \leqslant \sigma$ if and only if $\operatorname{graph}\pi \subseteq \operatorname{graph}\sigma$ in $\wp(A \times B)$. Note that a subset G of $A \times B$ is the graph of a partial map if and only if

$$(\forall s \in A)\,((s, x) \in G \;\&\; (s, x') \in G) \implies x = x'.$$

The examples above illustrate ways in which posets can model situations in which the relation $x \leqslant y$ has interpretations such as 'y is more defined than x' or 'y is a better approximation than x'. In each case, we have a notion of a **total object** (a completely defined, or idealized, element). These total objects are the infinite binary strings in the first example, the 1-point intervals in the second, and the total maps in the third. The most interesting examples from a computational point of view are those in which the total objects may be realized as limits (in an order-theoretic manner) of objects which are in some sense finite. A finite object should be one which encodes a finite amount of information: for example, finite strings, or partial maps with finite domains. These issues are taken up briefly in Section 8.

3.3 Diagrams

As we have already suggested, an attractive feature of posets is that, in the finite case at least, they can be 'drawn'. The diagram of a finite poset P is drawn in such a way that $x < y$ in P if and only if there is a sequence of connected line segments moving upwards from x to y. For our purposes, common sense will suffice to indicate what constitutes a legitimate diagram; the formal rules governing diagram-drawing are set out in ILO2, 1.15. As an example: Figure 4 gives a diagram for the subset of Σ^* consisting of strings of length $\leqslant 3$.

The same poset may have many different diagrams. Two valid alternative diagrams for the cube are shown in Figure 5. The first comes from a computer science text (do some computer scientists have twisted minds?). The second,

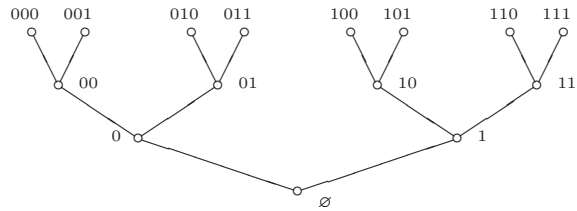


Fig. 4. Binary strings of length $\leqslant 3$ under the prefix order

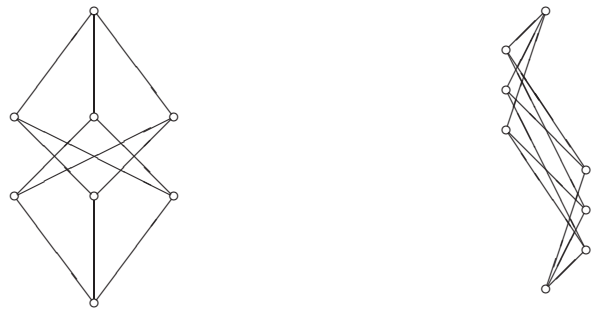


Fig. 5. Two 'bad' diagrams for a cube

while not having the maximal possible number of line-crossings **Mini-exercise**: prove that this number is 19) still serves to make the point that diagram-drawing is as much an art as a science. Good diagrams aid understanding.

3.4 Duality: Buy One, Get One Free

Given any poset P we can form a new poset P^∂ (the **dual** of P) by defining $x \leqslant y$ to hold in P^∂ if and only if $y \leqslant x$ holds in P. For P finite, we obtain a diagram for P^∂ simply by 'turning upside down' a diagram for P. Figure 6 provides a simple illustration.

Poset concepts and results hunt in pairs. Any statement about a poset P yields a corresponding (dual) statement about P^∂, obtained by interchanging $\leqslant$ and $\geqslant$ and making consequential changes to all other symbols (replacing $\not\geqslant$ by $\not\leqslant$, and so on). This **Duality Principle** permits us to prove just one of any pair of mutually dual claims.

3.5 Bottom and Top

Let P be a poset. We say P has a bottom element if there exists $\bot \in P$ (called **bottom**) with the property that $\bot \leqslant x$ for all $x \in P$. Dually, P has a top element if there exists $\top \in P$ such that $x \leqslant \top$ for all $x \in P$. As a simple instance of the

Fig. 6. A pair of mutually dual posets

Duality Principle note that the true statement '$\bot$ is unique when it exists' has as its dual version the statement '$\top$ is unique when it exists'.

In $\langle \wp(X); \subseteq \rangle$, we have $\bot = \varnothing$ and $\top = X$. A finite chain always has bottom and top elements, but an infinite chain need not have. For example, the chain $\mathbb{N}$ has bottom element 1, but no top, while the chain $\mathbb{Z}$ of integers possesses neither bottom nor top. Bottom and top do not exist in any antichain with more than one element.

In the context of information orderings, $\bot$ and $\top$ have the following interpretations: $\bot$ represents 'no information', while $\top$ corresponds to an overdetermined, or contradictory, element. None of the posets in 3.2 has a top element, except for $A \multimap\!\!\to B$ in very special cases. Each has a bottom element: $[-\infty, \infty]$ for interval approximations to real numbers, the empty string for Σ^{**} and the partial map with empty domain for $A \multimap\!\!\to B$. In each case $\bot$ is the least informative element. In modelling computations, a bottom element is also useful for representing and handling non-termination. Accordingly, computer scientists commonly choose as models posets which have bottom elements, but prefer these topless.

3.6 Lifting

It is tiresome that what may be thought of as the simplest posets of all, namely the antichains, fail to have bottoms (except in the 1-element case). Lack of a bottom element can be easily remedied by adding one. Given any poset P (with or without $\bot$), we form $P_\bot$ (called P 'lifted') as follows. Take an element $\bot \notin P$ and define $\leqslant$ on $P_\bot := P \cup \{\bot\}$ by

$$x \leqslant y \text{ if and only if } x = \bot \text{ or } x \leqslant y \text{ in } P.$$

For example, take the natural numbers $\mathbb{N}$ with the antichain order, $=$. Then $\mathbb{N}_\bot$ is as shown in Figure 7. $P_\bot$ is just $\{\bot\} \oplus P$. A poset of the form $S_\bot$, where S is an antichain, is called **flat**.

3.7 New Posets from Old: Sums and Products

Antichains and chains, and the lifting construction, are examples of constructing new posets from existing ones by forming suitable order-theoretic sums. Given

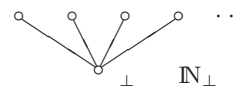


Fig. 7. Lifting

two disjoint posets P and Q we form their **linear sum** $P \oplus Q$ by stacking Q on top of P. Formally, we take $P \cup Q$ and order it by $x \leqslant y$ if and only if one of the following holds:

(i) $x, y \in P$ and $x \leqslant_P y$,
(ii) $x, y \in Q$ and $x \leqslant_Q y$,
(iii) $x \in P$ and $y \in Q$.

Given two disjoint posets P and Q we order their union $P \cup Q$ by setting $x \leqslant_{P \cup Q} y$ if and only if either (i) $x, y \in P$ and $x \leqslant_P y$ or (ii) $x, y \in Q$ and $x \leqslant_Q y$. The resulting poset is denoted $P \cup Q$. Note that in the formation of linear sums and disjoint sums it is essential that the posets P and Q be disjoint.

Given two posets P and Q, we can form their product $P \times Q$ by giving the set $P \times Q$ of all ordered pairs $\{ (p, q) : p \in P, q \in Q \}$ the co-ordinatewise order

$$(p_1, p_2) \leqslant_{P \times Q} (q_1, q_2) \iff p_1 \leqslant_P p_2 \text{ and } q_1 \leqslant_Q q_2.$$

See ILO2, 1.12, for comments on diagrams of products posets. A related **Mini-exercise**: what can you deduce from the diagram of $\wp(\{0, 1, 2, 3\})$ shown in Figure 2(b)?

Mini-exercise Let X be a set with n elements. Prove that $\wp(X) \cong \mathbf{2}^n$ (here $\wp(X)$ has the usual inclusion order, and $\mathbf{2}^n$ denotes the n-fold product of $\mathbf{2}$ with itself. (If stuck, consult ILO2, 1.26.)

3.8 Maps between Posets

It should come as no surprise at all that along with posets we also consider suitable structure-preserving maps between posets. Let P and Q be posets. A map $F \colon P \to Q$ is said to be

(i) **monotone** (or, alternatively, **order-preserving**) if $x \leqslant y$ in P implies $F(x) \leqslant F(y)$ in Q;
(ii) an **order-embedding** if $x \leqslant y$ in $P \iff F(x) \leqslant F(y)$ in Q;
(iii) an **order-isomorphism** if it is an order-embedding mapping P onto Q.

When there exists an order-isomorphism from P to Q, we say that P and Q are **order-isomorphic** and write $P \cong Q$. Order-isomorphic posets are essentially indistinguishable; in the finite case this happens if and only if they can be represented by the same diagram (see ILO2, 1.18). Frequently used properties of maps are contained in the next Mini-exercise.

Mini-exercise

(i) Any order-embedding is clearly monotone. Show that it is also one-to-one (you will need (po2), antisymmetry of $\leqslant$, in P). Show that not every one-to-one monotone map is an order-embedding (get an example using 2-element posets).
(ii) Let $F\colon P \to Q$ and $G\colon Q \to R$ be maps between posets P, Q, R. Show that if F and G are monotone (order-embeddings, order-isomorphisms) then so is the composite $G \circ F\colon P \to R$.
(iii) A monotone map $F\colon P \to Q$ is an order-isomorphism if and only if it has a monotone inverse $G\colon Q \to P$ (meaning that $G \circ F = \mathrm{id}_P$ and $F \circ G = \mathrm{id}_Q$). (Here $\mathrm{id}_S\colon S \to S$ denotes the **identity map** on S given by $\mathrm{id}_S(x) = x$ for all $x \in S$.)

The familiar poset $\wp(X)$ of subsets of a set X with its inclusion order is connected by an order-isomorphism to another important poset associated with X, namely the poset $\mathbb{P}$ of **predicates** on X. A **predicate** is a statement taking value **T** (true) or value **F** (false), or, more formally, a function from X to $\{\mathbf{T}, \mathbf{F}\}$; here we don't distinguish between different ways of specifying the same function. For example, the map $p\colon \mathbb{R} \to \{\mathbf{T}, \mathbf{F}\}$ given by $p(x) = \mathbf{T}$ if $x \geqslant 0$ and $p(x) = \mathbf{F}$ if $x < 0$ is a predicate on $\mathbb{R}$, which can be alternatively be specified by $p(x) = \mathbf{T}$ if $|x-1| \leqslant |x+1|$ and **F** otherwise. We order $\mathbb{P}(X)$ by implication: for $p, q \in \mathbb{P}(X)$,

$$p \Rrightarrow q \text{ if and only if } \{\, x \in X \,:\, p(x) = \mathbf{T} \,\} \subseteq \{\, x \in X \,:\, q(x) = \mathbf{T} \,\}.$$

Then $F\colon \mathbb{P}(X) \to \wp(X)$ given by $F(p) := \{\, x \in X \,:\, p(x) = \mathbf{T} \,\}$ sets up an order-isomorphism between $\langle \mathbb{P}(X); \Rrightarrow \rangle$ and $\langle \wp(X); \subseteq \rangle$. In the special case that X has just one element, $\mathbb{P}(X)$ is (isomorphic to) the poset of booleans.

3.9 Pointwise Ordering of Maps

Now let Q be a poset and X any set. Then the ordering on Q can be lifted, pointwise, to a partial order $\sqsubseteq$ on the set Q^X of all maps from X to Q: for $F, G\colon X \to Q$,

$$F \sqsubseteq G \iff (\forall x \in X)\, F(x) \leqslant G(x).$$

(We shall always use $\sqsubseteq$ rather than $\leqslant$ when ordering functions pointwise.) Thinking of predicates on X as maps from X into $\{\mathbf{T}, \mathbf{F}\}$ the pointwise order is just the implication order $\Rrightarrow$. When X is itself a poset, P say, the subset of Q^P consisting of the monotone maps from P to Q inherits the order $\sqsubseteq$; we denote this poset by $\langle P \to Q \rangle$.

Mini-exercise (Currying) Prove that, for all posets P, Q and R,

$$\langle P \to \langle Q \to R \rangle \rangle \cong \langle P \times Q \to R \rangle.$$

3.10 Up-Sets: An Inbred Example

Let P be a poset.

(i) Let $x \in P$. Then define $\uparrow x := \{\, y \in P : y \geqslant x \,\}$.
(ii) Let $Y \subseteq P$. Then Y is an **up-set** of P if $x \in P$, $x \geqslant y$, $y \in Y$ implies $x \in Y$.

Note that $\uparrow x$ is an up-set for each $x \in P$ (by (po3), the transitivity of $\leqslant$). Denote the family of up-sets of P by $\mathcal{U}(P)$, and order it by inclusion. Thus $\mathcal{U}(P)$ is itself a poset. We shall shortly see that it is much more than this. In particular, an elementary calculation shows that if $\{A_i\}_{i\in I}$ is any subset of $\mathcal{U}(P)$ then $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$ belong to $\mathcal{U}(P)$. **Mini-exercise**: check this statement (follow your nose). As an example, we show in Figure 8 a diagram of a poset P and of $\mathcal{U}(P)$.

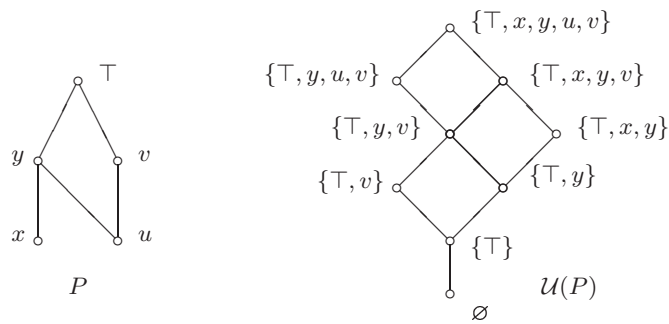


Fig. 8. A poset P and its poset $\mathcal{U}(P)$ of up-sets

Mini-exercise

(i) Prove that P is an antichain if and only if $\mathcal{U}(P) = \wp(P)$.
(ii) Analyse $\mathcal{U}(P)$ when P is (a) the chain $\mathbb{N}$, (b) the chain $\mathbb{R}$ (both with the usual order).

Mini-exercise Let P and Q be disjoint posets. Describe the up-sets of $P \dot\cup Q$ and prove that $\mathcal{U}(P \dot\cup Q) \cong \mathcal{U}(P) \times U(Q)$.

3.11 Monotone Maps and Up-Sets

Let P be a poset and recall that $\mathbf{2} = \{0, 1\}$ is the 2-element chain ordered by $0 < 1$. Then there is an order-isomorphism between $\mathcal{U}(P)$ and $\langle P \to \mathbf{2}\rangle$ ordered as always by the pointwise order $\sqsubseteq$. Under this isomorphism an up-set U is associated to its characteristic function χ_U (which takes value 1 on U and 0 otherwise). **Mini-exercise**: Verify this assertion.

3.12 Exercise (More on Monotone Maps and Up-Sets)

Let P and Q be posets and $F\colon P \to Q$ a map.

(i) Prove that $F\colon P \to Q$ is monotone if and only if

$$F^{-1}(Y) := \{\, x \in P \,:\, F(x) \in Y \,\}$$

is an up-set in P whenever Y is an up-set in Q.

(ii) Assume $F\colon P \to Q$ is monotone. Then, by (i), $F^{-1}\colon \mathcal{U}(Q) \to \mathcal{U}(P)$ is a well defined map.

(a) Prove that F is an order-embedding if and only if F^{-1} maps $\mathcal{U}(Q)$ onto $\mathcal{U}(P)$.

(b) Prove that F maps P onto Q if and only if $F^{-1}\colon \mathcal{U}(Q) \to \mathcal{U}(P)$ is one-to-one.

It is also instructive to re-formulate and re-work this exercise in a purely functional setting (see 3.11).

3.13 Down Is Nice Too

We can define down-sets of a poset P in just the same manner as we defined up-sets and form the poset down-sets, $\mathcal{O}(P)$, carrying the inclusion ordering. The symbol $\mathcal{O}$ is traditional here—$\mathcal{O}$ stands for 'order ideal', a synonym for 'down-set'.

Mini-exercise Formulate explicitly the analogues for down-sets of the definitions and Mini-exercise results in 3.10.

Mini-exercise Draw a labelled diagram of $\mathcal{O}(P)$ for the poset P in Figure 6.

Mini-exercise Let P be a poset. How do the down-sets of $P_\perp$ (as defined in 3.6) relate to those of P? Prove that $\mathcal{O}(P_\perp) \cong \mathcal{O}(P)_\perp$.

Mini-exercise Prove that the poset Σ^{**} of all binary strings is a **tree** (that is, a poset P with $\perp$ such that $\downarrow x$ is a chain for each $x \in P$).

3.14 Exercise (Turning Things Upside Down)

(i) For $Y \subseteq P$, prove that $Y \in \mathcal{O}(P)$ if and only if $P \setminus Y \in \mathcal{U}(P)$.

(ii) Prove that $\mathcal{O}(P) \cong \mathcal{U}(P)^\partial$.

(iii) Prove that $\mathcal{U}(P^\partial) \cong \mathcal{U}(P)^\partial$ and $\mathcal{O}(P^\partial) \cong O(P)^\partial$.

You might have doubts about how the orderings work out here. If so, refer to the example in Figure 8 and Mini-exercise 3.13.

3.15 The Down-Set Operator, $\downarrow$, and the Up-Set Operator, $\uparrow$

Let P be a poset and $x, y \in P$. Then we claim that the following are equivalent:

(a) $x \leqslant y$;
(b) $\downarrow x \subseteq \downarrow y$;
(c) $(\forall Y \in \mathcal{O}(P))\ y \in Y \implies x \in Y$.

This innocent little result says that the order $\leqslant$ on P is determined by the down-sets in P. The implication (a) $\implies$ (b) is needed in 5.6.

Mini-exercise Prove the claim. (a) $\implies$ (b) has already been noted; follow-your-nose for (b) $\implies$ (c). For (c) $\implies$ (a) take $Y := \downarrow y$. Note: you will need (po1) and (po3).

There is likewise an up-set operator, $\uparrow$, mapping each subset of P to the up-set it generates. Notice though that $x \leqslant y$ if and only if $\uparrow x \supseteq \uparrow y$ (check it!). This order reversal means that for many purposes down-sets and $\downarrow$ are more convenient to work with than up-sets and $\uparrow$. However up-sets relate better to monotone functions; see 3.11.

For any subset Y in a poset P there is a smallest down-set containing Y. This may be described in two equivalent ways:

(a) $\downarrow Y = \{\, z \in P : (\exists y \in Y)\, z \leqslant y \,\} = \bigcup \{\, \downarrow y : y \in Y \,\}$;
(b) $\downarrow Y = \bigcap \{\, Z : Z \in \mathcal{O}(P),\ Z \supseteq Y \,\}$.

Note that when $Y = \{x\}$, where $x \in P$, then $\downarrow\{x\}$, as defined in (a), is just $\downarrow x$ as defined in 3.10; we henceforth always write this down-set as $\downarrow x$. Observe that the operator $\downarrow\colon A \mapsto \downarrow A$ defines a map from $\wp(P) \to \wp(P)$ whose image is precisely $\mathcal{O}(P)$.

Mini-exercise Prove the equivalence of (a) and (b) above, by showing that each of the sets presented is contained in the other.

Exercise (Properties of the Operator $\downarrow$) Prove the following: for all $Y, Z \in \wp(P)$,

(i) $Y \subseteq \downarrow Y$;
(ii) $Y \subseteq Z \implies \downarrow Y \subseteq \downarrow Z$;
(iii) $\downarrow Y = \downarrow\downarrow Y$;
(iv) $Y = \downarrow Y$ if and only if $Y \in \mathcal{O}(P)$.

(Obviously, $\uparrow$ behaves analogously.)

3.16 Exercise (A Context Explored)

Let P be a poset and consider the context $(P, P, R_{\not\geqslant})$. Let $^{\triangleright}$ and $^{\triangleleft}$ be the associated polar maps.

(i) Show that $g^{\triangleright} = P \setminus {\downarrow}g$ and $m^{\triangleleft} = P \setminus {\uparrow}m$ for $g, m \in P$.
(ii) For $A, B \subseteq P$ show that $A^{\triangleright} = P \setminus {\downarrow}A$ and $B^{\triangleleft} = P \setminus {\uparrow}B$.
(iii) Show that $(A, B) \in \wp(P) \times \wp(P)^{\partial}$ is a concept if and only if $A \in \mathcal{O}(P)$ and $B \in \mathcal{U}(P)$, with $A = P \setminus B$.

3.17 Maximal and Minimal Elements

We next introduce some important special elements. Let P be a poset and let $S \subseteq P$. Then $a \in S$ is a **maximal** element of S if $a \leqslant x \in S$ implies $a = x$. We denote the set of maximal elements of S by $\operatorname{Max} S$. Note that $\operatorname{Max} S$ contains just one element if S (with the order inherited from P) has a top element, $\top_S$; in this case $\top_S$ is called the **greatest** element of S and denoted $\max S$. Note that then $\operatorname{Max} Y = \{\max S\}$. Note also that $x \in \operatorname{Max} P$ if and only if ${\uparrow}x = \{x\}$. A **minimal** element of $S \subseteq P$ and $\operatorname{Min} S$ and (where it exists) $\min S = \bot_S$ are defined dually, that is by reversing the order.

In general Y may have many maximal elements, or none. A subset of the chain $\mathbb{N}$ has a maximal element if and only if it is finite and non-empty. In the subset Y of $\wp(\mathbb{N})$ consisting of all subsets of $\mathbb{N}$ except $\mathbb{N}$ itself, there is no top element, but $\mathbb{N} \setminus \{n\} \in \operatorname{Max} Y$ for each $n \in \mathbb{N}$. The subset of $\wp(\mathbb{N})$ consisting of all finite subsets of $\mathbb{N}$ has no maximal elements. An important set-theorists' tool, **Zorn's Lemma**, guarantees the existence of maximal elements, under suitable conditions. Zorn's Lemma is discussed from an order theory viewpoint in ILO2, Chapter 10.

Referring to the examples in 3.2, we see that the maximal elements in Σ^{**} are the infinite strings and those in $A \multimap B$ are the total maps. This suggests that when an order relation models information we might expect a correlation between maximal elements and totally defined elements.

Mini-exercise Let P be a *finite* poset and let $\emptyset \neq Y \subseteq P$.

(i) Prove that $\operatorname{Max} Y$ is a non-empty antichain.
(ii) Prove that Y is a down-set if and only if $Y = {\downarrow}\operatorname{Max} Y$.

3.18 Stocktaking

In connection with a poset P and its subsets we have now met

- **binary relations**: $\leqslant$, its converse $\geqslant$, and their complements $\not\leqslant$, $\not\geqslant$;
- paired **polar maps**: $(^{\triangleright}, ^{\triangleleft})$ between $\wp(P)$ and $\wp(P)^{\partial}$, associated with any relation $R \subseteq P \times P$ and in particular with the relations $\leqslant$, $\geqslant$, $\not\leqslant$ and $\not\geqslant$;

- **families of sets** $\mathcal{U}(P)$ (**up-sets**) and $\mathcal{O}(P)$ (**down-sets**), and the family of **concepts** associated with the pair of polar maps $({}^{\triangleright}, {}^{\triangleleft})$ associated with a relation $R \subseteq P \times P$, all themselves posets;
- the **down-set operator**, $\downarrow$, and the **up-set operator**, $\uparrow$;
- **diagrams** for posets, in particular for the poset of **concepts** of a context.

A number of points should be clear from our examples: that the notions above are closely interconnected, and that the families of sets arising have nice properties not possessed by posets in general.

Our next task is to pursue order-theoretic ideas further, in order to have available the vocabulary needed to define and explore complete lattices and their relationship to Galois connections and closure operators. Afficionados of Galois connections will need to be a little patient: we put in place the other pieces of the jigsaw before slotting in this key piece.

4 Lattices in General and Complete Lattices in Particular

Many important properties of a poset P are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. Important classes of posets defined in this way are

- lattices,
- complete lattices,
- CPOs (complete partial orders).

These classes enter into different application areas to differing extents. Lattices, where we are dealing with *finitary* operations, are algebraic structures and their theory belongs to, and has a symbiotic relationship with, algebra. There are also close connections with logic. Complete lattices are the ordered structures of most interest to us in these notes. CPOs are more general, and provide an appropriate setting in which to study fixed point theorems; restricting somewhat to (Scott) domains, we have a much-studied class of semantic domains. Much of the motivation comes from the need to have semantic models supporting recursion.

4.1 Lattices

Consider the posets depicted in Figure 9. In (a) we have $\uparrow a \cap \uparrow b = \varnothing$. In (b) we find that $\uparrow a \cap \uparrow b = \{c, d\}$. Similar considerations apply to the down-set operator. For points x, y in a poset P there may be a point $z \in P$ such that $\downarrow x \cap \downarrow y = \downarrow z$, or this may fail either because the intersection is empty or because it is not of the form $\downarrow z$.

By contrast, if we look at a powerset $\wp(X)$ we find easily that, for any subsets A, B of X, there exists $C \in \wp(X)$, namely $C = A \cup B$, such that $\uparrow A \cap \uparrow B = \uparrow C$; and similarly for $\downarrow$.

(a) (b)

Fig. 9. Thwarted suprema

Let L be a non-empty poset. Then L is a **lattice** if, for $x, y \in L$, there exists elements $x \vee y$ and $x \wedge y$ in L such that

$$\uparrow x \cap \uparrow y = \uparrow(x \vee y) \quad \text{and} \quad \downarrow x \cap \downarrow y = \downarrow(x \wedge y);$$

the elements $x \vee y$ and $x \wedge y$ are called, respectively, the **join** (or **supremum**) and **meet** (or **infimum**) of x and y. Formally, $\vee\colon L \times L \to L$ and $\wedge\colon L \times L \to L$ are binary operations on L. Note that L^∂ is a lattice if and only if L is, with the roles of $\vee$ and $\wedge$ swapping.

Mini-exercise As a special kind of poset, a lattice is equipped with a partial order, $\leqslant$, as well as with the binary operations of join and meet. The link between $\vee$, $\wedge$ and $\leqslant$ (portentously called the **Connecting Lemma** in ILO2), is given by

$$x \wedge y = x \iff x \leqslant y \iff x \vee y = y$$

(note that this implies that either of $\vee$ and $\wedge$ determines $\leqslant$). Verify these implications.

Mini-exercise

(i) Show that any *finite* lattice possesses top and bottom elements.
(ii) Give an example of a poset with $\top$ and $\bot$ which is *not* a lattice.

Let L be a lattice. Then $(L; \vee, \wedge)$ may be viewed as an algebra, with $\vee$ and $\wedge$ satisfying certain laws (equations) capturing the properties that their order-theoretic ancestry gives them. For all $x, y, z \in L$,

(L1)	$(x \vee y) \vee z = x \vee (y \vee z)$	(associativity)
(L2)	$x \vee y = y \vee x$	(commutativity)
(L3)	$x \vee x = x$	(idempotency)
(L4)	$x \vee (x \wedge y) = x$	(absorption),

and their dual versions, $(\text{L1})^\partial$–$(\text{L4})^\partial$. It is an easy **Mini-exercise** to verify (L1)–(L3) and their duals. Note that only the absorption laws (L4) and $(\text{L4})^\partial$ involve *both* $\vee$ and $\wedge$. These laws, of course, capture exactly what the Connecting Lemma demands. In the opposite direction, the lattice laws are set up in such

a way that any (non-empty) structure $(L; \vee, \wedge)$ satisfying these laws gives rise to a poset $\langle L; \leqslant \rangle$: the partial order is (well-)defined by the equivalent conditions $x \leqslant y \iff x \vee y = y \iff x \wedge y = x$, and join and meet operations given by sup and inf are just the original $\vee$ and $\wedge$. This correspondence between lattice-as-algebra and lattice-as-poset is set out in more detail in Chapter 2 of ILO2.

The associative laws allow us unambiguously to define iterated joins and meets: if $F = \{a_1, \ldots, a_n\}$ is a finite non-empty subset of a lattice L then we write $\bigvee F$ as alternative notation for $a_1 \vee \cdots \vee a_n$, and $\bigwedge F$ for $a_1 \wedge \cdots \wedge a_n$.

We have demanded that a lattice be non-empty, whereas we allow posets to be empty. This reflects customary practice: algebras are non-empty but relational structures, such as posets, are allowed to have an empty underlying set.

Mini-exercise Let P and Q be non-empty posets. Prove that $P \times Q$, with the usual co-ordinatewise order, is a lattice if and only if both P and Q are lattices.

Mini-exercise Let L be a lattice. Prove that for all $a, b, c, d \in L$

(i) $a \leqslant b$ implies $a \vee c \leqslant b \vee c$ and $a \wedge c \leqslant b \wedge c$;
(ii) $a \leqslant b$ and $c \leqslant d$ imply $a \vee c \leqslant b \vee d$ and $a \wedge c \leqslant b \wedge d$. (Note that this says precisely that the binary operations $\vee \colon L \times L \to L$ and $\wedge \colon L \times L \to L$ are monotone.)

4.2 Examples of Lattices

(1) Every non-empty chain is a lattice in which $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Thus the real numbers, $\mathbb{R}$, and the natural numbers, $\mathbb{N}$, are lattices under the usual orderings. Note that $\mathbb{R}$ lacks both $\top$ and $\bot$ and that $\mathbb{N}$ lacks $\top$.
(2) For any set X, the powerset $\wp(X)$ is a lattice in which $\vee$ and $\wedge$ are just $\cup$ and $\cap$. Dually, $\wp(X)^\partial$ is a lattice, with $\vee$ as $\cap$ and $\wedge$ as $\cup$.
(3) Now let $\varnothing \neq \mathfrak{L} \subseteq \wp(X)$. Then $\mathfrak{L}$ is known as a **lattice of sets** if it is closed under finite unions and intersections. In a lattice of sets $\mathfrak{L}$ we have $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ for $A, B \in \mathfrak{L}$. This is not *quite* obvious; see 5.2. As examples, we see that, for any poset P, our old friends $\mathcal{U}(P)$ and $\mathcal{O}(P)$ are lattices of sets.

4.3 Distributive Lattices

In any powerset $\wp(X)$ we have, for $A, B, C \subseteq X$,

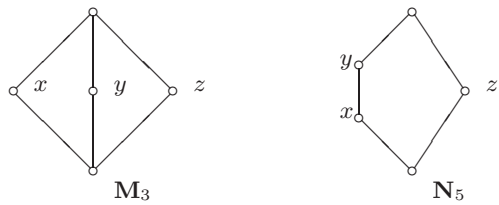
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{and} \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Therefore $\wp(X)$ satisfies the distributive laws: for all $x, y, z \in L$

(D) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(D)$^\partial$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

More generally, any lattice of sets satisfies these laws, and in particular the lattices $\mathcal{O}(P)$ and $\mathcal{U}(P)$ of down-sets and up-sets of a poset P are distributive.

We may ask whether *every* lattice is distributive. It is far from transparent algebraically whether or not this is true, and it is well known that in the pioneering days of lattice theory more than a century ago it was 'proved' that all lattices are distributive. However it can be seen extremely easily that both the lattices in Figure 10 fail (D)—the pictorial approach demonstrates its power!



$\mathbf{M}_3$ $\mathbf{N}_5$

Fig. 10. A pair of non-distributive lattices

We remark that it can be shown that, globally in a lattice L, (D) holds if and only if $(D)^\partial$ does. On the other hand, locally, for particular triples of elements x, y, z, the two conditions are not equivalent. See ILO2, Chapter 4.

Mini-exercise Consider again the context $(X, X, R_=)$ in Exercise 2.5.

(i) Show that if $|X| = 3$ the poset $\mathfrak{B}(X, X, R_=)$ is isomorphic to the lattice $\mathbf{M}_3$ in Figure 10.
(ii) Convince yourself that for arbitrary X the poset $\mathfrak{B}(X, X, R_=)$ is a lattice and that this is not distributive.

4.4 Boolean Algebras

The best-known lattices of all are the powersets, and these come ready equipped with an extra unary operation, $'$, of complementation satisfying $x \vee x' = \top$ and $x \wedge x' = \bot$. Such an operation is not available in arbitrary distributive lattices with $\bot$ and $\top$. In the chain $0 < 1 < 2$, for example, 1 has no complement and, in a down-set lattice $\mathcal{O}(P)$, the only complemented elements are those which are up-sets in P as well as being down-sets. The distributive lattices possessing nullary operations $\bot$, $\top$ and a unary operation $'$ are the **Boolean algebras**.

4.5 Lattices in Logic

It is no coincidence that the symbols adopted for join and meet in a lattice are the same as those used for disjunction and conjunction in logic. Consider PROP, the propositions of classical propositional calculus: it looks as if these should form a lattice with 'or' and 'and' acting as the lattice operations; $\neg$ as $'$;

F ('falsity') as $\bot$ and **T** ('truth') as $\top$. We would then expect $\rightarrow$ ('implies') to play the role of $\leqslant$. This doesn't quite work: we can have distinct propositions α and β for which $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ both hold. Thus $\rightarrow$ defines a pre-order rather than a partial order. To get a partial order we don't distinguish α and β when $\alpha \leftrightarrow \beta$. With this identification, which can be formalized in terms of the relation of logical equivalence on PROP, we do get a Boolean algebra. A brief account of the theory of Boolean algebras, including an elementary treatment of the role of lattice theory in propositional calculus, is given in ILO2, Chapter 4.

One point about logic and lattices is well worth stressing. It is by no means always the case that logics have a classical, Boolean, negation. Logics of various different kinds are extensively used in computer science as a means of reasoning about programs. In such a setting we may wish to model negation in a less restrictive way, retaining the property that $P \wedge \neg P = \mathbf{F}$ ('not both of P and $\neg P$ are true'), but discarding the **Law of the Excluded Middle**, $P \vee \neg P = \mathbf{T}$, and substituting something weaker. This is done, for example in intuitionistic logic, where the implication operation behaves differently from that in classical logic; see 6.5.

We may also wish to allow truth values other than the booleans **F** and **T**: for example, we might want to accommodate a third value, **P**, representing 'possible' or 'not yet determined'. Or we might, as in probabilistic models and in fuzzy logic, wish to allow truth values lying in the interval $[0, 1]$. We may also, as in modal and temporal logic, wish to allow for additional operations, such that $\Diamond$ and $\Box$. But, however we want our additional operations to behave, it is almost always the case that the usual laws will govern disjunction and conjunction, and then there will be an underlying distributive lattice associated with the logic. The study of logics from an algebraic point of view has benefitted from and driven forward the study of lattices, in particular of distributive lattices with additional operations. We do not have space to explore these ideas further here. For additional information see, for example, the article by Davey & Priestley [6] and Brink & Rewitzky [5].

4.6 Upper Bounds and Sups, Lower Bounds and Infs

So far we have discussed lattices, but not the complete lattices we have advertised. We now work towards remedying this omission. Let P be a poset and let $S \subseteq P$. We define

$$S^u := \{\, x \in P \,:\, (\forall s \in S)\, x \geqslant s \,\} \quad \text{and} \quad S^\ell := \{\, x \in P \,:\, (\forall s \in S)\, x \leqslant s \,\};$$

S^u and S^ℓ are, respectively, the sets of all **upper bounds** and all **lower bounds** of S. Notice that $\varnothing^u = \varnothing^\ell = P$. Now let $S \neq \varnothing$. Then it is easy to see that the sets of bounds can be alternatively described by

$$S^u := \bigcap \{\, \uparrow s \,:\, s \in S \,\} \quad \text{and} \quad S^\ell := \bigcap \{\, \downarrow s \,:\, s \in S \,\}.$$

In particular, for elements x, y of P,

$$\{x, y\}^u = \uparrow x \cap \uparrow y \quad \text{and} \quad \{x, y\}^\ell = \downarrow x \cap \downarrow y.$$

Accordingly, a lattice is a non-empty poset in which, for every pair of elements x, y, the set $\{x, y\}^u$ has a least (bottom) element and $\{x, y\}^\ell$ has a greatest (top) element.

For an arbitrary subset S of a poset P, we say that the **supremum** or **sup** (also known as the **least upper bound** or **join**), α, of S exists if

(sup1) $(\forall s \in S)\ s \leqslant \alpha$ (that is, $\alpha \in S^u$, so α is *an* upper bound of S);
(sup2) $(\forall x \in S^u)\, \alpha \leqslant x$ (that is, α is the *least* upper bound of S).

In this case we write $\bigvee S$ for α, or, when we need to keep track of the poset in which we are working, $\bigvee_P S$. When dealing with families of sets, we abuse notation slightly and write $\bigvee_{i\in I} A_i$ in place of $\bigvee\{ A_i : i \in I \}$, and similarly with other operators in place of $\bigvee$.

The supremum α of S is characterized by

(sup) $(\forall y \in P)((\forall s \in S)\, y \leqslant s \iff y \leqslant \alpha)$.

This is slicker, and more in the spirit of an equational approach, but is less transparent until the two-step definition has been fully mastered.

Likewise, the **infimum** or **inf** (also known as the **greatest lower bound** or **meet**), *beta*, of S exists if

(inf) $(\forall y \in P)((\forall s \in S)\, y \geqslant s \iff y \geqslant \beta)$.

and we write $\bigwedge S$ (or $\bigwedge_P S$) for β. Clearly sup and inf are dual notions, with sups in P translating into infs in P^∂.

Mini-exercise Let P be a poset, let $S, T \subseteq P$ and assume that $\bigvee S$, $\bigvee T$, $\bigwedge S$ and $\bigwedge T$ exist in P. Check the following oft-used elementary facts.

(i) For all $s \in S, s \leqslant \bigvee S$ and $s \geqslant \bigwedge S$.
(ii) Let $x \in P$; then $x \leqslant \bigwedge S$ if and only if $x \leqslant s$ for all $s \in S$.
(iii) Let $x \in P$; then $x \geqslant \bigvee S$ if and only if $x \geqslant s$ for all $s \in S$.
(iv) $\bigvee S \leqslant \bigwedge T$ if and only if $s \leqslant t$ for all $s \in S$ and all $t \in T$.
(v) If $S \subseteq T$, then $\bigvee S \leqslant \bigvee T$ and $\bigwedge S \geqslant \bigwedge T$.

(Compare with Mini-exercise 4.1.)

4.7 Much Ado about Nothing, and about Everything

Let P be a poset and $S = \varnothing$. As noted earlier, $\varnothing^u = P$ and hence $\sup \varnothing$ exists if and only if P has a bottom element, and in that case $\sup \varnothing = \bot$. Dually, $\inf \varnothing = \top$ whenever P has a top element.

It is easily seen that, if P has a top element, then $P^u = \{\top\}$ in which case $\sup P = \top$. When P has no top element, we have $P^u = \varnothing$ and hence $\sup P$ does not exist. By duality, $\inf P = \bot$ whenever P has a bottom element.

4.8 Complete Lattices

As we have already seen, in a lattice L, $x \vee y = \bigvee\{x, y\}$ and $x \vee y = \bigwedge\{x, y\}$. Further, $\bigvee F$ and $\bigwedge F$ exist for any non-empty finite subset of L, viewed either as iterated binary joins and meets or as an instance of sups and infs. We say that a non-empty poset P is a **complete lattice** if $\bigwedge S$ and $\bigvee S$ exist for all $S \subseteq P$. We do not exclude $S = \varnothing$ so that any complete lattice has $\top$ and $\bot$.

4.9 Completeness on the Cheap

Let P be a non-empty poset.

(i) Assume that $\bigwedge S$ exists in P for every non-empty subset S of P. Then $\bigvee S$ exists in P for every subset S of P which has an upper bound in P; indeed, $\bigvee S = \bigwedge S^u$.

(ii) The following are equivalent:
   (a) P is a complete lattice;
   (b) $\bigwedge S$ exists in P for every subset S of P;
   (c) P has a top element, $\top$, and $\bigwedge S$ exists in P for every non-empty subset S of P.

Proof. (i) Let $S \subseteq P$ and assume that S has an upper bound in P; thus $S^u \neq \varnothing$. Hence, by assumption, $\beta := \bigwedge S^u$ exists in P. But this means that $\bigvee S = \beta$.

(ii) It is trivial that (a) implies (b), and (b) implies (c) since the inf of the empty subset of P exists only if P has a top element (see 4.7). It follows easily from (i) that (c) implies (a). □

4.10 A Special Class of Complete Lattices

Any finite lattice L is automatically a complete lattice, because the supremum (infimum) of a non-empty subset is in fact an iterated join (meet) while the sup and inf of $\emptyset$ are $\bigwedge L$ and $\bigvee L$, respectively (see 4.7 and 4.8).

This completeness result extends to an important class of infinite posets. A poset P satisfies the **ascending chain condition**, (ACC), if given any sequence $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$ of elements of P, there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \ldots$. Any flat poset satisfies (ACC). As non-flat examples of posets satisfying (ACC) we present $\mathbb{N}^\partial$ and $\wp_{\mathrm{fin}}(\mathbb{N})^\partial$ (where $\wp_{\mathrm{fin}}(\mathbb{N})$ denotes the finite subsets of $\mathbb{N}$, ordered by inclusion). As an example of an infinite lattice in which (ACC) holds, take any infinite antichain with top and bottom adjoined.

Let L be a lattice satisfying (ACC). We assert that for $\varnothing \neq S \subseteq P$ there exists a finite subset F of S such that $\bigvee S = \bigvee F$ (which certainly exists). Consequently, if P is a lattice with $\bot$ which satisfies (ACC) then P is a complete lattice, by 4.9. The assertion that arbitrary suprema reduce to finite ones in the presence of (ACC) relies on an ancillary result of independent interest, stating that a poset P satisfies (ACC) if and only if $\operatorname{Max} S \neq \varnothing$ for $\varnothing \neq S \subseteq P$. The forward implication needs the Axiom of Choice; see 8.14 below.

4.11 Suprema, Infima, and Monotone Maps

Consider a little example. Let P be the linear sum $\mathbb{N} \oplus Q$, where $\mathbb{N}$ has its usual order and $Q = \{a, b\}$ is a 2-element chain with $a < b$. Note that $\bigvee_P \mathbb{N} = a$. Define $F\colon P \to P$ by letting F be the identity map on $\mathbb{N}$ and map both a and b to b. Then F is monotone. Also

$$F\big(\bigvee_P \mathbb{N}\big) = F(a) = b > a = \bigvee_P \mathbb{N} = \bigvee_P F(\mathbb{N}).$$

The moral is that monotone maps need not preserve suprema (or, dually, infima) even when these exist.

On the other hand, any map between lattices preserving $\vee$ (or $\wedge$) is automatically monotone. To see this, let $F\colon L \to M$ be such that $F(x \vee y) = F(x) \vee F(y)$ for all x, y. Then, for $x, y \in L$,

$$\begin{aligned} x \leqslant y &\implies x \vee y = y && \text{(by the Connecting Lemma, 4.1)} \\ &\implies F(x) \vee F(y) = F(y) && \text{(by assumption)} \\ &\implies F(x) \leqslant F(y) && \text{(by the Connecting Lemma).} \end{aligned}$$

All the more so, if F is a map between complete lattices P and Q such that $F(\bigvee_P S) = \bigvee_Q F(S)$ for any (non-empty) subset S of P, then F is monotone. Examining the argument above we see that all we have used is the fact that $F(\bigvee\{u, v\}) = \bigvee\{F(u), F(v)\}$ when u and v are comparable. Accordingly, any map preserving suprema of chains, when these exist, is monotone.

Although monotone maps need not preserve suprema or infima, we can, usefully, get half way: let $F\colon P \to Q$ be a monotone map between posets P and Q and let $S \subseteq P$. Then

$$F\big(\bigvee_P S\big) \geqslant \bigvee_Q F(S) \quad \text{and} \quad F(\bigwedge_P S) \leqslant \bigwedge_Q F(S)$$

whenever the sups and infs involved exist. To verify the first of these note that:

$$\begin{aligned} (\forall s \in S)\, s \leqslant \bigvee_P S &\implies (\forall s \in S)\, F(s) \leqslant F\big(\bigvee_P S\big) && \text{(since } F \text{ is monotone)} \\ &\implies F\big(\bigvee_P S\big) \in F(S)^u. \end{aligned}$$

For order-isomorphisms the situation is, not unexpectedly, better. To state the result we need a definition. Let P and Q be posets. A map $F\colon P \to Q$ is said to **preserve existing sups** if whenever $\bigvee_P S$ exists then $\bigvee_Q F(S)$ exists and $F\big(\bigvee_P S\big) = \bigvee_Q F(S)$. Preservation of existing infs is defined dually.

Mini-exercise Assume that P and Q are posets and that $F\colon P \to Q$ is an order-isomorphism. Then F preserves all existing sups and infs. In particular, the image of a (complete) lattice under an order-isomorphism is a (complete) lattice. Further, an order-isomorphism preserves $\top$ and $\bot$ when these exist.

The pay-off from the elementary observations in this subsection comes when we consider Galois connections and fixed points in Sections 7 and 8.

5 Complete Lattices, Concretely: Closure Systems and Closure Operators

Once we have recorded one elementary technical fact we will be able to exhibit many examples of complete lattices.

5.1 A Useful Technical Remark

Let K be a non-empty subset, with the inherited order, of some complete lattice L. For $S \subseteq K$, we claim that

$$\bigvee_L S \in K \implies \bigvee_K S \text{ exists and equals } \bigvee_L S,$$
$$\bigwedge_L S \in K \implies \bigwedge_K S \text{ exists and equals } \bigwedge_L S,$$

leaving as a **Mini-exercise** the verification that the elements asserted to be the sup and inf really do serve as the least upper bound and greatest lower bound.

5.2 Complete Lattices of Sets

Applied in a powerset, 5.1 tells us that any non-empty family $\mathcal{L}$ of subsets of a set X is a complete lattice if it is such that $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$ belong to $\mathcal{L}$ for any family of sets $\{A_i\}_{i\in I}$ in $\mathcal{L}$ is a complete lattice, in which $\bigvee_{\mathcal{L}}$ and $\bigwedge_{\mathcal{L}}$ are given by $\bigcup$ and $\bigcap$. A lattice of this type is known as a **complete lattice of sets**. Important examples are powersets and their duals, and the lattices $\mathcal{U}(P)$ and $\mathcal{O}(P)$ of up-sets and down-sets of a poset P.

5.3 Closure Systems

The down-set and up-set lattices $\mathcal{O}(P)$ and $\mathcal{U}(P)$ are defined within the powerset lattice $\wp(P)$ by reference to the order relation $\leqslant$ on P. Likewise, there are many other situations in which structure on a set X naturally leads to consideration of subsets of $\wp(X)$. Here are some mathematical examples, instructive for those with the requisite knowledge.

(1) Suppose that V is a vector space. Then $\mathrm{Sub}\,(V)$, the subspaces of V, form a subset of $\wp(V)$. Let $U_1, U_2 \in \mathrm{Sub}\,(V)$. Notice that the intersection $U_1 \cap U_2$ is always a subspace, but that the union $U_1 \cup U_2$ never is, unless either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. (For an example, look at the vector space $\mathbb{R}^2$, with U_1 and U_2 distinct lines through 0: the sum of non-zero vectors, one from U_1 and one from U_2, is in neither U_1 nor U_2.) Thus $\mathrm{Sub}\,(V)$ is *not* a complete lattice of sets, as defined in 5.2, nor even a lattice of sets.

(2) The example in (1) is one of a family of examples of similar type: substructures of some given algebraic structure. We might look for example at subgroups of a group, subrings of a ring, ... In very many such cases we have closure under intersections, but seldom closure under unions.

(3) Let X be a set equipped with a topology $\mathfrak{T}$. Then the family of **closed** subsets of the topological space $(X, \mathfrak{T})$ is closed under arbitrary intersections but not in general arbitrary unions. If we look at the open sets, then the position is reversed: open sets are closed under arbitrary unions but not in general under arbitrary intersections. (We remark in passing that every poset P has an associated topology in which we declare the open sets to be the up-sets; taking complements, the closed sets are then the down-sets. In this case, the family of open sets is closed under intersections.)

In all the preceding examples we are half way to having a complete lattice of sets, but only half way. This is where 4.9 comes to the rescue. We do get complete lattices, but with only one of the operations $\bigwedge$ and $\bigvee$ as the set-theoretic one. Here's how.

5.4 Closure Systems

Let X be a set and let $\mathfrak{L}$ be a family of subsets of X, ordered as usual by inclusion, and such that

(cs1) $\bigcap_{i\in I} A_i \in \mathfrak{L}$ for every non-empty family $\{A_i\}_{i\in I} \subseteq \mathfrak{L}$, and
(cs2) $X \in \mathfrak{L}$.

Then $\mathfrak{L}$ is a complete lattice in which

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i,$$
$$\bigvee_{i\in I} A_i = \bigcap \{ B \in \mathfrak{L} : \bigcup_{i\in I} A_i \subseteq B \}.$$

Proof. By 4.9, to show that $\mathfrak{L}$ is a complete lattice for the inclusion order it suffices to show that $\mathfrak{L}$ has a top element and that the inf of every non-empty subset of $\mathfrak{L}$ exists in $\mathfrak{L}$. By (cs2), $\mathfrak{L}$ has a top element, namely X. Let $\{A_i\}_{i\in I}$ be a non-empty subset of $\mathfrak{L}$; then (cs1) gives $\bigcap_{i\in I} A_i \in \mathfrak{L}$. By 5.1, $\bigcap_{i\in I} A_i$ is the inf of $\{A_i\}_{i\in I}$ in $\mathfrak{L}$, that is,

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i.$$

Thus $\mathfrak{L}$ is indeed a complete lattice when ordered by $\subseteq$.

Since X is an upper bound of $\{A_i\}_{i\in I}$ in $\mathfrak{L}$, 4.9(i) gives

$$\begin{aligned} \bigvee_{i\in I} A_i &= \big(\bigwedge_{i\in I} A_i\big)^u \\ &= \bigcap \{ B \in \mathfrak{L} : (\forall i \in I)\, A_i \subseteq B \} \\ &= \bigcap \{ B \in \mathfrak{L} : \bigcup_{i\in I} A_i \subseteq B \}. \quad \square \end{aligned}$$

If $\mathfrak{L}$ is a non-empty family of subsets of X which satisfies conditions (cs1) and (cs2) above, then $\mathfrak{L}$ is said to be a **closure system** (called a **topped intersection structure** in ILO2) on X. If $\mathfrak{L}$ just satisfies (cs1), it is referred to as an **intersection structure**. Intersection structures arising in computer science are usually topless while those in algebra are almost invariably topped.

5.5 Examples

(1) Consider $A \multimap B$, where A, B are non-empty sets. From the observations in 1.6 we saw that the map $\pi \mapsto \operatorname{graph}\pi$ is an order-embedding of $A \multimap B$ into $\wp(A \times B)$. Let $\mathfrak{L}$ be the family of subsets of $A \times B$ which are graphs of partial maps. To prove that $\mathfrak{L}$ is closed under intersections, use the characterization given in 1.6: if $S \subseteq A \times B$, then $S \in \mathfrak{L}$ if and only if $(s, x) \in S$ and $(s, x') \in S$ imply $x = x'$. Thus $\mathfrak{L}$ is an an intersection structure. It is not topped unless B has just one element.

(2) Each of the following is a closure system and so forms a complete lattice under inclusion: the subspaces of a vector space, the subgroups, or normal subgroups, of a group, the congruence relations on an algebra, the convex subsets of the Euclidean plane, ... These families all belong to a class of intersection structures closely related to Scott domains.

(3) The closed subsets of a topological space are closed under finite unions and finite intersections and hence form a lattice of sets in which $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. In fact, the closed sets form a closure system and consequently the lattice of closed sets is complete. Infs are given by intersection while the sup of a family of closed sets is not their union but is obtained by forming the *closure* of their union.

5.6 From a Complete Lattice to a Closure System

For any poset P, the map $x \mapsto {\downarrow}x$ from P to $\mathcal{O}(P)$ is an order-embedding (recall 3.15). So any poset P can be faithfully mapped into a complete lattice $\mathcal{O}(P)$. We can take this a bit further. Let L be an (abstract) complete lattice. We claim that L is isomorphic to a closure system.

Proof. Use $F\colon x \mapsto {\downarrow}x$ to map L into $\mathcal{O}(L)$. Then F maps L order-isomorphically onto $\mathcal{L} := \{{\downarrow}x : x \in L\}$. Now, $\mathcal{L}$ has top element ${\downarrow}\top$ and is closed under intersections:

$$\bigcap_{i\in I} {\downarrow}x_i = \bigwedge_{i\in I} {\downarrow}x_i$$

(see 5.1). □

5.7 Defining Closure Operators

Let P be a poset. Then a map $c\colon P \to P$ is called a **closure operator** (on P) if, for all $x, y \in P$,

(clo1) $x \leqslant c(x)$,
(clo2) $x \leqslant y \Longrightarrow c(x) \leqslant c(y)$,
(clo3) $c(c(x)) = c(x)$.

An element $x \in P$ is called **closed** if $c(x) = x$. The set of all closed elements of P is denoted by P_c.

As examples of closure operators we have

(i) the operators $\downarrow$ and $\uparrow$ on the subsets of a poset, for which the closed sets are respectively the down-sets and the up-sets (3.15);
(ii) the operator of topological closure defined on any topological space.

In these examples the lattices of closed sets form closure systems, and so are complete lattices. This is true more generally.

5.8 New Complete Lattices from Old: From a Closure Operator to a Complete Lattice

Let P is a complete lattice and let $c\colon P \to P$.

(i) (The **Prefix Lemma**) Assume that c is monotone. Then $Q := \{\, x \in P : c(x) \leqslant x \,\}$ is a complete lattice.
(ii) Assume that c is a closure operator on P. Then

$$c(P) = P_c := \{\, x \in P : c(x) = x \,\}$$

is a complete lattice in which

$$\bigwedge\nolimits_{P_c} S = \bigwedge\nolimits_P S \text{ and } \bigvee\nolimits_{P_c} S = c\big(\bigvee\nolimits_P S\big),$$

and $\top_{c(P)} = c(\top_P)$.

Proof. (i) To prove that Q is a complete lattice, it suffices, by 4.9, to show that arbitrary infs exist in Q. By 5.1, this happens if, for every $S \subseteq Q$, we have $\alpha := \bigwedge_P S \in Q$ (and then $\bigwedge_Q S = \alpha$). But

$$\begin{aligned} (\forall s \in S)\, s \geqslant \alpha &\Longrightarrow (\forall s \in S)\, c(s) \geqslant c(\alpha) && \text{(by definition of } Q\text{, and (po3))} \\ &\Longrightarrow \alpha \geqslant c(\alpha) && \text{(by definition of } \textstyle\bigwedge\text{).} \end{aligned}$$

Now consider (ii). Note first that $y \in P_c$ implies $y = c(y) \in c(P)$ while if $y = c(x)$ for some $x \in P$, then $c(y) = c(c(x)) = c(x) = y$, by (clo3), and so $y \in P_c$. Thus $c(P) = P_c$, and by (clo3) this set is just the set Q in (i). From the proof above, P_c is a complete lattice, whose infs coincide with those in P. We must now establish the formula for sups. Let $\beta := c(\bigvee_P S)$ where $S \subseteq P_c$. Certainly $\beta \in c(P) = P_c$. Also

$$\begin{aligned} s \in S &\Longrightarrow s \leqslant \textstyle\bigvee_P S && \text{(definition of } \textstyle\bigvee_P\text{)} \\ &\Longrightarrow s = c(s) \leqslant c(\textstyle\bigvee_P S) = \alpha && \text{(by (clo2)),} \end{aligned}$$

so β is an upper bound for S in P_c. Also, for any upper bound y for S with $y \in P_c$,

$$\begin{aligned} (\forall s \in S)\, s \leqslant y &\Longrightarrow \textstyle\bigvee_P S \leqslant y && \text{(definition of } \textstyle\bigvee_P\text{)} \\ &\Longrightarrow \beta = c(\textstyle\bigvee_P S) \leqslant c(y) = y && \text{(using (clo2)),} \end{aligned}$$

so β is indeed the least upper bound. Finally, $\top_P = c(\top_P)$, by (clo1). □

5.9 Closure Operators more Concretely

The most commonly occurring closure operators are those on powerset lattices. By specializing 5.8 we get (i) below. The proofs of (ii) and (iii) are straightforward.

(i) Let X be a set and C a closure operator on $\wp(X)$. Then the family

$$\mathfrak{L}_C := \{\, A \subseteq X \,:\, C(A) = A \,\}$$

of closed subsets of X is a closure system and so forms a complete lattice, when ordered by inclusion, in which

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i,$$
$$\bigvee_{i\in I} A_i = C\big(\bigcup_{i\in I} A_i\big).$$

(ii) Given a closure system $\mathfrak{L}$ on X the formula

$$C_{\mathfrak{L}}(A) := \bigcap \{\, B \in \mathfrak{L} \,:\, A \subseteq B \,\}.$$

defines a closure operator $C_{\mathfrak{L}}$ on X.

(iii) The relationship between closure systems and closure operators is a bijective one: the closure operator induced by the closure system $\mathfrak{L}_C$ is C itself, and, similarly, the closure system induced by the closure operator $C_{\mathfrak{L}}$ is $\mathfrak{L}$; in symbols,

$$C_{\mathfrak{L}_C} = C \quad \text{and} \quad \mathfrak{L}_{C_{\mathfrak{L}}} = \mathfrak{L}.$$

Of course, under the correspondence in (iii), the closure operator $\downarrow\colon \wp(P) \to \wp(P)$ corresponds to $\mathcal{O}(P)$, and $\uparrow$ to $\mathcal{U}(P)$, for any poset P.

6 Galois Connections: Basics

This section may profitably be read in parallel with the treatment of Galois connections in Chapter 4. The latter complements the mathematical discussion here by giving a detailed presentation of some computationally instructive examples and by recasting some core notions in a framework especially well suited to fixed point calculus. Galois connections are also explored both in Chapter 4 and in Chapter 9, with the emphasis in the latter being from the viewpoint of what are termed **Galois algebras**.

6.1 Introduction

Let P and Q be posets. A pair (F, G) of maps $F\colon P \to Q$ and $G\colon Q \to P$ is said to defines a **Galois connection** between P and Q if and only if

(Gal) $F(p) \leqslant q \iff p \leqslant G(q)$ for all $p \in P$, $q \in Q$,

Those familiar with maps between sets but less familiar with Galois connections may find it helpful to recognize that when $\leqslant$ is the equality relation $=$ then F and G are just set-theoretic inverses for each other: $F(p) = q$ holds if and only if $G(q) = p$, for $p \in P$, $q \in Q$. This set-theoretic example gives pointers to certain elementary facts about Galois connections. For example, just as invertible (bijective) maps $F_1 \colon P \to Q_1$ and $G \colon Q \to S$ compose to give an invertible map $G \circ F \colon P \to S$, so too can Galois connections be composed: if (F, G) is a Galois connection between P and Q and (H, K) is a Galois connection between Q and S, then $(H \circ F, K \circ G)$ is a Galois connection between P and S.

The symbols F and G don't actively assist one in remembering which map appears to the left of $\leqslant$ and which to the right. Accordingly, following the usage in ILO2, we shall in the first part of this section replace F and G by $^{\triangleright}$ and $^{\triangleleft}$ with these triangle maps written to the right of their arguments. In this notation, the Galois connection condition becomes

$$\text{(Gal)}\quad p^{\triangleright} \leqslant q \iff p \leqslant q^{\triangleleft} \text{ for all } p \in P \text{ and } q \in Q.$$

Just as there is no universally accepted notation for Galois connections there is also, regrettably, no uniformly adopted terminology, though generally nomenclature relates to the side of $\leqslant$ on which a map appears: we adopt the terms **lower adjoint** and **upper adjoint** for $^{\triangleright}$ and $^{\triangleleft}$, respectively; alternative terms are **left adjoint** and **right adjoint**. The notation in Gierz *et al.* [9]: d (for 'down') and g (for 'greater') seems, grammatically at least, a trifle odd.

We suggested at the start that ubiquitous concepts adopt different guises in different settings. Galois connections illustrate this all too well. There are two versions of the definitions: the one we adopt here, in which the paired maps are monotone (that is, order-preserving) and the other in which these are order-reversing. The literature seems to divide roughly equally between the two alternatives (for example, ILO2, Birkhoff [4] and Ganter & Wille [8] have order-reversing maps and Aarts [1] and Gierz *et al.* [9] order-preserving ones. Historically, and in algebra, there are arguments for order-reversal: Galois's own Galois connection between field extensions and subgroups of a Galois group is order-reversing. The two formulations are instances of, respectively, a categorical adjunction and dual adjunction. The difference is not significant: we can swap backwards and forwards between the two versions by swapping between Q and Q^{∂}.

6.2 Lattice Representation via Galois Connections

Assume that L is a lattice and let X be a subset of L which is **join-dense**, in the sense that every element of L is a (possibly empty) join of elements from X. Define $F \colon \mathcal{O}(X) \to L$ by $F(A) := \bigvee A$ and $g \colon L \to \mathcal{O}(X)$ by $G(a) := {\downarrow}a \cap X$. Then (F, G) is a Galois connection and it satisfies $G \circ F = \mathrm{id}_L$. **Mini-exercise**: verify these claims.

Now assume that L is finite. A natural choice for X here is $\mathcal{J}(L)$, the subset of L consisting of elements which are **join-irreducible**, that is, cannot be

obtained as a (finite) join of strictly smaller elements. With this choice, more can be said about the Galois connection (F, G). If—and necessarily only if—L is distributive, it can be shown that we additionally have $F \circ G = \mathrm{id}_{\mathcal{O}(X)}$. Thus F and G set up an isomorphism between L and $\mathcal{O}(X)$. This is **Birkoff's representation theorem**. The theorem shows that every finite distributive lattice can be concretely represented as the down-set lattice of a finite poset.

Consider the special case of the above in which X has the equality order. Then we have $\mathcal{O}(X) = \wp(X)$ (by the dual of 3.10). This occurs precisely when L has a complementation operation, $'$, and so is a Boolean algebra (as a finite lattice, L certainly has $\bot$ and $\top$). In the concrete representation of L as a powerset this negation is captured by set complement (recall Exercise 3.14). What we have here is the finite version of Stone's famous representation of Boolean algebras.

There are two ways in which these representations may be extended to the infinite case. The link between the two approaches lies in the theory of canonical extensions, famously pioneered in the Boolean case by Jónsson & Tarski; see Jónsson [10], in particular §3.2.

The first approach captures the finitary nature of $\vee$ and $\wedge$ by adding a compact topology to the structure X. We pursue this a little further in Section 9. Alternatively, we may consider infinitary disjunctions and conjunctions, and replace join-irreducible elements by completely join-irreducible ones. An element is **completely join-irreducible** if it is not the supremum of strictly smaller elements. We obtain a Galois connection (F, G) satisfying $F \circ G = \mathrm{id}_{\mathcal{O}(X)}$ between L and $\mathcal{O}(X)$, where X is the set $\mathcal{J}^\infty(L)$ of completely join-irreducible elements of L. Assume that L is complete and distributive. Then a variety of different but equivalent conditions can be imposed on L to make F and G mutually inverse isomorphisms. These conditions include strong distributivity conditions involving arbitary disjunction, $\bigvee$, and arbitrary conjunction, $\bigwedge$. The results reduce in the Boolean case to well-known characterizations of powerset algebras as those Boolean algebras which are, equivalently, either complete and completely distributive or complete and atomic. A self-contained treatment of both the distributive and Boolean cases can be found in ILO2, Chapter 10.

6.3 Galois Connections from Binary Relations: Method I

Let $R \subseteq G \times M$ be a binary relation, do that (G, M, R) is a context. As we indicated in 2.4 the maps ${}^{\triangleright} : \wp(G) \to \wp(M)^\partial$ and ${}^{\triangleleft} : \wp(M)^\partial \to \wp(G)$ given by

$$A^{\triangleright} := \{\, m \in M \,:\, (\forall g \in A)\, (g, m) \in R \,\},$$
$$B^{\triangleleft} := \{\, g \in G \,:\, (\forall m \in B)\, (g, m) \in R \,\}$$

define a Galois connection. Two special instances are worth noting; both are of order-theoretic interest.

Example 1. Let P be a poset. Take the relation R as $\not\geqslant$. Then (Exercise 3.16) we have

$$A^{\triangleright} = P \setminus {\downarrow}A \quad \text{and} \quad A^{\triangleleft} = P \setminus {\uparrow}A$$

and $({}^{\triangleright}, {}^{\triangleleft})$ establishes a Galois connection between $\wp(P)$ and $\wp(P)^{\partial}$. Further, we have

$$A^{\triangleright\triangleleft} = P \setminus \uparrow(P \setminus \downarrow A) = P \setminus (P \setminus \downarrow A) = \downarrow A$$

(using the fact that $P \setminus \downarrow A$ is always an up-set). Dually, $A^{\triangleleft\triangleright} = \uparrow A$. $\supseteq$,

Example 2. Our choice of $\not\geqslant$ as the relation in the preceding example may have seemed a trifle perverse. Now consider instead the $\leqslant$ relation of a poset P. Then for $A, B \subseteq P$ we see that $A^{\triangleright}$ and $B^{\triangleleft}$ are respectively the sets of upper bounds of A and lower bounds of B:

$$A^u := \{\, y \in P \,:\, (\forall x \in A)\, x \leqslant y \,\},$$
$$B^\ell := \{\, y \in P \,:\, (\forall x \in B)\, y \leqslant x \,\}.$$

It is easy to see directly that $({}^u, {}^\ell)$ is a Galois connection between $\wp(P)$ and $\wp(P)^{\partial}$:

$$\begin{aligned} A^u \supseteq B &\iff (\forall y \in B)((\forall x \in A)\, x \leqslant y) \\ &\iff (\forall x \in A)((\forall y \in B)\, y \geqslant x) \\ &\iff A \subseteq B^\ell. \end{aligned}$$

6.4 Galois Connections and Algebras—A Fleeting Glimpse

Any algebra $(A; F)$ gives rise to a fundamental (order-reversing) Galois connection, via the maps Inv and Pol. These arise as the polar maps associated with the binary relation of preservation between the operations, F, and the finitary relations, R, on A. Specifically,

- Pol(R) denotes the family of all finitary functions $f\colon A^n \to A$ ($n \geqslant 1$) which preserve the relations in R (this is known as a **clone**);
- The set of all relations s which are invariant under all functions $f \in F$ is denoted by Inv(F).

For illustrative examples, see Mckenzie *et al.* [14], pp. 51–53.

6.5 Galois Connections by Sectioning

Galois maps can be viewed as unary operations. However many operations of importance in, for example, algebra and logic, are binary. Bringing such operations within the scope of the theory of Galois connections requires a well-worn trick: treat one argument as a parameter and the other as the variable on which $^{\triangleright}$ or $^{\triangleleft}$ is to act.

Just as Boolean algebras model classical propositional logic, so Heyting algebras model intuitionistic propositional logic. A distributive lattice L with $\bot$ and $\top$ is a **Heyting algebra** if, for every $a, b \in L$, there exists $a \to b \in L$ characterized by

$$c \leqslant (a \to b) \iff a \wedge c \leqslant b.$$

Put another way, each map $\wedge_a : c \mapsto a \wedge c$ $(a \in L)$ has an upper adjoint, $b \mapsto (a \to b)$. Very many other examples of Galois connections arising by **sectioning** can be found in Chapters 4 and 9.

It is well known that, in a Heyting algebra,

$$a \to b = \max\{\, c \in L \,:\, a \wedge c \leqslant b \,\};$$

the existence of the maximum element in the set on the right-hand side is the condition for the existence of $a \to b$. This observation is not particular to this example. In 6.11 below, we give formulae for obtaining each of the maps $({}^{\triangleright}, {}^{\triangleleft})$ in any Galois connection from the other.

6.6 Galois Connections from Binary Relations: Method II

It may be thought really tiresome that our way of constructing a Galois connection from a binary relation $R \subseteq G \times M$ has Galois maps between $\wp(G)$ and $\wp(M)^\partial$ and so involves an order reversal on $\wp(M)$. There is an alternative approach which avoids this, while retaining monotonicity of the Galois maps. Define maps $F_R : \wp(G) \to \wp(M)$ and $G_R : \wp(M) \to \wp(G)$ by

$$\begin{aligned} F_R(A) &:= \{\, m \in M \,:\, (\exists g \in A)\,(g,m) \in R \,\}, \text{ and} \\ G_R(B) &:= \big\{\, g \in G \,:\, (\forall m \in M)\,\big((g,m) \in R \Rightarrow m \in B\big) \,\big\}, \end{aligned}$$

for all $A \subseteq G$ and $B \subseteq M$. Then (a straightforward **Mini-exercise**)

(i) (F_R, G_R) is a Galois connection between $\wp(G)$ and $\wp(M)$;
(ii) Conversely, given any Galois connection (F, G) between the powersets $\wp(G)$ and $\wp(M)$, define $R \subseteq G \times M$ by

$$R := \{\, (g,m) \in G \times M \,:\, m \in F(\{g\}) \,\}.$$

Then $(F, G) = (F_R, G_R)$.

We have already seen in particular instances how a context $(G.M, R)$ or, in other parlance, a binary relation $R \subseteq G \times M$, gives rise to a complete lattice. We seek to expose the relationship between Galois connections and complete lattices in general. But first we need to have available the fundamental properties of Galois connections.

6.7 Galois Connections: Basic Properties

It is immediate from the definition that $({}^{\triangleright}, {}^{\triangleleft})$ is a Galois connection between P and Q if and only if $({}^{\triangleleft}, {}^{\triangleright})$ is a Galois connection between Q^∂ and P^∂. Consequently we have a 'buy one, get one free' situation, and only need below to prove one from each pair of mutually dual assertions. The names attached to properties (Gal1) and (Gal3) derive from their interpretation in the case that $\leqslant$ is the relation $=$.

Assume $({}^{\triangleright}, {}^{\triangleleft})$ is a Galois connection between P and Q. Then

(Gal1) **Cancellation Rule:** $p \leqslant p^{\triangleright\triangleleft}$ for all $p \in P$ and $q^{\triangleleft\triangleright} \leqslant q$ for all $q \in Q$.
(Gal2) **Monotonicity Rule:** ${}^{\triangleright}$ and ${}^{\triangleleft}$ are both monotone.
(Gal3) **Semi-inverse Rule:** $p^{\triangleright\triangleleft\triangleright} = p^{\triangleright}$ and $q^{\triangleleft\triangleright\triangleleft} = q^{\triangleleft}$ for all $p \in P$ and $q \in Q$.

Proof. Throughout, p, p_1, p_2 and q denote elements of P and Q, respectively.
(Gal) $\implies$ (Gal1):

$$\begin{aligned} p \in P &\Longrightarrow p^{\triangleright} \leqslant p^{\triangleright} && \text{(by (po1))} \\ &\Longleftrightarrow p \leqslant p^{\triangleright\triangleleft} && \text{(by (Gal) with } q = p^{\triangleright}\text{)}. \end{aligned}$$

(Gal) $\implies$ (Gal2):

$$\begin{aligned} p_1 \leqslant p_2 &\Longrightarrow p_1 \leqslant p_2{}^{\triangleright\triangleleft} && \text{(by (Gal1), (po3))} \\ &\Longleftrightarrow p_1{}^{\triangleright} \leqslant p_2{}^{\triangleright} && \text{(instance of (Gal)).} \end{aligned}$$

(Gal) $\implies$ (Gal3):

$$p \in P \Longrightarrow p^{\triangleright} \leqslant p^{\triangleright\triangleleft\triangleright} \qquad \text{(instance of (Gal1))}$$

and

$$\begin{aligned} p \in P &\Longrightarrow p^{\triangleright\triangleleft} \leqslant p^{\triangleright\triangleleft} && \text{(instance of (po1))} \\ &\Longleftrightarrow p^{\triangleright\triangleleft\triangleright} \leqslant p^{\triangleright} && \text{(instance of (Gal)).} \end{aligned}$$

Hence, by (po2), $p^{\triangleright} = p^{\triangleright\triangleleft\triangleright}$. □

Note that (Gal 3) may be stated as $\mathrm{id}_P \sqsubseteq {\triangleright}\circ{\triangleleft}$ (in $\langle P \to Q\rangle$) and $\mathrm{id}_Q \sqsupseteq {\triangleleft}\circ{\triangleright}$ (in $\langle Q \to P\rangle$). In the special case that P and Q carry the antichain orders, the implication (Gal) $\implies$ (Gal3) is just a trivial fact about bijective maps.

Those who hanker after results stated in maximum generality may wonder how far the theory of Galois connections can be taken if posets are replaced by pre-ordered sets. Antisymmetry of $\leqslant$,(po2), was used above in the derivation of (Gal3) from (Gal). This signals that we could not expect more than a smattering of rudimentary properties still to hold in the wider setting of pre-ordered sets.

6.8 $^{\triangleright}$ and $^{\triangleleft}$ Have Isomorphic Images

Property (Gal3) in 6.7 deserves closer scrutiny. What it says is that, given a Galois connection $({}^{\triangleright}, {}^{\triangleleft})$ between posets P and Q, the map ${}^{\triangleleft} \circ {}^{\triangleright}$ acts as the identity when restricted to elements of the form $p^{\triangleright}$ for $p \in P$, and dually.

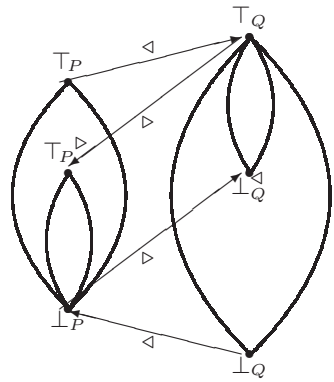


Fig. 11. Isomorphic images of Galois maps

Consequently the maps $^{\triangleright}$ and $^{\triangleleft}$ set up mutually inverse order-isomorphisms between the images

$$P^{\triangleright} := \{\, p^{\triangleright} \;:\; p \in P \,\} \quad \text{and} \quad Q^{\triangleleft} := \{\, q^{\triangleleft} \;:\; q \in Q \,\}$$

(recall Mini-exercise 3.8). Figure 11 illustrates the situation (for the case when P and Q have top and bottom elements, as they do in particular when these posets are complete lattices).

Look again at Example 1 in 6.3. In this case,

$$\wp(P)^{\triangleright} = \mathcal{U}(P)^{\partial} \quad \text{and} \quad (\wp(P)^{\partial})^{\triangleleft} = \mathcal{O}(P),$$

and these are indeed isomorphic: recall 3.16. This example is atypical, in that not just the images but also the domains of $^{\triangleright}$ and $^{\triangleleft}$ (namely $\wp(P)$ and $\wp(P)^{\partial}$) are also order-isomorphic. Certainly Galois maps do not have to have isomorphic domains in general: indeed, as 6.3–6.4 confirm, this need not be the case.

6.9 Equivalent Definitions for Galois Connections

Let P and Q be posets and let $^{\triangleright} : P \to Q$ and $^{\triangleleft} : Q \to P$ maps. Then the following are equivalent:

(i) $(^{\triangleright}, {}^{\triangleleft})$ is a Galois connection (that is, (Gal) holds) ;
(ii) $^{\triangleright}$ and $^{\triangleleft}$ are monotone and $p \leqslant p^{\triangleright\triangleleft}$ and $q^{\triangleleft\triangleright} \leqslant q$ for all $p \in P$, $q \in Q$ (that is, (Gal1) and (Gal2) are satisfied);
(iii) $^{\triangleright}$ and $^{\triangleleft}$ satisfy the following:
  (a) $^{\triangleright}$ is monotone,
  (b) $q^{\triangleleft\triangleright} \leqslant q$ for all $q \in Q$,
  (c) $p^{\triangleright} \leqslant q \Longrightarrow p \leqslant q^{\triangleleft}$ for all $p \in P$ and $q \in Q$.

Proof. We have proved in 6.7 that (i) implies (ii). We now prove that (ii) implies (iii). For $p \in P$ and $q \in Q$ we have

$$\begin{aligned} p \leqslant q^{\triangleleft} &\Longrightarrow p^{\triangleright} \leqslant q^{\triangleleft\triangleright} && \text{(by (Gal2))} \\ &\Longrightarrow p^{\triangleright} \leqslant q && \text{(by (Gal1), (po3)).} \end{aligned}$$

Finally, to prove that (iii) implies (i) we need to show that, when (a) and (b) in (iii) hold, then $p \leqslant q^{\triangleleft}$ implies $p^{\triangleright} \leqslant q$ for $p \in P$ and $q \in Q$. But

$$\begin{aligned} p \leqslant q^{\triangleleft} &\Longrightarrow p^{\triangleright} \leqslant q^{\triangleleft\triangleright} && \text{(by monotonicity of } {}^{\triangleright}\text{)} \\ &\Longrightarrow p^{\triangleright} \leqslant q && \text{(by (b) and (po3)),} \end{aligned}$$

as required. □

A few comments on the relative merits of the alternative characterizations in 6.9 are desirable. The initial definition via (Gal) is the easiest to remember, but in handling Galois connections it is frequently the properties (Gal1) and (Gal2) that are invoked. The third, asymmetric, formulation is occasionally useful too; it obviously has a dual version (**Mini-exercise**: formulate this) which focusses on $^{\triangleleft}$ instead of on $^{\triangleright}$.

There are many more triangle-juggling games that can be played. Here is a sample.

Mini-exercise Assume that $({}^{\triangleright}, {}^{\triangleleft})$ is a Galois connection between P and Q. Prove the following are equivalent for $p_1, p_2 \in P$:

(a) $p_1{}^{\triangleright} \leqslant p_2{}^{\triangleleft}$;
(b) $p_1{}^{\triangleright\triangleleft} \leqslant p_2{}^{\triangleright\triangleleft}$;
(c) $p_1 \leqslant p_2^{\triangleright\triangleleft}$.

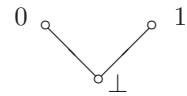
6.10 The Good (and Less Good) Behaviour of Galois Maps

One reason that Galois connections are so important is that they have preservation properties stronger than just monotonicity.

To motivate the general result, let us consider again Example 1 in 6.3. This example introduced the Galois connection on $\wp(P)$ associated with the binary relation $R_{\not\geqslant}$ of a given poset P. For any family $\{A_i\}_{i\in I}$ of subsets of P,

$$\textstyle\bigcap_{i\in I} A_i^{\triangleright} = P \setminus \bigcup_{i\in I} {\downarrow}A_i = P \setminus {\downarrow}(\bigcup_{i\in I} A_i) = \big(\bigcup_{i\in I} A_i\big)^{\triangleright}$$

This tells us that $^{\triangleright} : \wp(P) \to \wp(P)^{\partial}$ preserves infs. Dually, $^{\triangleleft}$ preserves sups. We cannot expect $^{\triangleright}$ to preserve sups or $^{\triangleleft}$ to preserve infs. To see this, take P to be the poset $\{\bot, 0, 1\}$ as shown.



Then

$$\{0\}^{\triangleright} \cup \{1\}^{\triangleright} = (P \setminus \{\bot, 0\}) \cup (P \setminus \{\bot, 1\}) = \{1\} \cup \{0\}$$

whereas

$$(\{0\} \cap \{1\})^{\triangleright} = \emptyset^{\triangleright} = P.$$

In fact, the left and right adjoint maps of a Galois connection $(^{\triangleleft}, ^{\triangleright})$ formed by taking the polar maps associated with a binary relation R preserve infs and sups, respectively. What makes this happen is the universal quantifier in the definition of $^{\triangleright}$. Because of the reversal of the inclusion order on the domain of $^{\triangleleft}$, this map preserves sups rather than infs. Similarly, the existential and universal quantifiers in the definitions of F_R and G_R in 6.3 impart opposite preservation properties to these left and right adjoints.

In complete generality we have the following result:

Let $(^{\triangleright}, ^{\triangleleft})$ be a Galois connection between posets P and Q. Then $^{\triangleright}$ $(^{\triangleleft})$ preserves existing sups (infs) in the sense defined in 4.11.

Proof. We first define $\alpha := \bigvee_P S$ and show that $\alpha^{\triangleright}$ is *an* upper bound for $S^{\triangleright}$. By (Gal2),

$$(\forall s \in S)\, s \leqslant \alpha \Longrightarrow (\forall s \in S)\, s^{\triangleright} \leqslant \alpha^{\triangleright}.$$

Now let q be any upper bound for $S^{\triangleright}$. Then

$$\begin{aligned} (\forall s \in S)\, s^{\triangleright} \leqslant q &\Longleftrightarrow (\forall s \in S)\, s \leqslant q^{\triangleleft} && \text{(by (Gal))} \\ &\Longrightarrow \textstyle\bigvee_P S \leqslant q^{\triangleleft} && \text{(by definition of } \textstyle\bigvee_P S) \\ &\Longleftrightarrow \left(\textstyle\bigvee_P S\right)^{\triangleright} \leqslant q && \text{(by (Gal)).} \end{aligned}$$

We conclude that $\alpha^{\triangleright}$ is the *least* upper bound of $S^{\triangleright}$. □

6.11 Uniqueness of Adjoints: $^{\triangleright}$ from $^{\triangleleft}$ and $^{\triangleleft}$ from $^{\triangleright}$

We have seen that Galois connections, and equations and inequalities concerning them, come in pairs. We show next the important fact that in a Galois connection $(^{\triangleright}, ^{\triangleleft})$ between posets P and Q each of $^{\triangleright}$ and $^{\triangleleft}$ uniquely determines the other. This rests on the ultra-elementary fact that if a non-empty subset Q of a poset has a greatest element, then this greatest element provides $\bigvee_P Q$.

By (Gal), for any $p \in P$,

$$p^{\triangleright} \text{ is an upper bound for } S := \{\, q \in Q \,:\, p \leqslant q^{\triangleleft} \,\},$$

and $p^{\triangleright} \in S$, by (Gal3). Consequently

$$p^{\triangleright} = \min\{\, q \in Q \,:\, p \leqslant q^{\triangleleft} \,\}$$

and likewise

$$q^{\triangleleft} = \max\{\, p \in P \,:\, p^{\triangleright} \leqslant q \,\}.$$

(Recall that $\min S$, $\max S$ denote respectively the least, greatest elements of a subset S of a poset, when these exist.)

We have already met one instance of the latter formula, in 6.5. For another illustration, we return yet again to Example 1 in 6.3. For $A \in \wp(P)$,

$$\begin{aligned} A^{\triangleright} \supseteq B &\iff P \setminus {\downarrow}A \supseteq {\uparrow}B && \text{(since the LHS is an up-set)} \\ &\iff {\downarrow}A \subseteq P \setminus {\uparrow}B && \text{(taking complements)} \\ &\iff A \subseteq P \setminus {\uparrow}B && \text{(since the RHS is a down-set)} \end{aligned}$$

and the largest set A satisfying the final condition is clearly $P \setminus {\uparrow}B$.

It is worth contrasting the formula for $^{\triangleleft}$ above with the formula in (iii) in the following exercise.

6.12 Exercise (Surjective and Injective Galois Maps)

Let $(^{\triangleright}, {}^{\triangleleft})$ be a Galois connection. Prove that the following are equivalent:

(i) $^{\triangleright}$ is a surjective map;
(ii) $^{\triangleleft}$ is an injective map;
(iii) $q^{\triangleleft} = \max\{\, s \in P \,:\, s^{\triangleright} = q \,\}$;
(iv) $^{\triangleleft} \circ {}^{\triangleright} = \mathrm{id}_Q$.

(What we have here is what is known in order theory as a **retraction** of P onto Q with retraction map $^{\triangleright}$ and coretraction map $\triangleleft$. Retraction and coretraction pairs are however not Galois connections in general.)

Formulate also the dual statement.

6.13 A Look Ahead

Let P and Q be posets. A valuable strategy for showing that a map $F\colon P \to Q$ is inf-preserving is to show that it is the left adjoint of some Galois connection (F, G). We know from 6.11 that there is a unique candidate for G and from 6.10 that unless G preserves all existing sups then it cannot be a right adjoint. Monotonicity is also necessary (recall (Gal2)). This leads us to ask the question: does a monotone map preserving existing sups possess an upper adjoint (and dually)? We shall prove in 6.15 that the answer is affirmative provided the domain of the map is a complete lattice, so that *all* sups (and infs) exist. En route we give the order-theoretic version of the result we seek. The content of the assertion in (ii) is that $F^{-1}({\downarrow}q)$ has a *greatest* element; by the dual of 6.14 this inverse image is always a down-set.

6.14 Existence of Adjoints: A Technical Lemma

Let P and Q be posets and $F\colon P \to Q$ a monotone map. Then the following are equivalent:

(i) F is the lower adjoint in a Galois connection, that is, there exists a monotone map $F^\sharp\colon Q \to P$ such that both $F^\sharp \circ F \sqsupseteq \mathrm{id}_P$ and $F \circ F^\sharp \sqsubseteq \mathrm{id}_Q$;
(ii) for each $q \in Q$ there exists a (necessarily unique) $s \in P$ such that $F^{-1}(\downarrow q) = \downarrow s$.

Proof. Assume (i). We claim that $F^{-1}(\downarrow q) = \downarrow F^\sharp(q)$. We have

$$\begin{aligned} p \in F^{-1}(\downarrow q) &\iff F(p) \leqslant q \\ &\implies (F^\sharp \circ F)(p) \leqslant F^\sharp(q) \quad \text{(since } F^\sharp \text{ is monotone)} \\ &\implies p \leqslant F^\sharp(q) \qquad\quad \text{(from } F^\sharp \circ F \sqsupseteq \mathrm{id}_P \text{ \& (po3))} \\ &\iff p \in \downarrow F^\sharp(q). \end{aligned}$$

For the other direction, we have

$$\begin{aligned} p \in \downarrow F^\sharp(q) &\implies F(p) \leqslant (F \circ F^\sharp)(q) \\ &\implies F(p) \leqslant q \\ &\implies p \in F^{-1}(\downarrow q). \end{aligned}$$

Therefore (ii) holds.

Now assume (ii). For each $q \in Q$ we have a unique element $s \in P$, depending on q, such that $F^{-1}(\downarrow q) = \downarrow s$. Define $F^\sharp(q) := s$. Restated, this means that

$$(\forall q \in Q)(\forall p \in P)\ F(p) \leqslant q \iff p \leqslant F^\sharp(q).$$

We now see that the pair $(F, F^\sharp)$ is a Galois connection between P and Q, so that the properties in (i) follow from 6.7. □

6.15 Existence Theorem for Adjoints

Let P and Q be posets and $F\colon P \to Q$ be a map.

(i) Assume P is a complete lattice. Then F possesses an upper adjoint $F^\sharp$ (that is, $(F, F^\sharp)$ is a Galois connection) if and only if F preserves arbitrary sups.
(ii) Assume Q is a complete lattice. Then G possesses a lower adjoint $G^\flat$ (that is, $(G^\flat, G)$ is a Galois connection) if and only if G preserves arbitrary infs.

Proof. We only need to prove (i). The forward implication comes from Proposition 6.10. For the backward implication, we shall use 6.14. Assume that F preserves arbitrary sups. Note first that F is monotone, by 4.11. Let $q \in Q$. We claim that

$$s := \bigvee_P \{\, p \in P : F(p) \leqslant q \,\} \ \left(= \bigvee_P F^{-1}(\downarrow q)\right)$$

is such that $F^{-1}(\downarrow q) = \downarrow s$. It is immediate that $F^{-1}(\downarrow q) \subseteq \downarrow s$. Since F preserves arbitrary joins,

$$F(s) = \bigvee_Q \{\, F(p) \in P \,:\, F(p) \leqslant q \,\}$$

and hence $F(s) \leqslant q$. For any $p \in \downarrow s$, we have $F(p) \leqslant q$, because F is monotone and $\leqslant$ is transitive. Therefore $\downarrow s \subseteq F^{-1}(\downarrow q)$. □

6.16 Postscript

Most of our concrete examples have been of Galois connections between powersets and their duals. In the next section we complete the circle of ideas linking such Galois connections with closure operators, closure systems and complete lattices.

A different focus can be seen in Chapter 4. There the term **pair algebra**, is used for a binary relation $R \subseteq X \times Y$ such that

$$(x, y) \in R \iff F(x) \leqslant y \iff x \leqslant G(y),$$

with (F, G) a Galois connection between X and Y. Unlike the construction of a Galois connection via polars, this imposes restrictions on R: precisely those needed for the necessary sup- and inf-preservation properties to hold. There are however important cases in which a pair algebra arises from the restriction of a Galois connection from powersets to their singleton members.

7 Making Connections, Conceptually

We are now ready to wheel on the machinery of Galois connections and closure operators to develop the basic theory of concept lattices. In the opposite direction this provides a concrete framework within which to interpret the abstract ideas of the preceding section. We shall also thereby put in place the remaining connections between the notions in Figure 1.

7.1 From a Galois Connection to a Closure Operator

Once again, let $({}^{\triangleright}, {}^{\triangleleft})$ be a Galois connection between posets P and Q. The composite maps $c := {}^{\triangleright} \circ {}^{\triangleleft}$ maps P to P and, by (Gal1)–(Gal3), it is a closure operator on P. Its closed sets are the members of

$$P_c := \{\, p \in P \,:\, p^{\triangleright\triangleleft} = p \,\}$$

and, as a set, this is exactly $P^{\triangleright}$ (use (Gal3)). Note however that, while this is a complete lattice, suprema are *not* in general the restricted suprema from P; see 5.8. At this point we encounter a minor irritation. (Gal3) tells us that $q \geqslant q^{\triangleleft\triangleright}$ for all $q \in Q$. So the inequality is the wrong way round for $k := \triangleleft \circ \triangleright$ to be a closure operator (it is an **i**nterior operator instead). But we do have that

$$Q_k := \{\, q \in Q \,:\, q^{\triangleleft\triangleright} = q \,\}$$

is a complete lattice, isomorphic to P_c, via the mutually inverse order-isomorphisms $^\triangleleft$ and $^\triangleright$. We see the closure operator, interior operator

By way of illustration, refer yet again to 6.3. In Example 1, we have $A^{\triangleright\triangleleft} = {\downarrow}A$, for $A \subseteq P$. So here the closure operator c is simply the down-set operator. Just as 5.8 leads us to expect, its image is indeed a complete lattice: it is just the family $\mathcal{O}(P)$ of down-sets, a complete lattice of sets. The up-set operator is also a closure operator on $\wp(P)$. But, when viewed on $\wp(P)^\partial$, it is an interior operator. All this fits with facts which are easy to establish directly.

Example 2 in 6.3, by contrast, shows that the theory of Galois connections may yield results which are not transparent. This theory tells us that $A \mapsto A^{u\ell}$ is a closure operator, c, on $\wp(P)$. The image of this closure operator is a complete lattice (by 5.8). In addition, this lattice contains the sets

$$\{p\}^{u\ell} = {\downarrow}p^\ell = {\downarrow}p \quad (\forall p \in P).$$

The map $p \mapsto {\downarrow}p$ is therefore an order-embedding of P into the complete lattice $c(P)$. This lattice is known as the **D**edekind–MacNeille completion $\mathbf{DM}(P)$ of P. This construction is most familiar as one route by which the rationals can be extended to the reals (with $\pm\infty$ adjoined as top and bottom elements).

7.2 From a Closure Operator to a Galois Connection

In a somewhat contrived way we can recognize that every closure operator arises as the composite of the left and right maps of a Galois connection. To see this, let $c\colon P \to P$ be a closure operator. Define $Q := P_c$, $^\triangleright : P \to P_c$ to be such that $p^\triangleright = c(p)$, and $^\triangleleft : P_c \to P$ to be the inclusion map. Then $c = {}^\triangleright \circ {}^\triangleleft$.

7.3 Contexts and Concepts: Re-setting the Scene

Let us consider again a context (G, M, R). Let $(^\triangleright, ^\triangleleft)$ be the associated Galois connection between $\wp(G)$ and $\wp(M)^\partial$:

$$A^\triangleright := \{\, m \in M : (\forall g \in A)\,(g,m) \in R \,\}, \quad \text{for } A \subseteq G,$$
$$B^\triangleleft := \{\, g \in G : (\forall m \in B)\,(g,m) \in R \,\} \quad \text{for } B \subseteq M.$$

We recall that, given $A \subseteq G$ and $B \subseteq M$, the pair $(A, B) \in \wp(G) \times \wp(M)^\partial$ is called a **c**oncept if

$$A = B^\triangleleft \quad \text{and} \quad A^\triangleright = B.$$

The set of all concepts is denoted by $\mathfrak{B}(G, M, R)$.

7.4 Ordering Concepts

The fewer objects we consider, the more shared attributes they are likely to possess. More precisely, for concepts $(A_1, B_1), (A_2, B_2)$,

$$\begin{aligned} A_1 \subseteq A_2 &\iff A_1 \subseteq B_2^\triangleleft && ((A_2, B_2) \text{ a concept}) \\ &\iff A_1^\triangleright \supseteq B_2 && (\text{by (Gal)}) \\ &\iff B_1 \supseteq B_2 && ((A_1, B_1) \text{ a concept}). \end{aligned}$$

So we have:

$$(A_1, B_2) \leqslant (A_2, B_2) \text{ in } \wp(G) \times \wp(M)^\partial \iff A_1 \leqslant A_2 \text{ in } \wp(G) \iff B_1 \leqslant B_2 \text{ in } \wp(M)^\partial.$$

As a by-product we have that, for concepts, $(A, B_1) = (A, B_2)$ if and only if $B_1 = B_2$ and, likewise, any concept is uniquely determined by its second component. Not surprising: recall 6.11.

7.5 Three for the Price of One: A Trinity of Complete Lattices

We define

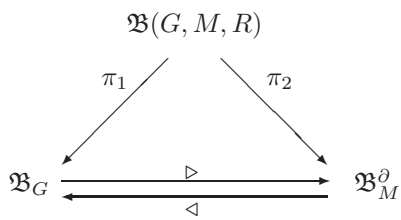
$$\mathfrak{B}_G := \{\, A \in \wp(G) \,:\, (\exists B \subseteq M)\,(A, B) \in \mathfrak{B}(G, M, R) \,\},$$
$$\mathfrak{B}_M := \{\, B \in \wp(M)^\partial \,:\, (\exists A \subseteq G)\,(A, B) \in \mathfrak{B}(G, M, R) \,\}.$$

Note that 7.5 tells us that the natural projections $\pi_1 \colon \mathfrak{B}(G, M, R) \to \wp(G)$ and $\pi_2 \colon \mathfrak{B}(G, M, R) \to \wp(M)^\partial$ are order-embeddings.

The polar maps $^\triangleright$ and $^\triangleright$ take us to and fro between $\mathfrak{B}_G$ and $\mathfrak{B}_M$. Now 7.1 tells us immediately that, as sets,

$$\mathfrak{B}_G := c(\wp(G)), \quad \text{where } c := {}^\triangleleft \circ {}^\triangleright,$$
$$\mathfrak{B}_M := k(\wp(M)), \quad \text{where } k := {}^\triangleright \circ {}^\triangleleft,$$

and we have a commutative diagram



with the indicated maps setting up order-isomorphisms.

Further, 7.1 tells us that $\mathfrak{B}_G$ and $\mathfrak{B}_M$ are complete lattices. Explicitly we have

$$\bigwedge_{\mathfrak{B}_G} \{A_j\}_{j\in J} = \bigcap_{j\in J} A_j,$$
$$\bigvee_{\mathfrak{B}_G} \{A_j\}_{j\in J} = \Big(\bigcup_{j\in J} A_j\Big)^{\triangleright\triangleleft};$$
$$\bigvee_{\mathfrak{B}_M} \{B_j\}_{j\in J} = \bigcap_{j\in J} B_j,$$
$$\bigwedge_{\mathfrak{B}_M} \{\, B_j \,:\, j \in J \,\} = \Big(\bigcup_{j\in J} B_j\Big)^{\triangleleft\triangleright}.$$

This implies that the concepts $\mathfrak{B}(G,M,R)$ form a complete lattice, in which sups and infs are given by

$$\bigvee_{j\in J}(A_j,B_j) = \big((\bigcup_{j\in J} A_j)^{\triangleright\triangleleft}, \bigcap_{j\in J} B_j\big),$$
$$\bigwedge_{j\in J}(A_j,B_j) = \big(\bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\triangleleft\triangleright}\big).$$

Our next objective is to see how the lattice $\mathfrak{B}(G,M,R)$ encodes G, M and R.

7.6 Manufacturing Concepts

For any sets A and B in G and M respectively, $(A^{\triangleright}, A^{\triangleright\triangleleft})$ and $(B^{\triangleleft\triangleright}, B^{\triangleleft})$ are concepts, by (Gal3). In particular, for each $g \in G$ and $m \in M$,

$$\gamma(g) := (g^{\triangleright\triangleleft}, g^{\triangleright}) \quad \text{and} \quad \mu(m) := (m^{\triangleleft}, m^{\triangleleft\triangleright})$$

are concepts, and we have maps $\gamma\colon G \to \mathfrak{B}(G,M,R)$ and $\mu\colon M \to \mathfrak{B}(G,M,R)$. Further, we can capture the relation R using these maps, as follows:

$$\begin{aligned}
(g,m)\in R &\iff g \in m^{\triangleleft} && \text{(definition of } {}^{\triangleleft}\text{)}\\
&\iff \{g\} \subseteq m^{\triangleleft} && \text{(set notation)}\\
&\iff g^{\triangleright\triangleleft} \subseteq m^{\triangleleft\triangleright\triangleleft} && \text{(by (Gal2) for } \Rightarrow\text{, (Gal1) for } \Leftarrow\text{)}\\
&\iff g^{\triangleright\triangleleft} \subseteq m^{\triangleleft} && \text{(by (Gal1))}\\
&\iff \gamma(g) \subseteq \mu(m) && \text{(see 7.3).}
\end{aligned}$$

7.7 Density: Generating all Concepts via γ or μ

For all $(A,B) \in \mathfrak{B}(G,M,R)$,

$$\pi_2(\textstyle\bigvee\gamma(A)) = \pi_2(\bigvee_{g\in A}\gamma(g)) = \bigvee_{g\in A} g^{\triangleright} = \bigcap_{g\in A} g^{\triangleright} = (\bigcap_{g\in A} g)^{\triangleright} = A^{\triangleright} = B.$$

This shows that every element of $\mathfrak{B}_M$ is obtained by taking intersections of sets of elements of the form $g^{\triangleright}$. Dually, every element of $\mathfrak{B}_G$ is obtained by taking unions of sets of elements of the form $m^{\triangleleft}$. Expressed another way, this says that every element of $\mathfrak{B}(G,M,R)$ is generated from $\gamma(G)$ by taking sups and from $\mu(M)$ by taking infs. Note that the empty sup and inf are allowed here. They give, respectively, $\bot$ and $\top$ in $\mathfrak{B}(G,M,R)$ (recall 4.7).

We label $\mathfrak{B}(G,M,R)$ by giving a label g to each concept $\gamma(g)$ and a label m to each concept $\mu(m)$. Then to find $g^{\triangleright}$ we simply look for those M-labels in $\uparrow g$, and to find $m^{\triangleleft}$ we look for the G-labels in $\downarrow m$.

Now look back to the example in 2.2. By taking intersections of sets of the form $Q^{\triangleleft}$ for Q ranging over the output states and of the form $p^{\triangleright}$ for p ranging over the input states we were generating all the concepts, in two ways. What we drew in Figure 3 was, of course, the concept lattice, labelled in this way.

Table 2. Contexts for Mini-exercise 7.7

	A	B	C
a	×	×	×
b		×	×
c			×

(i)

	A	B	C
a		×	×
b	×		×
c	×	×	

(ii)

Mini-exercise Draw and label the concept lattices for the contexts shown in Table 2 (take care over the top and bottom elements):

The contexts in the preceding Mini-exercise are small enough for the concept lattices to be obtained quite easily. However for contexts which are sufficiently large for concept lattices to be of genuine benefit in analysing them, a systematic method for obtaining the lattices becomes necessary. Chapter 3 of ILO2 presents an algorithm for obtaining all the concepts of a context.

7.8 From a Complete Lattice to a Concept Lattice

To complete our circle of ideas, we note that the properties we have derived in the preceding subsections characterize concept lattices. Assume that L is a complete lattice and that $\gamma\colon G \to L$ and $\mu\colon M \to L$ are mappings such that $\gamma(G)$ is sup-dense and $\mu(M)$ is inf-dense in L and $R \subseteq G \times M$ is such that $(g, m) \in R$ if and only if $\gamma(g) \leqslant \mu(m)$. Then it can be proved that $\mathfrak{B}(G, M, R) \cong L$. We refer to ILO2, Chapter 3 or Ganter & Wille [8], pp. 21–22.

Given any complete lattice L, we can choose $G = M = L$ and $R = R_{\leqslant}$, with $\gamma = \mu = \mathrm{id}_L$. The conditions of the preceding paragraph are clearly satisfied and hence $L \cong \mathfrak{B}(L, L, R_{\leqslant})$. We conclude that *every* complete lattice can be regarded as a concept lattice.

Taking a poset P rather than a complete lattice L, $\mathfrak{B}(L, L, R_{\leqslant})$, or more conveniently its image under π_1 (recall 7.5) yields the Dedekind–MacNeille completion of P. The well-known fact that P order-embeds into its completion as a subset which is both sup- and inf-dense is an instance of general properties of concept lattices. On the other hand, the other important feature of the embedding, *viz.* that it preserves all sups and infs which exist in P, depends on properties particular to this context. For further details see ILO2, Chapter 7.

7.9 The Case for the Defence

A criticism that is sometimes levelled at concept analysis is that it lacks depth. As we have presented it here, it interacts in an illuminating way with Galois connections, closure operators and complete lattices on a theoretical level. But does it go deeper than pretty diagrams? Certainly good diagrams of big concept lattices reveal dependencies and relationships not visible at all from a context table: a nice example is provided in Ganter & Wille [8], pp. 28–30, concerning

political groupings of more than 100 developing countries. The evangelists claim with justification that the analysis of data via the associated concept lattices goes beyond this. As a good piece of propaganda we draw attention to the analysis of binary relations to be found in Ganter & Wille [8], pp. 86–90. A list of properties of such relations is presented. Some of these (symmetric, reflexive, transitive, for example) appear frequently in mathematical and computer science contexts. Others have arisen in more detailed studies of relations, for example within measurement theory (see Schader [17]). Examples are **c**onnex (xRy or yRx for all $x \neq y$) and **n**egatively transitive (xRy true and yRz false imply xRz false, for all x, y, z). The objective is to expose which implications between the various properties are universally valid and which are not. This is done by seeking a set of 'test' examples, on small sets, and taking these as the objects for a context, and a list of properties as the attributes. The objective is to set up this context in such a way that the lists of properties and attributes are, in a sense which can be made precise, complete and irredundant. This is achieved by the theory developed by Wille *et al.* for attribute exploration. The resulting concept lattice (Ganter & Wille [8]), p. 90) displays the implications between the properties and the examples which witness the non-implications. Concept analysis allows the measurement theorists' empirical treatment to be illuminated by an approach which is systematic and backed up by theory.

7.10 Summing Up

We have shown

- how every closure system is a complete lattice and how every complete lattice can be realized concretely as a closure system (5.4 and 5.6);
- how closure operators give rise to closure systems and vice versa (5.9);
- how Galois connections give rise to closure operators, and vice versa (7.1 and 7.2);
- how every context, or equivalently every binary relation, gives rise to a Galois connection via its polars (2.4).
- how every context, or equivalently every binary relation, gives rise to a complete lattice, its concept lattice (7.5);
- how every complete lattice can be realized as a concept lattice (7.8).

All the arrows in Figure 1 are now in place, some of them by transitivity.

8 The Existence of Fixed Points

Fixed points, especially as models of recursive programs, are so ubiquitous in computer science that it would be gratuitous to include here any discussion of their provenance. The study of fixed points may be regarded as having two aspects: (i) the existence of fixed points and (ii) calculus with fixed points (and prefix points). Since so many computational models have an underlying partial order, posets of suitably special kinds provide a natural setting in which to

investigate fixed point theory. Here we concentrate on the existence of fixed points and highlight in particular two famous fixed point theorems:

- the Knaster–Tarski Fixed Point Theorem, in complete lattices, asserting that every monotone endofunction on a complete lattice has least and greatest fixed points;
- the Fixed Point Theorem for CPOs (with which the name of Park is frequently associated), asserting that a monotone endofunction on a CPO has a least fixed point, and which has a very simple, algorithmic, proof if F is continuous, in the sense of 8.10 below.

Lassez *et al.* [11] delve into the rather obscure and complicated history of the various fixed point theorems and their attributions.

8.1 Fixed Points and Least Fixed Points: Definitions

For completeness we record the following well-known definitions. Let P be a poset and let $F\colon P \to P$ be a map. We say $x \in P$ is a **fixed point** of F if $F(x) = x$, and a **prefix point** if $F(x) \leqslant x$. The set of all fixed points (prefix points) of F is denoted by $\operatorname{fix} F$ ($\operatorname{pre} F$); both carry the inherited order. The least element of $\operatorname{fix} F$, when it exists, is denoted by $\mu(F)$. In computational contexts it is usually the *least* fixed point of a map $F\colon P \to P$ which is the one sought. Very frequently the poset P is some CPO of partial maps $S_\perp \multimap S_\perp$ on a flat CPO $S_\perp$. For example, that favourite example of a recursive definition, the factorial function **fact**, arises as the least fixed point of the map $\pi \mapsto \overline{\pi}$, for π a partial map on the natural numbers (including 0), where $\overline{\pi}(0) = 1$ and $\overline{\pi}(k) = k\pi(k-1)$ for $k > 0$.

8.2 Characterizing Least Fixed Points via Least Prefix Points

Let P be a poset and let F be a monotone endofunction on P. Assume that F possesses a least prefix point $\mu_*(F)$. Then the least fixed point $\mu(F)$ exists and satisfies

$$F(x) \leqslant x \Rightarrow \mu(F) \leqslant x \qquad \text{(the } \textbf{Induction Rule}\text{)}.$$

Indeed, $\mu(F) = \mu_*(F)$. This is the case whenever P is a complete lattice.

Proof. Assume that $\mu_*(F)$ exists. We want to show that $\mu_*(F) \in \operatorname{fix} F$. Since $\operatorname{fix} F \subseteq \operatorname{pre} F$ we must then have $\mu(F) = \mu_*(F)$.

As $\mu_*(F) \in \operatorname{pre} F$ by definition, we have $F(\mu_*(F)) \leqslant \mu_*(F)$. Applying the monotone map F we get $F(F(\mu_*(F))) \leqslant F(\mu_*(F))$, so that $F(\mu_*(F)) \in \operatorname{pre} F$. Since $\mu_*(F)$ is the *least* element of $\operatorname{pre} F$ we therefore have $\mu_*(F) \leqslant F(\mu_*(F))$. Hence, $\mu_*(F)$ is indeed a fixed point of F.

In case P is a complete lattice there is an obvious candidate for $\mu_*(F)$, namely $\bigwedge \operatorname{pre} F$. It follows easily from the monotonicity of F that $\bigwedge \operatorname{pre} F \in \operatorname{pre} F$. □

8.3 The Knaster–Tarski Fixed Point Theorem

Let P be a complete lattice and $F\colon P \to P$ a monotone map. Then

$$\bigwedge \{\, x \in P \,:\, F(x) \leqslant x \,\}$$

is the least fixed point of F. Dually, F has a greatest fixed point, given by

$$\bigvee \{\, x \in P \,:\, F(x) \geqslant x \,\}.$$

Proof. This is immediate from 8.2 and its dual. □

8.4 Exercise

Let P be a complete lattice and let $F\colon P \to P$ be a monotone map. Stated in the terminology of this section, the Prefix Lemma in 5.8 says that $\operatorname{pre} F$ is a complete lattice. Prove that $\operatorname{fix} F$ is a complete lattice. Specifically, for X a subset of P, define

$$Y := \{\, y \in P \,:\, (\forall x \in X)\, y \leqslant F(y) \leqslant x \,\}$$

and let $\alpha := \bigvee_P Y$. Prove that $\alpha = \bigwedge_{\operatorname{fix} F} X$ and then appeal to 4.9.

8.5 Exercise

Let P be a complete lattice and define $F\colon \wp(P) \to \wp(P)$ by $F\colon A \mapsto {\downarrow}\bigvee A$. Show that F is monotone and that $\operatorname{fix} F \cong P$. (Consequently, *every* complete lattice is, up to isomorphism, a lattice of fixed points—this last fact is actually a triviality (why?).)

8.6 From Complete Lattices to CPOs

The Knaster–Tarski Theorem is mathematically slick, but it has certain deficiencies as regards computer science. First of all, in a semantic domain a top element will represent an overdetermined, or inconsistent, state, and topless models are often to be preferred. Also, the Knaster–Tarski proof identifies a fixed point, but does not supply an algorithm for computing it.

Let P be a poset with $\bot$ and let $F\colon P \to P$ be a monotone endofunction. Then a natural way to try to construct a fixed point is to consider the sequence $\bot, F(\bot), F(F(\bot)), \ldots$ and to hope that 'in the limit' we arrive at a fixed point. To make this precise, we let, recursively, $x_0 = \bot$ and $x_{n+1} := F(x_n)$ for $n \geqslant 0$. Then an easy proof by induction shows that we have a chain

$$x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant x_{n+1} \leqslant \ldots,$$

each member of which is half way to being a fixed point. Make the assumptions

(a) $\alpha := \bigvee \{\, x_n \,:\, n = 0, 1, 2, \ldots \}$ exists, and
(b) $F(\bigvee C) = \bigvee F(C)$ for any non-empty chain C.

Then

$$\begin{aligned} F(\alpha) &= \bigvee_{n\geqslant 0} F(x_n) && \text{(by (b), with } C \text{ as } \{x_n\}) \\ &= \bigvee_{n\geqslant 0} x_{n+1} && \text{(by definition of } x_{n+1}) \\ &= \bigvee_{n\geqslant 0} x_n && \text{(since } x_0 = \bot) \\ &= \alpha && \text{(by definition of } \alpha). \end{aligned}$$

So our assumptions allow us to assert that F has a fixed point. Note that infima play no role in the argument. The condition on F is stronger than monotonicity (recall 4.11); the condition on P is strictly weaker than lattice completeness (look at $\mathbb{N}_\bot$). This suggests that we should look at posets in which suprema exist for all chains (the supremum of the empty chain supplies $\bot$; see 4.7). It turns out to be no more restrictive, and in some ways more natural, to consider sets more general than chains.

8.7 A Sense of Direction

In the context of information orderings, an element is frequently the supremum of approximations below it: partial maps approximating total maps, finite strings approximating infinite strings, and so on. Take the example of a map $F\colon \mathbb{N} \to \mathbb{N}$. We may obtain F as the supremum of the partial maps which are the restrictions of F to finite subsets of $\mathbb{N}$. These approximations are mutually compatible, in the sense that any two of them give consistent information about F at any $k \in \mathbb{N}$ at which both are defined. Conversely, suppose we have a family D of elements of $\mathbb{N} \multimap \mathbb{N}$ with the property that, for any pair π_1 and π_2 in D, we have $\pi \in D$ with $\pi_1 \leqslant \pi$ and $\pi_2 \leqslant \pi$ in the ordering of $\mathbb{N} \multimap \mathbb{N}$ (so π extends both of π_1, π_2). Then the union of the graphs of the elements of D defines a partial map, which is $\bigvee D$.

Let S be a non-empty subset of a poset P. Then S is said to be **directed** if each pair of elements $x, y \in S$ has an upper bound lying in S. An easy induction shows that S is directed if and only if, for each non-empty finite subset F of S, there exists $z \in S$ such that $z \geqslant y$ for all $y \in F$, that is, $z \in F^u$. This definition should be compared with another which naturally arises in computer science contexts, namely that of a **consistent** set: S is consistent if for every non-empty finite subset F of S there exists $z \in P$ such that $z in F^u$.

Clearly any non-empty chain in a poset is directed. In a poset P satisfying (ACC) (as defined in 4.10) a non-empty subset D of P is directed if and only if D has a greatest element. We leave the easy verification as a **Mini-exercise**.

8.8 Exercise (Sups and Directed Sups Related)

Let P be a complete lattice. Assume that $\emptyset \neq S \subseteq P$. Prove that

$$\bigvee S = \bigsqcup \left\{ \bigvee F \,:\, \varnothing \neq F \subseteq S,\ F \text{ finite} \right\}.$$

8.9 CPOs

We say that a poset P is a **CPO** (an abbreviation for **complete partially ordered set**) if

(i) P has a bottom element, $\bot$,
(ii) $\bigvee D$ exists for each directed subset D of P.

We shall write $\bigsqcup D$ in place of $\bigvee D$ when D is directed, as a reminder of the directedness.

An appropriate setting for the fixed point existence proof in 8.6 can now be seen to be a CPO. The need for $\bot$ in that proof is clear. In other contexts posets in which directed sups exist, but which do not necessarily possess $\bot$, are important. Often, such a poset is called a **dcpo** (and a CPO given the name **dcppo**, the extra 'p' signifying 'pointed').

Here are some simple examples.

(1) Any complete lattice is a CPO.
(2) Let S be an antichain. Then $S_\bot$ is a CPO. In particular, $\mathbb{N}_\bot$ is a CPO. Indeed, this example is a motivation for the introduction of the lifting construction. More generally, any poset P with $\bot$ satisfying (ACC) is a CPO.
(3) Any closure system which is closed under directed unions is a CPO, with $\bigsqcup$ coinciding with $\bigcup$ (recall reflat-sets).

Our examples justify the introduction of directed sets. However it turns out, highly non-trivially, that a poset with $\bot$ is a CPO if and only if it is **chain-complete** (that is, $\bigsqcup C$ exists for every non-empty chain C).

8.10 Directed Sets and Continuity

It is natural to regard $\bigvee S$ as the limit of S precisely when S is directed. Accordingly a map F between CPOs preserving directed sups is called **continuous**: this means that $\bigsqcup F(D) = F(\bigsqcup D)$ for any directed set D. The following facts are elementary: if D is a directed subset in P and if F is monotone then the set $F(D)$ is directed and $\bigsqcup F(D) \leqslant F(\bigsqcup D)$ (see 4.11). Also, since $\{x, y\}$ is directed when $x \leqslant y$, it is easily seen that every continuous function is monotone.

Mini-exercise Define $F\colon \wp(\mathbb{N}) \to \wp(\mathbb{N})$ by

$$F(S) = \begin{cases} \varnothing & \text{if } S \text{ is finite,} \\ \mathbb{N} & \text{otherwise.} \end{cases}$$

Prove that F is monotone but fails to preserve the (directed) supremum of the family of finite subsets of $\mathbb{N}$. So not every monotone map is continuous.

Mini-exercise Prove that a monotone endofunction $F\colon P \to P$ is continuous whenever P satisfies (ACC).

The topological connotations of continuity are more than superficial analogies. The symbiotic relationship between topology and order has been extensively researched and exploited; see Gierz *et al.* [9], and Abramsky & Jung [2] and the references therein.

8.11 New CPOs from Old

A significant virtue of CPOs as a class in which to work is that it has excellent closure properties. Given CPOs P and Q the following are also CPOs:

$$(P \dot{\cup}\ Q)_\perp, \qquad P \times Q, \qquad [P \to Q];$$

Here $[P \to Q]$ denotes the set of of all continuous maps from P to Q, with the customary pointwise ordering. There is an alternative to the **separated sum** above. We can also form the **coalesced sum**) by taking $P \dot{\cup} Q$ and identifying $\perp_P$ and $\perp_Q$. The proofs that all these constructions do indeed yield CPOs are notation chases, of varying difficulty. See ILO2, Chapter 8, for guidance on the harder ones.

8.12 Fixed Point Theorem for a Continuous Function on a CPO

With the appropriate terminology in place we can record as a theorem what we proved in 8.6, and claim a bit more.

Let P be a CPO, let $F\colon P \to P$ be a continuous map and let

$$\alpha := \bigsqcup_{n \geqslant 0} F^n(\perp).$$

Then the least fixed point $\mu(F)$ exists and equals α. **Mini-exercise**: prove the leastness assertion.

8.13 From Continuity to Monotonicity

Continuity is often an undesirably strong assumption. Relaxing continuity to monotonicity is feasible but requires more powerful mathematical machinery. One approach builds on the elementary proof in 8.6, using ordinals and transfinite induction. Several other approaches to proving that any monotonic endofunction on a CPO has a least fixed point have been devised, most notably the recent elegant proof obtained by D. Pataraia. This proof is presented, with Pataraia's permission, in ILO2. Unlike earlier proofs it does not rely on first proving the independently interesting result that a fixed point exists for any increasing endofunction on a CPO (F being **increasing** if $x \leqslant F(x)$ for all x). The method we adopt below is much simpler, but necessitates calling on a property of CPOs which we shall take as an axiom: it says that any CPO has a maximal element.

8.14 An Assumption: Zorn's Lemma (ZL), CPO Form

Let P be a CPO. Then $\operatorname{Max} P \neq \varnothing$.

To indicate that this is very plausible, we consider first the special case that the CPO P satisfies (ACC). Let $x_0 = \bot$. If $x_0 \notin \operatorname{Max} P$, pick $x_1 > x_0$. If $x_1 \notin \operatorname{Max} P$, pick $x_2 > x_1$, and so on. This process terminates, at a maximal element, because we cannot have an infinite strictly ascending chain in P; the maximal element is the largest element of our chain, and necessarily its supremum. Thus for CPOs satisfying (ACC) we cannot climb for ever strictly upwards via a chain. For a general CPO the existence of suprema for directed sets, and in particular for non-empty chains, suggests that every non-empty chain is 'capped' by a maximal element of P. Those content to accept (ZL) for CPOs may without detriment skip over the next paragraph.

Replacing the informal 'and so on' above by a rigorous argument requires invocation of what is known as the Axiom of Choice. This famous optional add-on to ZF set theory is one of a cluster of equivalent statements. One of these is Zorn's Lemma in the form it is most often stated, another is (ZL) for CPOs. We do not wish to digress to present this background material here. A self-contained account of the equivalences most useful in the context of posets and lattices is given in ILO2, Chapter 10. Mathematics, and in particular order theory and algebra, are considerably impoverished if we deny ourselves (AC) and (ZL). However it can be proved that a set-theoretic world in which the *negation* of (AC) operates is not internally inconsistent.

8.15 The Fixed Point Theorem for a Monotone Endofunction on a CPO

Assume that P is a CPO and that $F\colon P \to P$ is monotone. Then F has a least fixed point.

Proof. (with (ZL); due to A.W. Roscoe, [15], p. 175) Let

$$Q := \{\, x \in P \,:\, x \leqslant F(x) \;\&\; (\forall y \in \operatorname{fix} F)\, x \leqslant y \,\}.$$

It is important to realize that in the definition of Q we are not claiming that F has a fixed point: The second clause in the definition is satisfied vacuously if $\operatorname{fix} F = \emptyset$. It serves to ensure that a fixed point of F in Q is the *least* fixed point of F.

We first claim that Q is a CPO (in the inherited order). Certainly $\bot_P \in Q$. Now let $D \subseteq Q$ be directed and denote $\bigsqcup_P D$ by α. Then, because $F(D)$ is directed and $\bigsqcup F(D) \leqslant F\big(\bigsqcup D\big)$ (recall the remarks in 8.9),

$$\begin{aligned} x \in D &\Longrightarrow x \leqslant F(x) && \text{(by the definition of } Q\text{)} \\ &\Longrightarrow x \leqslant \bigsqcup F(D) && \text{(the join existing as } F(D) \text{ is directed)} \\ &\Longrightarrow x \leqslant F\big(\bigsqcup D\big) = F(\alpha) && \text{(by (po3)))}. \end{aligned}$$

Hence $\alpha \leqslant F(\alpha)$. Also

$$\begin{aligned} y \in F &\Longrightarrow (\forall x \in D)\, x \leqslant y && \text{(since } D \subseteq Q\text{)} \\ &\Longrightarrow \alpha \leqslant y && \text{(by definition of } \alpha\text{)}. \end{aligned}$$

Therefore $\alpha \in Q$. This proves that Q is a CPO. (Compare this with the proof in 5.6).

By (ZL) for CPOs, Q has a maximal element, β say. By definition of Q we have $\beta \leqslant F(\beta)$. Clearly
(a) $F(\beta) \leqslant F(F(\beta))$ (since F is monotone), and
(b) $y \in \operatorname{fix} F \implies \beta \leqslant y$, so that $F(\beta) \leqslant F(y) = y$.
Hence $F(\beta) \in Q$, so maximality of β gives $\beta = F(\beta)$. By the definition of Q, we must have $\beta = \mu(F)$. □

8.16 Exercise

This exercise indicates a proof that the set of fixed points $\operatorname{fix} F$ of a monotone endofunction F on a CPO P is itself a CPO; compare with 8.4).)

Let D be a directed set in $\operatorname{fix} F$ and let $\alpha := \bigsqcup D$. Prove, with the aid of 8.2, that F maps $\uparrow\alpha$ to $\uparrow\alpha$. Deduce that G, the restriction of F to $\uparrow\alpha$, has a least fixed point, which is given by $\bigsqcup_{\operatorname{fix} F} D$.

8.17 Exercise

We have used (ZL) for CPOs to prove the fixed point theorem in 8.15. This exercise gives another application of (ZL) within order theory. Let X be a set. Let

$$\mathcal{S} := \{\, R_< \in \wp(X \times X) : < \text{ is a strict partial order on } X \,\}.$$

(i) Prove that $\mathcal{S}$ is a CPO.
(ii) Prove that any maximal element of $\mathcal{S}$ is a linear order (that is, it makes X a chain).
(iii) Deduce that any strict partial order is contained in a strict linear order.

(This is hard. Part (iii) is a well-known result from order theory (Szpilrajn's Theorem) whose proof requires Zorn's Lemma in some form.

8.18 Concluding Remarks

The theorems in 8.3 and 8.15 are in a sense optimal. The following converses exist. Both are difficult to prove; see Markowsky [13].

(i) Let L be a lattice and assume that every monotone endofunction $F\colon L \to L$ has a fixed point. Then L is a complete lattice.
(ii) Let P be a poset and assume that every monotone endofunction $F\colon P \to P$ has a least fixed point. Then P is a CPO.

9 Speaking Categorically

9.1 Categories

It is uncommon for a class of structures of a given type to be introduced without an associated class of structure-preserving maps following hard on its heels. Posets + monotone maps is one example. Others are complete lattices + maps preserving $\bigvee$ and $\bigwedge$, groups + group homomorphisms, topological spaces + continuous maps, etc., etc., etc. The recognition that an appropriate unit for study is a class of structures all of the same type, together with a class of structure-preserving maps between these, leads to category theory. Computer scientists have embraced this with such enthusiasm that it seems unnecessary to repeat the formal definitions here. Loosely, a **category** $\mathcal{C}$ consists of a collection $\mathrm{Obj}(C)$ of **objects** and a collection $\mathrm{Arr}(\mathcal{C})$ of **arrows** (also called **morphisms**). Each arrow $f \in \mathrm{Arr}(\mathcal{C})$ has a source $A \in \mathrm{Obj}(\mathcal{C})$ and a target B in $\mathrm{Obj}(\mathcal{C})$, and there is an operation of composition of arrows satisfying a rudimentary set of conditions demanding

(cat1) the existence of an **identity arrow** $\mathrm{id}_A \colon A \to A$ for each object A,
(cat2) arrows can be composed in the same way that maps between sets can be.

The set of arrows in $\mathcal{C}$ with source A and target B will be denoted by $\mathrm{Arr}(A, B)$. Commuting diagrams of objects and arrows expressing properties of categories and *their* structure-preserving maps (called **functors**) form the basis of category theory. Natural constructs within categories—products, limits, and so on—can be described in terms of commuting diagrams, with the philosophy 'everything is an arrow' perhaps sometimes taken to excess.

Every poset (or more generally pre-ordered set) P gives rise to a category we shall call $\mathcal{C}_p$: the objects are just the elements of P, and, for $p, q \in P$, there is an arrow from p to q if and only if $p \leqslant q$, and just such arrow; so we identify arrows with inequality statements. The existence of the arrow id_p for each p is just the reflexivity condition, (po1); the composition law on arrows is guaranteed by transitivity, (po3). Figure 12 indicates the interpretations in categories $\mathcal{C}_P$ of some fundamental categorical notions.

To illustrate, we consider the most complicated of these correlations: Galois connections as adjunctions. An adjunction is a symmetric relation between categories $\mathcal{A}$ and $\mathcal{X}$ set up by a pair of covariant functors $F \colon \mathcal{A} \to \mathcal{X}$ and $G \colon \mathcal{X} \to \mathcal{A}$ which are 'mutually inverse'. Explicitly, suppose that for each $A \in \mathcal{A}$ and each $X \in \mathcal{X}$ there are arrows

$$e_A \colon A \to GF(A) \quad \text{and} \quad \varepsilon_X \colon X \to FG(X).$$

It is said that $\langle F, G, e, \varepsilon \rangle$ sets up an **adjunction** between $\mathcal{A}$ and $\mathcal{X}$ if:

(adj1) for $u \colon A \to B$ in $\mathcal{A}$ and $\varphi \colon X \to Y$ in $\mathcal{X}$, the two square diagrams in Figure 13 commute;

P	$\mathcal{C}_P$
$\bot$	initial object
$\top$	terminal object
dual poset	opposite category
product	product
disjoint sum	coproduct
inf	limit
sup	colimit
monotone map	functor
order-embedding	monomorphism
Galois connection	adjunction
$(\forall p)\, p \leqslant p^{\triangleright\triangleleft}$	unit
$(\forall q)\, q^{\triangleleft\triangleright} \leqslant q$	counit
closure operator	monad

Fig. 12. From posets to categories

(adj2) for $A \in \mathcal{A}$ and $X \in \mathcal{X}$ there is a bijection between $\mathrm{Arr}_{\mathcal{A}}(A, G(X))$ and $\mathrm{Arr}_{\mathcal{X}}(X, F(A))$ associating u and φ as given in the two commuting triangle diagrams of Figure 13.

Of course, $\mathrm{id}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$ and $GF \colon \mathcal{A} \to \mathcal{A}$ are covariant functors and the left-hand square of Figure 13 says precisely that $e \colon \mathrm{id}_{\mathcal{A}} \to GF$ is a **natural transformation**. Similarly for the right-hand square.

All this may look complicated to beginners. But in the special case of a Galois connection $({}^{\triangleright}, {}^{\triangleleft})$ between posets P and Q it just sums up the basic calculus for Galois connections: put

objects	*arrows*	*functors*
$A := p_1$	$u := p_1 \leqslant p_2$	$F := {}^{\triangleright}$
$B := p_2$	$F := q_1 \leqslant q_2$	$G := {}^{\triangleleft}$
$X := q_1$	$e_A := p_1 \leqslant {p_1}^{\triangleright\triangleleft}$	
$Y := q_2$	$\varepsilon_X := q_1 \geqslant {q_1}^{\triangleleft\triangleright}$	

and relabel the diagrams!

Mini-exercise (For those more at home with categories than with posets)

(i) Convince yourself that disjoint sums in P correspond to coproducts in $\mathcal{C}_P$.
(ii) Convince yourself that the notion of supremum in a poset P corresponds to the notion of colimit in the category $\mathcal{C}_P$.

All this is well-trodden territory, mapped out in Maclane [12], for example. Less familiar is the way in which Table 9.1 can be extended by observing that the

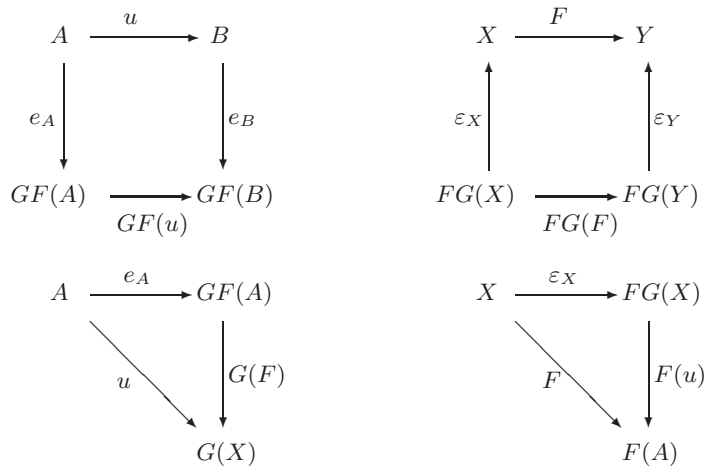


Fig. 13. An adjunction

poset concepts of prefix points and postfix points, match up in the categorical setting with algebras and coalgebras. Fixed point calculus in posets lifts to a fixed point calculus in categories whose calculational rules (the **Fusion Rule**, and so on) are highly useful in the context of coalgebras; Backhouse *et al.* [3] show how the transition from posets to categories can be made.

There is another way in which adjunctions enter into the theory of lattices. A particularly profitable kind of adjunction between categories $\mathcal{A}$ and $\mathcal{X}$ is one in which the categories $\mathcal{A}$ and $\mathcal{X}$ are of different types and we are able to use the adjunction as a tool for studying $\mathcal{A}$ by exploiting its properties as a category and the way that the adjunction mirrors these in $\mathcal{X}$. The most famous such adjunctions are dual adjunctions (set up by contravariant rather than covariant functors) and are dual equivalences rather than just adjunctions. They include those which set up dualities between

Boolean algebras	and	Boolean spaces	(Stone)
Finite distributive lattices	and	Finite posets	(Birkhoff; see 6.2)
Distributive lattices	and	Priestley spaces	(Priestley)

in each case with a natural class of morphisms. For the second of these, the functor from finite posets to finite distributive lattices is, on objects, simply the map taking a poset P to its up-set lattice $\mathcal{U}(P)$. Because of the way that distributive lattices with additional operations provide algebraic models for them, these dualities give a useful means of studying logics of various kinds; for a recent survey see Davey & Priestley [6].

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